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ON THE CONSTANCY ALONG CYLINDERS OF THE ANGULAR VELOCITY IN THE SOLAR CONVECTION ZONE

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ABSTRACT

If, in the absence of rotation, the Sun's convection zone is adiabatic and if in the radial and latitudinal equations of motion the main balance of forces is between pressure gradients, Coriolis forces, and buoyancy forces (which is a good approximation if differential rotation is important over the entire convection zone and the large-scale velocities are not too large), then the perturbations in the convective flux and the pole-equator differences in flux ($\Delta\mathscr{F}$) are very large in the lower half of the convection zone, unless the angular velocity is constant along cylinders. The meridional velocities associated with this rotation law are not small, however, and could generate a significant $\Delta\mathscr{F}$. In this analysis compressibility was taken into account, but the latitudinal and radial dependence of the stabilizing effect of rotation on turbulent convection was neglected.

Subject headings: convection—stars: interior—Sun: rotation

I. INTRODUCTION

One of the theories of the solar differential rotation that have received most attention is the one which assumes that the variations of the Sun's angular velocity with depth and latitude is due to the interaction between large-scale convection and rotation (Durney 1968, 1970; Busse 1970; Yoshimura and Kato 1971). This interaction gives rise to two main effects: (*a*) there is a transport of angular momentum toward the equator by the Reynold stresses, and (*b*) large-scale convection is preferentially stabilized by rotation at the poles. As a consequence of effect (*b*), large pole-equator differences in flux, $\Delta\mathscr{F}$, develop (cf. the discussion of Durney 1970; Gilman 1972; Durney 1974, hereinafter referred to as Paper I). A very crude order-of-magnitude estimate, based on the Boussinesq approximation, gives (cf. Paper I) $\Delta\Omega/\Omega \approx \sigma^{-2}\,\Delta\mathscr{F}/\mathscr{F}$, where $\Delta\Omega$ is the pole-equator difference in angular velocity and σ is the Prandtl number.

The solar convection zone is strongly stratified with depth. The properties of the large-scale convection in a strongly stratified medium are not known. (Large-scale convection is defined as follows: in a highly turbulent convective fluid, the small-scale turbulence acts as a viscosity and organizes fluid motions on a much larger scale.) It is not known whether the large-scale convection extends over the entire depth of the convection zone or whether it is significant only in the lower part. Unless the large-scale convective velocities are large only in the Sun's surface layers, a considerable fraction of the Sun's energy flux has to be carried by the large-scale convection, if this is to generate the solar differential rotation by the angular momentum transport toward the equator. Important pole-equator differences in flux seem therefore unavoidable; these would then drive meridional motions which could contribute significantly (through the action of Coriolis forces) to the solar differential rotation. In Paper I, the angular velocity was evaluated as a function of the radial distance and polar angle under the assumption that it is entirely generated by such a meridional circulation. A "high viscosity" approximation was used to simplify the momentum equation, and the meridional motions were chosen so as to minimize the perturbations in the convective flux. The variation of Ω with depth, as given in Paper I, is in good agreement (cf. Durney 1975) with the dependence of Ω on r found by Yoshimura (1975) from the requirement that his solar cycle model reproduce the observed butterfly diagram.

Parker (1975) has recently shown that the solar dynamo cannot be driven efficiently in the upper part of the convection zone. Differential rotation, not only in the radial but also in the latitudinal direction, appears to be essential to obtain agreement with the observations of the magnetic field at the Sun's surface (Yoshimura 1975). Therefore, a very important question needs to be answered: Is differential rotation a surface phenomenon, or does it penetrate deeply into the convection zone?

The object of this paper is to study the consequences implied by a differential rotation which is significant over the entire convection zone. (For a critical review of theories of differential rotation the reader is referred to the paper by the author [Durney 1976].)

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II. THEORY

We assume that

$$\Omega = \Omega_\odot[1 + \omega_0(r) + \omega_2(r)P_2(\cos\theta)]\,, \tag{1}$$

where $P_2(\cos\theta)$ is the second-order Legendre polynomial. Howard and Harvey (1970) measured the solar angular velocity at the surface and expressed it in the form $\Omega = a + b\cos^2\theta + c\cos^4\theta$, where θ is the polar angle. It is readily seen that $1 + \omega_0 = (a + c/5 + b/3)/\Omega_\odot$, $\omega_2 = (2b/3 + 4c/7)/\Omega_\odot$, and $\omega_4 = 8c/35\ \Omega_\odot$ (the coefficient of P_4 in the expansion for Ω is $\omega_4\ \Omega_\odot$). In Figure 1 we have plotted $\Omega_\odot(\omega_2 P_2 + \omega_4 P_4)$ and $\Omega_\odot\omega_2 P_2$ (*dashed line*) versus θ for the average values of a, b, and c ($a = 2.78$, $b = -0.351$, $c = -0.443$ microradians s^{-1}). It is clear that ω_4 is small and that equation (1) is a good approximation to Ω at the surface.

