NOTE ON THE EARTH-FIGURE PERTURBATIONS IN THE LUNAR THEORY

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Abstract. The $j=2$ lunar ephemeris is based on a flattening of the Earth which is slightly different from the 1964 IAU recommended value. The total effects of Earth-oblateness on the analytical lunar theory are determined, and the corrections, which are about 0.″02 in lunar longitude and latitude, are derived.

When the current system of astronomical constants was adopted by the IAU in 1964, formulas and tables for correction of ephemerides to the new system were provided (Supplement 1966). It now appears that there may be a slight defect in the correction of the lunar ephemeris to the 1964 IAU system. The relevant paragraphs from (Supplement 1966) are reproduced below.

"The value of the ellipticity, f, of the Earth's figure as used in Brown's tables (Brown 1919), and thus in the *Improved Lunar Ephemeris* (1954), is 1/294. The perturbations in the motion of the Moon are, however, proportional to the dynamical form factor J_2, where

$$\tfrac{3}{2}J_2 = f - \tfrac{1}{2}\sigma$$

and σ is approximately the ratio of the centrifugal force to gravity on the Earth's equator.

"According to the new system of constants the coefficients should be calculated with $\tfrac{3}{2}J_2 = 0.001\,624\,05$ (constant 6). It would appear that the values given in the Tables are based on $\tfrac{3}{2}J_2 = 0.001\,667\,36$ corresponding to a value of $\sigma = 0.003\,468$, slightly different from the new value (constants 6 and 16) of 0.003 457 8."

The value 0.003 468 used by Brown is mentioned, for example, in (Brown 1904) on p. 531. However, it is evident both from the context and from numerical calculation that Brown used this value to represent *exactly* the ratio of the centrifugal force to gravity on the Earth's equator, and did not include in it the corrections for higher order terms which are included in σ (as defined by the above equation). Letting the exact ratio (as used by Brown) be represented by m, we have the relation

$$\begin{aligned}\sigma &= m + (\tfrac{3}{2}J_2)^2 + \tfrac{5}{7}(\tfrac{3}{2}J_2)m - \tfrac{39}{28}m^2 + \cdots \\ &\simeq m - 0.000\,010.\end{aligned}$$

Therefore, the value of σ used by Brown was 0.003 458 rather than 0.003 468, in much better agreement with modern values (as is expectable at this level of precision).

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This implication is confirmed by an explicit statement in (Brown 1910), p. 79, that the value of $\frac{3}{2}J_2$ (or $\alpha^2\mu$ in Brown's notation, where $\alpha = 60.31847$) used in that location is 0.001 658 39, which is consistent with the true value of σ and the adopted flattening in (5), $1/f = 292.9$. The same conclusion had been reached earlier in (Eckert 1965). This fact was partially hidden by an error in the first article (Brown 1910), which he himself corrected (Brown 1909).

We see then that the value of $\frac{3}{2}J_2$ used by Brown in his theory (Brown 1910) was 0.001 685 39, and in his tables (Brown 1919) it was therefore 0.001 672 27, rather than 0.001 667 36 as assumed in (Supplement 1966). Hence (Supplement 1966) has undercorrected the lunar ephemeris; and the resulting current lunar ephemeris (known as the $j = 2$ lunar ephemeris) is actually based upon $\frac{3}{2}J_2 = 0.001\,628\,96$ and $1/f = 297.81$ rather than the intended $\frac{3}{2}J_2 = 0.001\,624\,05$ and $1/f = 298.25$.

Let us now derive the corrections to the lunar ephemeris required to make it consistent with the 1964 IAU system of constants. We first need to determine the total effect on the lunar ephemeris of the Earth's oblateness. After applying the corrections in (Brown 1909), (Brown 1910) gives a clear statement of the total effect of the Earth-figure perturbations on the Moon. Converting these to the 1964 IAU system correctly, we see that the total perturbation is

$$\begin{aligned}\delta\lambda = &-0''.0193 \sin(l-2D)\\ &-0.0039 \sin(l-2F)\\ &-0.0366 \sin 2\Omega\end{aligned}$$

$$\begin{aligned}\delta\beta = &+0''.0800 \sin(2\Omega+F)\\ &-0.0029 \sin(2\Omega+F-2D)\\ &+0.0048 \sin(2\Omega+l+F)\\ &-0.0048 \sin(2\Omega-l+F)\\ &-0.0164 \sin(\Omega+F)\\ &-0.0067 \sin(\Omega+F-2D)\end{aligned}$$

