Astron. Astrophys. 55, 239—243 (1977)

ASTRONOMY AND ASTROPHYSICS

# Analytical Study of Magneto-acoustic Gravity Waves

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Received July 5, revised July 27, 1976

**Summary.** An analytical study of magneto-acoustic gravity waves propagating in an isothermal atmosphere along the gravity field has been done; we obtain the equation giving the cut-off frequency and we establish the conditions for the existence of a cross-over in the dispersion equation.

**Key words:** waves — linear coupling — magnetic field — gravity — solar atmosphere

## I. Introduction

The propagation of linear waves in an isothermal atmosphere in which a constant magnetic field is embedded (i.e. magneto-acoustic gravity waves = M.A.G. waves) was recently received some attention in the context of the physics of the solar atmosphere (Bel and Mein, 1971; Chen and Lykoudis, 1972; Michalitsamos, 1973; Nakagawa et al., 1973). As a consequence of the complexity of the dispersion equation, all this work has concerned either special case of the propagation (work done numerically) or the existence of trapped modes. Since the combined effect of magnetic pressure gradient and gravity restoring forces could be of importance in atmospheric heating, it would be of interest to study the transfer of energy between gravity waves and M.A.G. waves or between fast and slow M.A.G. waves; an analytical form of the equation relating the phase to the frequency will lead to considerable simplification in these further studies. This equation will be also a useful way to study the phase-lags which have been observed in the solar atmosphere. In this paper, we restrict ourselves to the propagation of M.A.G. waves along the gravitational field and we obtain the analytical form of the equation of the phase propagation of such M.A.G. waves.

## II. Basic Equations

The notation is as used in Bel and Mein's paper (referred to as Paper I). The constant gravitational field is along the $z$-axis, $\boldsymbol{g}=(0,0,-g)$ with $g>0$ and the magnetic field can take any angle with respect to the gravitational one, $\boldsymbol{B}=(B_x,0,B_z)$; the magnetic field is assumed uniform.

The local dispersion equation of M.A.G. waves which propagate along the gravitational field is:

$$\omega^4-(i\gamma g\,k_z+(a^2+V_a^2)\,k_z^2)+a^2\,k_z^2\,V_{az}^2+i\gamma g\,k_z^3\,V_{az}^2=0\,. \quad (1)$$

This is Equation (6) of Paper I with $k_\perp=0$; $k_z$, the $z$-component of the wave vector is put in the form:

$$k_z=\alpha+i\beta\,. \quad (2)$$

In dimensionless variables used in Paper I:

$$\omega=\frac{a}{H}\omega' \quad \alpha=\frac{x}{H} \quad \beta=\frac{y}{H},$$

real and imaginary parts of Equation (1) can be written as:

$$\omega'^4+\omega'^2\,(y-(1+l^2)\,x^2-y^2)+m^2\,((x^2-y^2)^2-4x^2\,y^2)$$
$$-m^2\,y(3x^2-y^2)=0 \quad (3)$$

$$(1+2(1+l^2)\,y)\,\omega'^2-4m^2\,y(x^2-y^2)-m^2(x^2-3y^2)=0. \quad (4)$$

We recall that $l=(V_a/a)$ and $m=(V_a/a)\cos\theta$ where $V_a$ and $a$ are respectively the Alfvèn speed and the sound speed; these equations are Equations (13) and (17) of Paper I.

The elimination of $x^2$ between Equations (3) and (4) leads to Equation (18) of Paper I, which is the relation between the frequency and the growth rate of the amplitude of the wave. It is now necessary to eliminate $y$ from (3) and (4) in order to obtain the phase propagation i.e. the relation between $\omega'$ and $x$. The tedious calculations involved are made in the Appendix.