For the meridional motions we take

$$u_r = 2\psi P_2(\cos\theta)/r^2\rho\,; \qquad u_\theta = -\psi'\sin\theta\cos\theta/r\rho\,, \tag{2}$$

where ρ is the density and ψ is the stream function. We assume that the turbulent viscosity η behaves as a molecular viscosity (Durney and Roxburgh 1970). In the Sun, η is in the first approximation proportional to ρ ($\eta = \rho\nu$); that is, the (turbulent) kinematic viscosity ν is approximately constant.

We proceed now to find approximate expressions for the equations of motion, assuming that $\omega_0(r)$ and $\omega_2(r)$ are slowly varying functions of r.

a) *The Azimuthal Equation of Motion*

The equations for $\omega_0(r)$ and $\omega_2(r)$ are equations (2.16) and (2.17) of Paper I. It is important that the approximations used to simplify these equations be as clear as possible. We write the equation for $\omega_0(r)$ (eq. [2.16a]) and equation (2.17b) of Paper I in the following forms:

$$\omega_0' = \frac{2}{r^4\rho}\int_{r_\odot}^{r}\left\{\omega_2\rho r^2 + \frac{\psi}{3\nu}\frac{d}{dr}[r^2(1 + \omega_0 + 2\omega_2/5)]\right\}dr - \frac{2\psi}{3\nu r^2\rho}(1 + \omega_0 - \omega_2/5)\,, \tag{3a}$$

$$2\psi/\nu = r^2\rho(\omega_2' - 5\omega_0')/(1 + \omega_0 - 5\omega_2/7)\,. \tag{3b}$$

In equations (3) it has been assumed that the azimuthal stresses and the radial velocities vanish at $r = r_0$, the outer radius of the convection zone. With the help of (3b), equation (3a) can be written

$$\omega_0' = \frac{2}{r^4\rho}\int_{r_0}^{r}\rho r^2\left[\omega_2 + \frac{r(\omega_2' - 5\omega_0')}{3(1 + \omega_0 - 5\omega_2/7)}\right]dr - 2\psi(1 + \omega_0 - \omega_2/5)/3\nu r^2\rho\,. \tag{4}$$

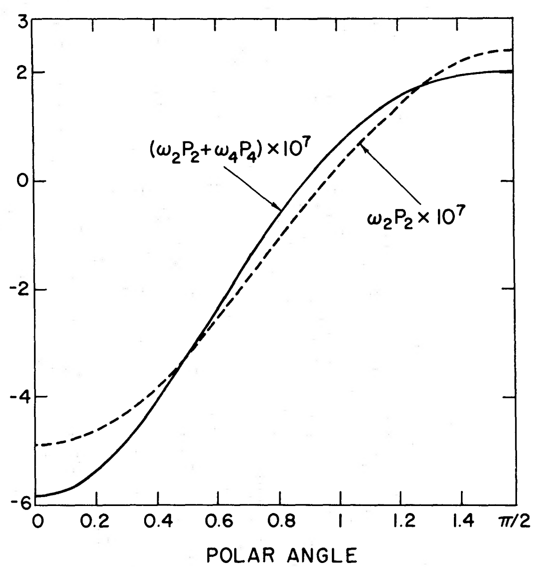

FIG. 1.—The Sun's angular velocity at the surface is expressed in the form $\Omega = \omega_0 + \omega_2 P_2 + \omega_4 P_4$. The curves give $\omega_2 P_2$ and $\omega_2 P_2 + \omega_4 P_4$ for the average values of a, b, c as measured by Howard and Harvey ($\Omega = a + b\cos^2\theta + c\cos^4\theta$; θ is the polar angle).

Since ω_0 and ω_2 are considered to be slowly varying functions of r, we have neglected $d[r^2(\omega_0 + 2\omega_2/5)]dr$ with respect to $2r$ in equation (4); that is,

$$\left|\frac{d}{dr}[r^2(\omega_0 + 2\omega_2/5)]\right| < 2r \tag{5}$$

Furthermore, $1 + \omega_0 + 2\omega_2/5$ was set equal to one. If at the lower boundary of the convection zone (r_c) the azimuthal stresses and radial velocity vanish ($\omega_0' = \psi = 0$), then the integral in equation (4) must be zero for $r = r_c$. Since $\omega_2 + r(\omega_2' - 5\omega_0')/3(1 + \omega_0 - 5\omega_2/7)$ is a slowly varying function of r, it can be taken out of the integral, and we obtain

$$\omega_2 \approx \frac{r}{3}(5\omega_0' - \omega_2')/(1 + \omega_0 + 5\omega_2/7)\,. \tag{6}$$