$$\delta L = +7''.0507 \sin\Omega$$

$$\delta\tilde{\omega} = +633''.2T - 2''.016 \sin\Omega$$

$$\delta\Omega = -592''.2T + 93''.171 \sin\Omega$$

$$\delta n = -0''.0087 \cos\Omega$$

$$\delta e = +0''.0019 \cos\Omega$$

$$\delta\gamma = 4''.1926 \cos\Omega$$

where λ = true longitude, β = latitude, (Ω, l, l', F, D) are the usual fundamental arguments of the lunar theory, L = lunar mean longitude, $\tilde{\omega}$ = mean longitude of lunar perigee, Ω = mean longitude of lunar ascending node, n = centennial lunar mean motion, e = mean lunar eccentricity, and γ = lunar inclination constant. T is time in centuries.

From the above we can calculate the corresponding total periodic additions to ecliptic longitude and latitude which are implied. For simplicity we here choose $L = F + \Omega$ in preference to Ω itself as a fundamental argument.

$$
\begin{aligned}
\delta\lambda = &+7''.051 \sin(L-F)\\
&+0.498 \sin(L-F+l)\\
&+0.498 \sin(L-F-l)\\
&+0.358 \sin(L+F)\\
&+0.081 \sin(L+2D-F)\\
&+0.081 \sin(L-2D-F)\\
&+0.056 \sin(L-2D-F+l)\\
&+0.056 \sin(L+2D-F-l)\\
&+0.047 \sin(L-2D+F)\\
&+0.039 \sin(L+F+l)\\
&-0.038 \sin(2L-2F)\\
&+0.035 \sin(L+F-l)\\
&+0.034 \sin(L-F+2l)\\
&+0.034 \sin(L-F-2l)\\
&+0.020 \sin(2D-2l)\\
&-0.014 \sin(L-3F)\\
&+0.011 \sin(L+2D-F+l)\\
&+0.011 \sin(L-2D-F-l)
\end{aligned}
$$

$$
\begin{aligned}
\delta\beta = &-8''.051 \sin(L)\\
&+0.455 \sin(L-l)\\
&-0.418 \sin(L+l)\\
&+0.326 \sin(L-2F)\\
&+0.284 \sin(L-2D)\\
&-0.086 \sin(L+2D-l)\\
&+0.083 \sin(2L-F)\\
&+0.074 \sin(L-2D+l)\\
&-0.047 \sin(L+2D)\\
&+0.040 \sin(L-2F-l)\\
&-0.024 \sin(L+2l)\\
&+0.015 \sin(L-2D-l)\\
&+0.014 \sin(L-2l)\\
&+0.013 \sin(L-2D+l')
\end{aligned}
$$

With the above and a correction factor of $-0.003023 = (0.001\,624\,05 - 0.001\,628\,96)/0.001\,624\,05$, we can readily list the additional corrections needed to update the present $j=2$ lunar ephemeris to place it fully on the 1964 IAU system:

$$\begin{aligned}\delta\lambda = &-0\overset{''}{.}0213 \sin \Omega\\ &-0.0015 \sin(\Omega + l)\\ &-0.0015 \sin(\Omega - l)\\ &-0.0011 \sin(\Omega + 2F)\\ &-0.0002 \sin(\Omega + 2D)\\ &-0.0002 \sin(\Omega - 2D)\\ &-0.0002 \sin(\Omega - 2D + l)\\ &-0.0002 \sin(\Omega + 2D - l)\end{aligned}$$

$$\begin{aligned}\delta\beta = &+0\overset{''}{.}0243 \sin L\\ &-0.0014 \sin(L - l)\\ &+0.0013 \sin(L + l)\\ &-0.0010 \sin(L - 2F)\\ &-0.0009 \sin(L - 2D)\\ &+0.0003 \sin(L + 2D - l)\\ &-0.0003 \sin(2L - F)\\ &-0.0002 \sin(L - 2D + l)\end{aligned}$$

The only omissions from either of the preceding sets of $\delta\lambda$, $\delta\beta$ expressions are the mixed-secular terms arising from corrections to the mean motions of the lunar nodes and perigee. Since the analytical lunar theory contains observed rather than theoretical rates, these are not subject to correction. However, if the full expressions were used to form analytical partial derivatives to differentially correct, say, a numerical integration of the Moon to some alternate value of J_2, then the effects of the secular terms in $\tilde{\omega}$ and Ω would have to be included along with the above: $\delta l = -633\overset{''}{.}2T$ and $\delta F = +592\overset{''}{.}2T$.

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