## III. Results

The relation between $\omega'$ and $x$ is found to be:

$$\omega'^2\,(\omega'^2-(1+l^2)\,x^2)+m^2\,x^4-v^2\,(\omega'^2-m^2\,x^2)=0 \quad (5)$$

where $v^2=v^2(l,m)$ is the square of the dimensionless cut-off frequency and is the real solution of the cubic equation:

$$A_0\,X^3+A_1\,X^2+A_2\,X+A_3=0 \quad (6)$$

[cf. Eq. (I.14) of the Appendix]

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with:

$$A_0 = 16(4m^2 - (1+l^2)^2)^2$$
$$A_1 = -4[((1+l^2)^2 - 4m^2)((1+l^2)(1+m^2) - 12m^2) + 32m^2(1+l^2)(l^2-m^2)]$$
$$A_2 = -m^2[(m^2-l^2)((1+l^2)+27(1+m^2)) + 8((1+l^2)(1+m^2)-6m^2)]$$
$$A_3 = -4m^2 . \tag{7}$$

Using the dimensioned variables and parameters, Equation (5) can be written as:

$$\omega^2(\omega^2 - (a^2+V_a^2)\alpha^2) + a^2 V_a^2 \cos^2\theta\, \alpha^4 - \omega_0^2(\omega^2 - V_a^2 \cos\theta\, \alpha^2) = 0 \tag{8}$$

where

$$\omega_0 = \frac{a}{H} v .$$

This is the phase propagation of the M.A.G. waves along the gravitational field. It is easily seen that Equation (8) reduces to the usual magneto-acoustic wave equation as the scale height $H$ tends to infinity and to the usual internal gravity wave equation for $V_a = 0$ (See below the limiting value of $v$ for $l=m=0$). $\omega_0$ is the cut-off frequency.

*1. Cut-off Frequency*

The dimensionless squared cut-off frequency $v^2(l, m)$ is the real solution of Equation (6). The explicit solutions of Equation (6) can be written easily but these expressions are somewhat complicated and we have preferred to find a numerical result; this is of no great importance since $v^2$ enters only as a parameter in Equation (8) and it is rather the form of this equation which will be useful for further studies.

The dimensionless cut-off frequency is plotted in Figure 1 as a function of $l$, for $\theta=0$ ($l=m$, the magnetic field is parallel to the gravitational field), $\theta = \frac{\pi}{6}$, $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{3}$, $\theta = \frac{\pi}{2}$ ($m=0$, the magnetic field is normal to the gravitational field). It can be easily shown that for $l=m$, Equation (6) can be written as:

$$16\left(X - \frac{1}{4}\right)((l^2-1)^2 X^2 + l^2)^2 = 0$$

i.e. $X = v^2 = \frac{1}{4}$, $\omega_0 = \frac{a}{2H}$: the Lamb frequency is not affected by the magnetic field when it is parallel to the gravitational field (in particular for $l=m=0$, $\omega_0$ is, as expected, the usual Lamb frequency).

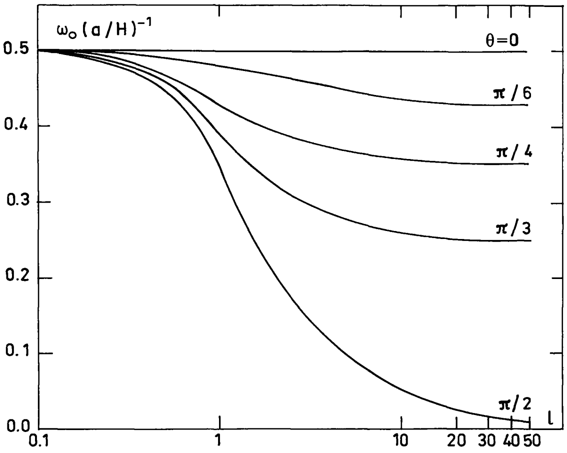


**Fig. 1.** Cut-off frequency $\omega_0$ in units of $a/H$, as a function of $l$, the ratio between the Alfvèn velocity and the velocity of sound. $\theta$ is the angle between the magnetic and gravitational fields.