Equations (6) and (3b) then give

$$\omega_2 \approx -2\psi/3r\rho\nu\,. \tag{7}$$

Equation (7) gives an estimate of the values of the meridional motions needed to generate the latitudinal differential rotation, ω_2. For $\omega_2 \approx -0.2$ we obtain $\psi/r^2\rho \approx 10$ cm s^{-1} in excellent agreement with the results of Paper I (cf. Fig. 2*a*). Equation (6) is essentially equivalent to equation (10) of Durney (1975). The present derivation shows clearly, however, the assumptions (eq. [5] and zero azimuthal stresses and radial velocities at the upper and lower boundary of the convection zone) on which equation (6) is based. An estimate of ψ that involves boundary conditions only at the top of the convection zone can be obtained as follows: if $|u_r| < |u_\theta|$ and $|\rho'\omega_2'/\rho| > 10\omega_2/r^2$, the equation for $\omega_2(r)$ (eq. [2.16b]) of Paper I) reduces to $\rho'\omega_2'/\rho = -4\psi'/3r^2\rho\nu$. Integrating this equation and neglecting the variations of ω_2' and r^2 with respect to r, we obtain $\omega_2' = -4\psi/3r^2\nu$, that is, equation (7) if $\omega_2' \approx \omega_2/r$. Equation (6) follows then from equation (3b).

To proceed further, we need an estimate of the integral $\rho^{-1}\int_{r_\odot}^{r}\rho dr$ which appears in equation (4). We take an adiabatic polytrope ($p = p_0(\rho/\rho_0)^\gamma$, $\gamma = 5/3$), then $\rho = \rho_0[1 - (r - r_\odot)/l]^{3/2}$ with $l = \gamma RT_0/(\gamma - 1)\mu g$. ($R$ is the gas constant, μ the molecular weight, g gravity, and ρ_0, T_0 the values of the density and temperature at $r = r_\odot$.) For $r > l$ (that is, over most of the convection zone with the exception of a thin layer at the top) we have $\rho^{-1}\int_{r_\odot}^{r}\rho dr = \frac{2}{5}(r - r_\odot)$. The first term in the right-hand side of equation (4) is then not larger than

$$4|\omega_2|(r_\odot - r)/5r^2 < 4|\omega_2|(r_\odot - r_c)/5r^2 = 8|\omega_2|/25r\,;$$

whereas the second term is of the order of $|\omega_2|/r$ (cf. eq. [7]). This suggests neglecting the integral in equation (4) which then becomes

$$\omega_0' = -2\psi/3r^2\rho\nu\,; \qquad \omega_2' - 4\psi/3r^2\rho\nu\,, \tag{8}$$

where the second equality follows from the first and from equation (3b). Equations (8) are identical with equations (2.18) of Paper I. However, what has been shown here is the following: *if ω_0 and ω_2 are slowly varying functions of r and if the radial velocities and azimuthal stresses vanish at the boundary of the convection zone, then equations (7) and (8) are good estimates of ω_2 and ω_0', ω_2'.*

It is readily seen that equations (7) and (8) satisfy the inequality (5). It is also of interest to note that from equations (7) and (8) it follows that

$$\omega_2(r) \approx \omega_2(r_\odot)(r/r_\odot)^2\,. \tag{9}$$

At $r = r_c$, we obtain $\omega_2(r_c) \approx -0.1$, in excellent agreement with the results of Paper I.

b) The Radial Equation of Motion

The radial equation of motion can be written

$$\rho\left[u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta^2 + u_\phi^2}{r}\right] = -\frac{\partial p}{\partial r} - \rho g + R_r\,, \tag{10}$$

where R_r is the viscous term. An order-of-magnitude estimate of the terms involved in the radial and latitudinal equations of motion has been given by Gierasch (1974). For completeness we present here a brief discussion; equations (7) and (8) will be used as giving the order of magnitude of the relations of ω_0', ω_2', and ω_2 with ψ. Since the turbulent viscosity is proportional to ρ, the largest viscous term is $(d\eta/dr)\,(\partial u_r/\partial r)$; with the help of equations (8) and (9) it can be written

$$\nu\frac{d\rho}{\partial r}\frac{\partial u_r}{\partial r} \approx \nu\frac{d\rho}{dr}\frac{\partial}{\partial r}\left(\frac{\psi}{r^2\rho}\right) \approx \nu^2\frac{d\rho}{dr}\frac{d}{dr}(\omega_2') \approx \frac{\nu^2}{r_\odot^2}\frac{d\rho}{dr}\omega_2(r_\odot)\,.$$

It is, of course, not entirely justified to take the derivative of equation (9). However, $d(\psi/r^2\rho)/dr$ as calculated from equations (8) and (9) is in good agreement (in the lower part of the convection zone) with the results of Paper I. It is readily seen that the above term is very small with respect to the largest inertial term $(=\Omega_\odot^2\omega_2 r\rho)$: $\nu^2\rho'\omega_2(r_\odot)/r_\odot^2\ \Omega_\odot^2\omega_2 r\rho < 10^{-7}r\rho'/\rho$. The largest of the remaining inertial terms is u_θ^2/r. An estimate of u_θ can be obtained as follows: from equation (8), $\psi \approx -r^2\rho\nu\omega_2'$; therefore, $\psi' \approx -r^2\rho'\nu\omega_2'$ and $u_\theta \approx r\rho' u_r/\rho$; in consequence $u_\theta/u_\phi \approx (r\rho'/\rho)\nu\omega_2'/\Omega_\odot r\omega_2 \approx (r\rho'/\rho)\nu/\Omega_\odot r^2 < 10^{-4}r\rho'/\rho < 1$. Equation (10) therefore reduces to

$$\rho u_\phi^2/r - \rho g = \partial p/\partial r\,. \tag{11}$$

Equation (11), as well as the latitudinal equation of motion, will be solved by expansions of the form

$$p = p_u + p_0 + p_2 P_2 + p_4 P_4\,; \qquad \rho = \rho_u + \cdots\,; \qquad T = T_u + \cdots\,, \tag{12}$$

where P_2 and P_4 are Legendre polynomials; p_u is the undisturbed pressure; and $p_0, p_2, \ldots$ arise because of the term $\rho u_\phi^2/r$. Differential rotation will be furthermore assumed to be small; that is, ω_0^2, ω_2^2, and $\omega_0\omega_2$ will be neglected (as well, of course, as the products of ρ_i, p_i, T_i $[i = 0, 2, 4]$). Equation (11) gives, then,

$$dp_u/dr = -\rho_u g\,; \qquad \frac{dp_0}{dr} = -\rho_0 g + \tfrac{2}{3}\,\Omega_\odot^2 r\rho_u(1 + 2\omega_0 - 2\omega_2/5)\,;$$

$$\frac{dp_2}{dr} = -\rho_2 g - \tfrac{2}{3}\,\Omega_\odot^2 r\rho_u(1 + 2\omega_0 - 10\omega_2/7)\,;$$

$$dp_4/dr = -\rho_4 g - \tfrac{24}{35}\,\Omega_\odot^2 r\rho_u\omega_2\,. \tag{13}$$

c) *The Latitudinal Equation of Motion*

This equation can be written

$$\rho\left[u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial\theta} + R_\theta \tag{14}$$

The largest viscous term is $\nu\rho'\partial u_\theta/\partial r$. We evaluate the ratio $\nu\rho'\partial u_\theta/\partial r/\rho\Omega^2{}_\odot\omega_2 r$. In an evident notation $\partial u_\theta/\partial r \approx u_\theta/L_{u_\theta} \approx r\rho' u_r/\rho L_{u_\theta} \approx r\rho'\nu\omega_2/rL_{u_\theta}$. Therefore, the above ratio is given by $(r\rho'/\rho)^3\nu^2/(r^2\Omega_\odot)^2 \approx 4(r\rho'/\rho)^3 10^{-8}$, where we have taken a conservatively small value for $L_{u_\theta}(\sim\rho/\rho')$. Therefore, with the exception of a thin layer at the top of the convection zone, the viscous terms are small. The largest of the remaining inertial terms is $\sim u_\theta^2/r$; and as shown above, it can be neglected also. Equation (14) reduces then to

$$\rho u_\phi^2\frac{\cot\theta}{r} = \frac{1}{r}\frac{\partial p}{\partial\theta}\,. \tag{15}$$

Substituting expansions (12) into the equation (15), we obtain

$$p_2 = -\tfrac{1}{3}\rho_u\Omega_\odot^2 r^2(1 + 2\omega_0 + 2\omega_2/7)\,; \qquad p_4 = -\tfrac{6}{35}\rho_u\Omega_\odot^2 r^2\omega_2\,. \tag{16}$$

These nonviscous approximations of the radial and latitudinal equations of motion (eqs. [11] and [15]) were used by Gierasch (1974) in his work on differential rotation.

With the help of equations (13), (16), and the gas equation $p_i = R\mu^{-1}(\rho_i T_u + \rho_u T_i)$, $i = 2, 4$, we can readily evaluate T_2 and T_4. We find

$$T_2 = -\tfrac{1}{3}\Omega_\odot^2 r^2(\mu/R + \rho_u' T_u/g\rho_u)(1 + 2\omega_0 + 2\omega_2/7) - \frac{2}{21}\frac{\Omega_\odot^2 r}{g}T_u(7r\omega_0' + r\omega_2' + 12\omega_2)\,, \tag{17}$$

$$T_4 = -\tfrac{6}{35}\Omega_\odot^2 r^2\omega_2(\mu/R + \rho_u' T_u/g\rho_u) - \frac{6}{35}\frac{\Omega_\odot^2 T_u r}{g}(r\omega_2' - 2\omega_2)\,.$$