For $m=0$ and any $l$ $\left(\theta = \frac{\pi}{2}\right)$, Equation (6) can be written as:

$$X^2\left(X - \frac{1}{4(1+l^2)}\right) = 0 ;$$

the cut-off frequency is simply $\omega_0 = \frac{a}{2H}\frac{a}{\sqrt{a^2+V_a^2}}$.

The Figure 1 shows the decrease of the cut-off frequency due to the magnetic field when $\theta \neq 0$.

*2. Form of the Curve $\omega(\alpha)$*

It can be shown that the curve $\omega(\alpha)$ defined by Equation (8) is the one drawn in Figure 1 of Paper I, i.e. one positive root for a given $\omega$ smaller than the cut-off frequency $\omega_0$ and two positive roots for a given $\omega > \omega_0$. ($\omega$ and $x$ axes are, of course, still axes of symmetry.) It is interesting to investigate when the curve $\omega(\alpha)$ has a cross-over. This point exists if the discriminant $\Delta$ of Equation (8) in term of $\alpha^2$ vanishes, i.e.:

$$\Delta = \omega'^4((1+l^2)^2 - 4m^2) + 2m^2 v^2(1-l^2) + v^2 m^4 = 0 .$$

The discriminant of this quadratic equation in $\omega^2$ is:

$$\delta = 4m^2 v^2 (m^2 - l^2)$$

which is negative for $m \neq l$ (the particular case $m=0$ is degenerate; see below).

If $l=m$, $\Delta$ writes:

$$\Delta = \left(\omega'^2(1-l^2) + \frac{l^2}{4}\right)^2$$

$\Delta$ vanishes only when $l > 1$ for

$$\omega_c'^2 = \frac{l^2}{4(l^2-1)} = \frac{H^2}{a^2}\omega_c^2 .$$

The abscissa of the cross-over is given by $x_c^2 = \frac{1}{4(l^2-1)} = H^2 \alpha_c^2$.

*3. The Particular Case $m=l$ ($\theta=0$)*

The cross-over of the two branches of the curve $\omega(\alpha)$ was found in Paper I for values of $l$ which were, in fact, greater than 1; however, $m$ was slightly different from $l$: this was probably due to the fact that the computations were made in single precision. In dimensioned variables and for $V_a^2 > a^2$ ($l>1$), the two modes defined by Equation (8) can be represented in the ($\alpha, \omega$) plane by one hyperbola and two straight lines:

$$\omega^2 = \omega_1^2 = a^2\alpha^2 + \frac{a^2}{4H^2}$$

$$\omega^2 = \omega_2^2 = V_a^2\alpha^2 .$$

For $\alpha^2 < \alpha_c^2$, one mode is dominated by the restoring of the pressure force; the gravity makes the mode dispersive; it is the slow-mode. The other mode is dominated by the magnetic forces and remains non-dispersive; it is the fast-mode. For $\alpha^2 > \alpha_c^2$, the two modes switch around. (For $V_a^2 < a^2$, the two modes are the fast mode $\omega^2 = \omega_1^2$ and the slow mode $\omega^2 = \omega_2^2$ and they do not cross over; for $V_a^2 = a^2$, the two modes $\omega_1^2 = a^2\alpha^2 + (a^2/4H^2)$ and $\omega_2^2 = a^2\alpha^2$ become asymptotically identical).

*4. The Particular Case $m=0$*

For $m=0$ and any $l$ (i.e. $\theta=0$, the magnetic field is normal to the gravitational field and the wave vector), Equation (8) is then degenerate and gives only one mode, the usual acoustic mode which exists alone when the magnetic field is normal to the wave vector and for which the gravity introduces a cut-off frequency:

$$\omega^2 = (a^2 + V_a^2)\,\alpha^2 - \frac{a^2}{4H^2}\frac{a^2}{a^2+V_a^2}$$

[the same result appears in Chen and Lykoudis (1972) for their $k_y$ tending to zero].