The expansion of u_ϕ^2/r and $u_\phi^2\cot\theta/r$ used in deriving equations (13) and (16) are given below:

$$u_\phi^2/r = \tfrac{2}{3}\Omega_\odot^2 r[1 + 2\omega_0 - 2\omega_2/5 - P_2[1 + 2\omega_0 - \tfrac{10}{7}\omega_2] - \tfrac{36}{35}\omega_2 P_4]\,, \tag{18a}$$

$$u_\phi^2\cot\theta/r = \Omega_\odot^2 r(1 + 2\omega_0 + 2\omega_2 P_2)\sin\theta\cos\theta\,. \tag{18b}$$

In the discussion above only the viscous terms have been considered. Velocities due to the large-scale convection could also be present. If these velocities are not larger than the turbulent convective velocities, then equations (11) and (15) remain valid (cf. Durney 1976).

d) *The Perturbations in the Convective Flux*

For constant values of c_v and γ the energy equation becomes

$$\operatorname{div} \mathscr{F} = c_p \rho \boldsymbol{u} \cdot \nabla \Delta T \tag{19}$$

with

$$c_p = R\gamma/(\gamma - 1)\mu \quad \text{and} \quad \nabla \Delta T = T[(1 - 1/\gamma)\nabla p/p - \nabla T/T];$$

$\boldsymbol{u}$ is the meridional velocity, and $\mathscr{F}$ is the convective flux. We use for $\mathscr{F}$ the mixing length expression, $\mathscr{F} = \frac{1}{2}Lc_p\rho v_c \nabla \Delta T$; therefore since the turbulent convective velocity, v_c, is given by $v_c = \frac{1}{2}L(g \cdot \nabla \Delta T/T)^{1/2}$ we can write

$$\mathscr{F}_r = \frac{L_u^2}{4} g^{1/2} \rho_u c_p A_r^{3/2}/T_u^{1/2}; \qquad \mathscr{F}_\theta = \frac{L_u^2}{4} g^{1/2} \rho_u c_p A_{ru}^{1/2} A_\theta / T_u^{1/2}, \tag{20a}$$

with

$$A_r = \left(1 - \frac{1}{\gamma}\right) \frac{T}{p} \frac{\partial p}{\partial r} - \frac{\partial T}{\partial r}; \qquad A_\theta = \left(1 - \frac{1}{\gamma}\right) \frac{T}{pr} \frac{\partial p}{\partial \theta} - \frac{1}{r} \frac{\partial T}{\partial \theta}. \tag{20b}$$

Above, L_u is the mixing length, and the subindex, u, refers to the unperturbed state in the absence of rotation. The superadiabatic gradient is small. Therefore, we expect the perturbations (δ) in the superadiabatic gradient to be the ones that are the most important, i.e., $\delta \nabla \Delta T / (\nabla \Delta T)_u \gg \delta p/p_u$, $\delta \rho/\rho_u$, $\delta T/T_u$. This is why we neglected the term $\frac{1}{4}L_u^2 g^{1/2} \rho c_p A_{ru}^{3/2}/T^{1/2}$ in the expression for $\mathscr{F}_r$ in equation (20a); we did not include quadratic terms in the perturbation either.

The quantities A_r and A_θ defined by equation (20) can be written with the help of equations (11) and (15) as follows:

$$A_r = [-g + u_\phi^2/r]/c_p - \partial T/\partial r, \tag{21a}$$

$$A_\theta = u_\phi^2 \cot\theta / rc_p - \frac{1}{r} \frac{\partial T}{\partial \theta}. \tag{21b}$$

The term $-g/c_p - \partial T_u/\partial r$ gives the unperturbed flux. The perturbed spherically symmetric part of A_r, i.e.,

$$\left(-\frac{2}{3} \frac{\Omega_\odot^2 r}{c_p} (1 + 2\omega_0 - 2\omega_2/5) - \partial T_0/\partial r\right),$$

must be zero since rotation cannot change the emerging flux. This is just an equation for $\partial T_0/\partial r$. To evaluate the remaining terms of A_r and A_θ we approximate the unperturbed state by a polytrope:

$$\rho_u = \rho_c[1 - (r - r_c)/l]^{1/(\gamma-1)}; \qquad p_u = p_c[1 - (r - r_c)/l]^{\gamma/(\gamma-1)}; \qquad T = T_c[1 - (r - r_c)/l], \tag{22}$$

where $l = \gamma R T_c/(\gamma - 1)\mu g$, and ρ_c, p_c, T_c are the values of ρ_u, p_u, and T_u at $r = r_c$, the lower boundary of the convection zone. For a polytrope the following equality holds:

$$\mu/R + \rho_u' T_u / g\rho_u = 1/c_p. \tag{23}$$

With the help of equations (17), (18), and (23) it is readily found that A_θ and the perturbed part of A_r are given by

$$A_r = \frac{2}{7} \frac{\Omega_\odot^2 T_u}{g} \left[\frac{1}{3} \frac{d}{dr}(rB_1)P_2 + \frac{3}{5} \frac{d}{dr}(rB_2)P_4\right], \tag{24a}$$

$$A_\theta = \frac{2}{7} \frac{\Omega_\odot^2 T_u}{g} \left[\tfrac{1}{3}B_1 \frac{\partial P_2}{\partial \theta} + \tfrac{3}{5}B_2 \frac{\partial P_4}{\partial \theta}\right], \tag{24b}$$

where

$$B_1 = 12\omega_2 + r\omega_2' + 7r\omega_0', \qquad B_2 = r\omega_2' - 2\omega_2. \tag{24c}$$