## Conclusion

We have obtained an equation for the phase velocity of a magneto-acoustic gravity wave propagating in an isothermal atmosphere along the direction of the gravitational field. We are in this way able to study analytically problems which have been handled numerically in the past; our expression leads to a clearer picture of the physical nature of the wave and of the interplay between the pressure, gravitational and magnetic restoring forces.

We have established the equation for the cut-off frequency $\omega_0$ of the "slow mode": this cut-off is due to the presence of the gravitational field. The absolute value of the magnetic field $\boldsymbol{B}$ and the angle $\theta$ between it and the gravitational field affect the decrease of the cut-off frequency. For a given magnetic field (i.e. for a given value of the ratio of the Alfvèn velocity and the velocity of sound), the cut-off is lower the closer $\theta$ is to $\frac{\pi}{2}$: low frequencies ($\omega \leqq \omega_{\mathrm{Lamb}}$) are propagated when the magnetic field is nearly horizontal. For a given $\theta$, the cut-off frequency is lower the bigger the ratio $V_a/a$. When the magnetic field is vertical, we have shown that the cut-off frequency is independent of the value of the field.

In the case of sunspots, for which one can take typically $a \approx 7\,10^5$ cm s$^{-1}$, $H \approx 10^7$ cm, we find that $T_0 = (2\pi/\omega_0) = (4\pi H/a) = 179$ s: this result is to be compared to observational work (Becker, 1975), from which one has deduced that the period of the photospheric oscillations can be as low as 180 s.

We have also established the necessary and sufficient conditions for the appearance of a cross-over point in the $\omega(\alpha)$ curve. For a wave propagating along the gravitational field, this cross-over appears if $\boldsymbol{B}$ is parallel to $\boldsymbol{g}$ and $V_a > a$. This latter condition leads to a linear coupling between the slow and fast modes, on the one hand, and between the acoustic and Alfvèn modes, on the other. This generalizes the condition already suggested by Pikel'ner and Livshits (1965) for the coupling in the neighbourhood of $V_a \approx a$, and shows that energy may be transferred from one mode to another. In particular, as a consequence of the slow mode, which is only slightly dissipated and refracted, a significant fraction of the chromospheric oscillations can penetrate into the corona along $B$. The condition $V_a \approx a$ is satisfied in the lower chromosphere (1300 km above the photosphere). At the edges of the chromospheric networks, $B \approx 100$ Gauss, $a \approx 7$ km s$^{-1}$, $V_a \approx$ 30–500 km s$^{-1}$, which leads to periods of 200–300 s. These values are quite comparable to observation, which renders the hypothesis of linear coupling highly probable.

*Acknowledgment.* One of us (B.L.) acknowledges the Deutsche Forschung Gemeinschaft for financial support.

## Appendix

*The relation between the frequency $\omega$ and the phase $\alpha$ of the M.A.G. waves.*

The derivation is made with dimensionless variables and parameters $\omega'$, $x$, $y$, $l$ and $m$. The equation relating $\omega'$ to $\alpha$ is obtained by eliminating $y$ between the real and imaginary parts of the dispersion Equation (1), i.e. between Equations (3) and (4) which (re-arranging terms in decreasing powers of $y$) can be written as:

$$P = m^2y^4 + m^2y^3 + y^2((1+l^2)\,\omega'^2 - 6m^2x^2) + (\omega'^2 - 3m^2x^2)\,y + \omega'^2(\omega'^2 - (1+l^2)\,x^2) + m^2x^4 = 0 \tag{I.1}$$

$$Q = 4m^2y^3 + 3m^2y^2 + 2y((1+l^2)\,\omega'^2 - 2m^2x^2) + (\omega'^2 - m^2x^2) = 0 . \tag{I.2}$$