The radial and latitudinal components of the energy flux are given by equation (20a). In Table 1 we list the values of

$$\Delta\mathscr{F} = \frac{L_u^2}{4} g^{1/2} \rho_u c_p A_{ru}^{1/2} \frac{3}{2} \left(\frac{2}{7} \frac{\Omega_0^2 T_u}{g}\right) \Big/ T_u^{1/2} = \tfrac{3}{2}\mathscr{F}\left[\frac{2}{7} \frac{\Omega_0^2 T_u}{g} \Big/ (\nabla \Delta T)_u\right]$$

as a function of depth; the values of ρ_u, T_u, and v_c were taken from Baker and Temesvary's tables; $c_p = R\gamma/\mu(\gamma - 1)$ with $\mu = 0.6$ and $\gamma = 5/3$, and the superadiabatic gradient was evaluated from the mixing length expression for the convective flux, $\mathscr{F} = \frac{1}{2}Lv_c\rho c_p \nabla \Delta T$. It is readily seen that $\Delta\mathscr{F}$ gives an indication of the pole-equator differences

TABLE 1

VALUES OF $\Delta\mathscr{F} = \frac{3}{2}\mathscr{F}[2\Omega_0{}^2 T_u/7g(\nabla\Delta T)_u]$ AS A FUNCTION OF DEPTH d

d (cm)................	3.4(6)	1.3(7)	3.7(7)	1.5(8)	3.6(8)	1.0(9)	3.3(9)	1.1(10)	1.8(10)	1.9(10)
$\Delta\mathscr{F}$ (ergs cm^{-2} s^{-1}).....	2.4(1)	5.3(2)	1.2(3)	1.4(4)	1.8(5)	8.8(6)	6.0(8)	2.8(10)	7.8(10)	6.3(10)

in flux if the terms inside the bracket of equation (24a) do not vanish. Furthermore, $\mathscr{F}_\theta \approx \Delta\mathscr{F}$ if the terms inside the bracket of equation (24b) do not vanish. It appears impossible that such large values of $\Delta\mathscr{F}$ and $\mathscr{F}_\theta$ could exist inside the convection zone. We conclude therefore that B_1 and B_2 given by equation (24c) must be small, or

$$\omega_2(r) \approx \omega_2(r_c)(r/r_c)^2\,; \qquad \omega_0(r) + \omega_2(r) \approx \text{constant}\,. \tag{25}$$

The angular velocity is then given by

$$\Omega = \Omega_0[1 - \tfrac{3}{2}\omega_2(r_c)(r/r_c)^2 \sin^2\theta]\,, \tag{26}$$

where we have taken as zero the constant appearing in equation (25). Therefore, the angular velocity is constant along cylinders. What has been shown is the following: *If in the absence of rotation the convection zone is adiabatic, and if equations (11) and (15) are good approximations for the radial and latitudinal equations of motion, then in the lower half of the convection zone the perturbations in the convective flux (produced by differential rotation) are unacceptably large unless the angular velocity is constant along cylinders.* (Of course, this result holds only if the value of the latitudinal differential rotation is not too small in the lower half of the convection zone; also the effect of rotation on turbulent convection has been neglected.)

If there is no rotation, it is well known that the superadiabatic gradient A_r has to be small; otherwise the convective flux is too large. The equations $T_u' = -g/c_p$, $p_u' = -g\rho_u$, and $p_u = R\mu^{-1}\rho_u T_u$ can then be satisfied only if $p_u = p_{uc}(\rho_u/\rho_{uc})^\gamma$ $[c_p = R\gamma/(\gamma - 1)\mu]$, that is, if the convection zone is adiabatic. If there is rotation and we impose the condition that the superadiabatic gradient be small, equations (20) give

$$\sin\theta \frac{\partial}{\partial\theta}(u_\phi{}^2) - r\cos\theta\frac{\partial}{\partial r}(u_\phi{}^2) = 0\,;$$

that is, the angular velocity must be constant along cylinders.