If $y$ can be eliminated from Equations (I.1) and (I.2), then the polynomials $P$ and $Q$ share a root; consequently, there are two polynomials $p$ and $q$ at degree 3 and 2 respectively such that:

$$\frac{P}{Q}=-\frac{p}{q} \tag{I.3}$$

where

$$p=a_1 y^3+a_2 y^2+a_3 y+a_4$$

$$q=b_1 y^2+b_2 y+b_3$$

and the $a_i$ $(i=1,4)$ and the $b_j$ $(j=1,3)$ are to be determined. Equation (I.3) is equivalent to a sixth order polynomial in $y$ whose coefficients must vanish. This leads to seven linear and homogeneous equations which allow us to find the $a_i$ and $b_j$:

$$\begin{aligned}
&b_1+4a_1=0\\
&b_1+b_2+3a_1+4a_2=0\\
&\gamma b_1+\beta b_2+\beta b_3+2\beta a_1+3\beta a_2+4\beta a_3=0\\
&\varepsilon b_1+\gamma b_2+\beta b_3+Ca_1+2Ba_2+3\beta a_3+4\beta a_4=0\\
&\tau b_1+\varepsilon b_2+\gamma b_3+Ca_2+2Ba_3+3\beta a_4=0\\
&\tau b_2+\varepsilon b_3+Ca_3+2Ba_4=0\\
&\tau b_3+Ca_4=0
\end{aligned} \tag{I.4}$$

where

$$\begin{aligned}
&\beta=m^2\\
&\gamma=(1+l^2)\,\omega'^2-6m^2x^2\\
&\varepsilon=\omega'^2-3m^2x^2\\
&\tau=\omega'^2(\omega'^2-(1+l^2)\,x^2)+m^2x^4\\
&B=(1+l^2)\,\omega'^2-2m^2x^2\\
&C=\omega'^2-m^2x^2\,.
\end{aligned} \tag{I.5}$$

The first three equations in (I.4) allow us to express the $b_j$ as functions of the $a_i$:

$$\begin{aligned}
&b_1=-4a_1\\
&b_2=a_1-4a_2\\
&b_3=\frac{a_1}{\beta}(4\gamma-2B-\beta)+a_2-4a_3\,.
\end{aligned} \tag{I.6}$$

Substituting these values of $b_j$ in the last four equations of (I.4), we obtain four equations which are linear and homogeneous in $a_i$, and these equations have a solution if:

$$\begin{vmatrix}
5\gamma-2B-\beta+C-4\varepsilon & -4\gamma+2B+\beta & -1 & 4\\
\beta(\varepsilon-4\tau)+\gamma(4\gamma-2B-\beta) & \beta(-4\varepsilon+\gamma+C) & 2(B-2\gamma) & 3\beta\\
\beta\tau-\varepsilon(4\gamma-2B-\beta) & \beta(\varepsilon-4\tau) & C-4\varepsilon & 2B\\
\tau(4\gamma-2B-\beta) & \beta\tau & -4\tau & C
\end{vmatrix}=0\,. \tag{I.7}$$

This is the relation we have been looking for. After some straightforward but cumbersome algebra, we find that (I.7) can be written as:

$$T_6+T_5+T_4+T_3=0 \tag{I.8}$$

where the $T_i$ are homogeneous polynomials in $\omega'^2$ and $x^2$ of the $i^{\text{th}}$ degree and

$$\begin{aligned}
T_6&=16\,[\omega'^2(\omega'^2-(1+l^2)\,x^2)+m^2x^4]\\
&\quad\cdot[4m^2\omega'^4-((1+l^2)\,\omega'^2-4m^2x^2)^2]^2\\
T_3&=-4m^4(\omega'^2-m^2x^2)\,(\omega'^2-4m^2x^2)^2;
\end{aligned}$$

$T_5$ and $T_4$ are much more complicated and in any case we do not absolutely need them for what follows.