III. MERIDIONAL MOTIONS ASSOCIATED WITH AN ANGULAR VELOCITY OF THE FORM $\Omega = f(r\sin\theta)$

We assume that the angular velocity is given by equation (26) and that the meridional motions are axisymmetric:

$$u_r = \frac{\partial\psi}{\partial\theta}\Big/\rho r^2\sin\theta\,; \qquad u_\theta = -\frac{\partial\psi}{\partial r}\Big/\rho r\sin\theta\,.$$

The azimuthal equation of motion can then be written

$$\frac{\partial\psi}{\partial\theta}\sin\theta - \frac{\partial\psi}{\partial r}r\cos\theta = -\frac{3\omega_2(r_c)\nu\rho_u'(r\sin\theta)^4}{2r_c{}^2[1 - \frac{3}{2}\omega_2(r_c)(r/r_c)^2\sin^2\theta]}\,. \tag{27}$$

In equation (27) we have kept all inertial terms but have retained only the largest viscous term ($\nu\rho_u' r\sin\theta\,\partial\Omega/\partial r$). A sufficient insight into the type of motions described by equation (27) can be obtained by assuming that the pressure and density are related by a polytropic relation $[p_u = p_{uc}(\rho_u/\rho_{uc})^\gamma]$ with $\gamma = 3/2$, which is very close to the adiabatic ($\gamma = 5/3$)value. (We neglect perturbations in pressure and density due to the motions). Noting that $r\sin\theta = \text{constant}$ is a particular integral of the characteristic system of equations corresponding to the partial differential equation (27), it is readily found that the general solution is given by

$$\psi = -\frac{3\omega_2(r_c)\nu\rho_c(r\sin\theta)^4}{(r_c l)^2(1 - \frac{3}{2}\omega_2(r_c)(r\sin\theta/r_c)^2)}\left[(l + r_c)\log\frac{1 + \cos\theta}{\sin\theta} - r\cos\theta\right] + \phi(r\sin\theta)\,, \tag{28}$$

where l has been defined following equation (22) and $\phi(r\sin\theta)$ is an arbitrary function of $r\sin\theta$. (If $\gamma = 3/2$, eq. [27] can be solved analytically.) An idea of the type of motions described by the stream function ψ (with $\phi = 0$) can be obtained from Table 2; we took $\omega_2(r_c) = -0.1$, which gives the observed value of the latitudinal differential rotation at the surface if $\omega_2(r) = \omega_2(r_c)(r/r_c)^2$, and $\nu = 10^{13}$ cm^2 s^{-1}.

IV. THE ENERGY EQUATION

It should be stressed that it has not been shown that rotation in cylinders is a consequence of the energy and momentum equation. In this section we investigate in a very qualitative way whether this is the case. Let $\delta\mathscr{F}$ be the perturbation in the energy flux ($\delta\mathscr{F} = \frac{3}{2}\mathscr{F}A/(\nabla\Delta T)_u$ with A_r and A_θ given by equations (24a), (24b), and (24c). It is readily seen that

$$\delta\mathscr{F}_r = \frac{45}{28}\frac{\nu\Omega_0{}^2p_u}{g}B_r\,, \tag{29a}$$

$$\delta\mathscr{F}_\theta = \frac{45}{28}\frac{\nu\Omega_0{}^2p_u}{g}B_\theta\,, \tag{29b}$$

where B_r and B_θ are the bracketed expressions in equations (24a) and (24b). If B_r and B_θ are not small ($B_r \approx B_\theta \approx 1$), then since the pressure is a strongly varying function of r,

$$\mathrm{div}\,\delta\mathscr{F} = -\tfrac{45}{28}\nu\Omega_0{}^2\rho_u B_r\,, \tag{30}$$

and the energy equation (eq. [19]) becomes

$$-\tfrac{45}{28}\nu\Omega_0{}^2\rho_u B_r \sim c_p\rho_u u_r(\nabla\Delta T)_u = \frac{\mathscr{F}u_r}{\frac{3}{2}\nu} \tag{31}$$

At $r = 6 \times 10^{10}$ cm, $\nu \approx 10^{13}$ cm^2 s^{-1} and $\rho_u \approx 7.6 \times 10^{-2}$ g cm^{-3}. For the left- and right-hand sides of equation (30) we obtain $10B_r$ and 0.23 cgs units, respectively (we took $u_r = 40$ cm s^{-1}; cf. Table 2). Therefore, for values of u_r of the order of 40 cm s^{-1}, the energy equation appears to favor small values of B_r. If B_r is small, then rB_1 and rB_2 must be slowly varying functions of r. We take

$$rB_2 = \alpha r_c\,; \qquad rB_1 = \beta r_c \tag{32}$$

with α and β two constants; $\omega_0(r)$ and $\omega_2(r)$ are then given by

$$\omega_2(r) = -\frac{\alpha}{3}(r_c/r) + \xi(r/r_c)^2\,; \qquad \omega_0(r) = -(11\alpha + 3\beta)r_c/21r - \xi(r/r_c)^2\,, \tag{33}$$

where ξ is a constant.