It is now necessary to factorise (I.8), since we require the equation of propagation of two upward waves and to downward waves, i.e. a second order equation in $\omega'^2$ and $x^2$.

We recall now that Equation (20) of Paper I and Equation (4) [from which $\omega'$ is eliminated using Equation (20) of Paper I] are the parametric representation of the curve $\omega'(x)$. Now, $\omega'=0$ if $y=0$ or $y=-\frac{1}{2}$; it follows from Equation (3) that $x$ is equal to zero only when $y=0$; in the neighbourhood of this point, we have:

$$\omega'^2-m^2x^2=0 \tag{I.9}$$

(this result is consistent with the numerical analysis of Paper I). Since we know that the curve in question passes through the origin, (I.8) has to be factorised in the form:

$$T_6+T_5+T_4+T_3=(S_2+S_1)\,(R_4+R_3+R_2) \tag{I.10}$$

where $S_i$ and $R_i$ are homogeneous polynomials of $i^{\text{th}}$ degree in $\omega'^2$ and $x^2$: $S_2+S_1=0$ will be the equation $\omega'(x)=0$ for which we are searching.

When $g$ tends to zero and $H$ to infinity (the product $Hg$ being kept constant), the equation $S_2+S_1=0$ written in dimensioned variables reduces to $S_2=0$: this equation must, of course, be the dispersion equation of the magneto-acoustic waves:

$$\omega'^2(\omega'^2-(1+l^2)\,x^2)+m^2x^4=0 \tag{I.11}$$

which is precisely the first factor in the expression for $T_6$. $S_2$ is consequently proportional to the left-hand side of Equation (I.11).

With the form found for $T_3$, we see that the other term $S_1$ is proportional either to $\omega'^2-m^2x^2$ or to $\omega'^2-4m^2x^2$; as it represents the tangents at the origin of the curve $\omega'(x)$, it must be proportional to the left-hand side of Equation (I.9). Finally, the equation for which we are searching has the form:

$$\omega'^2(\omega'^2-(1+l^2)x^2)+m^2x^4-v^2(\omega'^2-m^2x^2)=0 \quad \text{(I.12)}$$

where $v^2$ is a coefficient which depends on $l$ and $m$ and which is to be determined.

If we put $x^2=0$ in Equation (I.12), we find:

$$\omega'^2(\omega'^2-v^2)=0\,. \quad \text{(I.13)}$$

$v^2$ is the square of the dimensionless cut-off frequency and must be identical with one of the solutions of the equation obtained by putting $x^2=0$ in Equation (I.8), i.e.:

$$(\omega'^2)^3[A_0(\omega'^2)^3+A_1(\omega'^2)^2+A_2(\omega'^2)+A_3]=0 \quad \text{(I.14)}$$

where

$$A_0=16(4m^2-(1+l^2)^2)^2$$

$$A_1=-4[((1+l^2)^2-4m^2)((1+l^2)(1+m^2)-12m^2)+32m^2(1+l^2)(l^2-m^2)]$$

$$A_2=-m^2[(m^2-l^2)((1+l^2)+27(1+m^2))+8((1+l^2)(1+m^2)-6m^2)]$$

$$A_3=-4m^4\,.$$

Apart from the trivial solution $\omega'^2=0$, Equation (I.14) is a cubic equation in $\omega'^2$ whose solutions are easily obtained but are tedious to write out; we verified numerically that there is only one real and positive solution; the two complex solutions are, of course, the solutions of $R_4+R_3+R_2=0$, which is of second order in $\omega'^2$ with $x^2=0$.

It was also verified numerically that Equation (I.8) has only two positive solutions in $x^2$ for a given $\omega'^2$, where $v^2$ is the real positive solution of (I.14); consequently, all the solutions of $R_4+R_3+R_2=0$ are complex and so there is only one equation of the phase propagation $\omega'(x)$, which is as one would expect. This last equation is Equation (I.12) where $v^2$ is the real positive solution of (I.14).

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