If $\omega_0(r)$ and $\omega_2(r)$ are given by equations (33), then the energy equation reduces to

$$\delta\mathscr{F}_\theta/r \approx \mathscr{F}u_r/\tfrac{3}{2}\nu\,. \tag{34}$$

At $r = 6 \times 10^{10}$ cm, the left- and right-hand sides of equation (34) are equal to $0.6B_\theta$ and 0.23 cgs units, respectively ($u_r = 40$ cm s^{-1}). These quantities are of the same order if $B_\theta \approx 1$. In the present very crude analysis we have assumed that $\boldsymbol{u}\cdot\nabla\Delta T \approx u_r(\nabla\Delta T)_u$ with $u_r \sim 40$ cm s^{-1}, and it is clear that a much more careful study of this problem is needed before we can resolve whether the momentum and energy equation indeed favor a constant angular velocity along cylinders.

In the calculations above we have assumed that the mixing length L is independent of θ; *we have neglected the latitudinal and radial dependence of the stabilizing effect of rotation on convection.* An estimate of the magnitude of this effect is an important problem that remains to be solved and it is not known at present if therein lies the solution to the heat flux problem. It is appealing, of course, to think that *differential rotation arises as a way for the convection zone to minimize the constraints imposed by rotation on convection and so to maintain a uniform heat flux.*

TABLE 2

VALUES OF u_r AND u_θ AT $r = 6.2 \times 10^{10}$ CENTIMETERS CORRESPONDING TO THE STREAM FUNCTION DEFINED BY EQUATION (28): $\omega_2(r_c) = -0.1$; $\phi(r \sin\theta) = 0$

θ (radians)	u_r (cm s^{-1})	u_θ (cm s^{-1})
0	0	0
0.157	24	−3.9
0.314	49	−17
0.471	56	−32
0.628	45	−41
0.785	24	−40
0.942	2.1	−31
1.1	−14	−20
1.26	−24	−9.6
1.41	−28	−3.4
$\pi/2$	−30	0

It is possible, perhaps, to give a nonrigorous argument against the existence of large $\Delta\mathscr{F}$'s inside the convection zone: in the surface layers ($d < \sim 3 \times 10^9$ km) the effect of rotation on turbulent convection becomes negligible and the convective flux should be well described by equations (20). The values of $\Delta\mathscr{F}$ listed in Table 1 are small for $d < \sim 10^9$ km. The existence of larger values of $\Delta\mathscr{F}$ would imply that equations (11) and (15) are not good approximations at these depths; therefore, large velocities would have to be present in the Sun's surface layers.

V. DISCUSSION

The expressions for ω_2 given by equations (9) and (25) are identical; however, the expressions for ω_0, as given by equations (8) and (25), are quite different. In the derivation of equations (8) it was assumed that the radial velocities and azimuthal stresses vanished at the boundaries (especially at the upper boundary) of the convection zone. It is clear that it is not possible to chose $\phi(r \sin \theta)$ in equation (28) so that

$$u_r = \frac{\partial \psi}{\partial \theta}\bigg/\rho r^2 \sin \theta$$

vanishes at a specified value of r; boundary layers must develop.

It could be thought that an angular velocity which decreases inward is in contradiction to theories of the solar cycle. However, this is not necessarily so. To ensure that the solar dynamo displays the correct butterfly diagram (i.e., that the mean toroidal field migrates toward the equator), the product $\alpha\partial\Omega/\partial r$ must be negative (Stix 1974). According to Yoshimura (1975) the α-term (due to the global convection) changes sign in the lower part of the convection zone. To obtain the observed properties of the butterfly diagram (propagation of the toroidal magnetic field toward the equator), the angular velocity must decrease inward. Therefore, rotation in cylinders could be compatible with Yoshimura's model of the solar cycle (Yoshimura 1975) if the observed toroidal magnetic field is generated in the *lower* part of the convection zone as it has been recently argued by Parker (1975). There can be little doubt that this is the most appealing region for the dynamo to operate in; and as a consequence, differential rotation should penetrate deeply inside the convection zone. However, it appears, at present, that this would imply a major perturbation in the thermodynamics of the convection zone. It would be of great interest, therefore, to establish independently that $\omega_2(r) \approx \omega_2(R_0)$ for $r \approx r_c$. Assume that differential rotation is generated, by an unspecified mechanism, in the surface layers of the Sun. From observations, $\Delta\mathscr{F}$ and ΔT are very small at $r = R_0$. Assume that the momentum and energy equation are solved with boundary conditions expressing that $\Delta\mathscr{F}$ and ΔT are small at the surface and that $\omega_2(R_0) = -0.189$. If differential rotation is a surface phenomenon we expect that Ω will behave as follows: with increasing depth, $\Delta\mathscr{F}$'s will appear (cf. Table 1) that will drive meridional motions. These meridional motions should be such as to decrease $|\omega_2(r)|$ and make it eventually vanish.

Preprints of the author's review paper on differential rotation (Durney 1976) are available upon request.

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