Obliquity and Precession for the Last 5000000 Years

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Summary. A new solution is numerically determined for the obliquity of the ecliptic and the longitude of the perihelion measured from the equinox of date. For the eccentricity, the longitude of the perihelion measured from the equinox of epoch, the inclination and the longitude of the node, the solution by Bretagnon has been used. It includes terms to the "second" order to the disturbing masses and to the third degree with respect to the planetary eccentricities and inclinations. For the obliquity and the precession in longitude, Shara's analytical expansions are used, which means that terms to the second degree with respect to the Earth's eccentricity have been kept. In this paper their numerical determination takes into account all terms of Bretagnon. Parameters of the expansions are tabulated and a graphical representation extends back to 5000000 years before present and forewords to 1000000 years after present.

Key words: planetary theory—long-term variation—paleoclimatology

1. Introduction

In a recent paper (Berger, 1976), the author has demonstrated the importance of terms generally neglected in the solution giving the long-term variations of the Earth's orbital elements. Table 1 (Berger, 1976) recalls the successive improvements which have been made to calculate these solutions. The results obtained when comparing all these solutions may be summarized in the following way with regard to the influence of different terms on the accuracy of the expansions used. First, further improvements in planetary masses will not have significant influence. Second, for the system in eccentricity $e$ and longitude of the perihelion $\pi$ measured from the equinox of epoch, terms depending upon the second order as to the disturbing masses are more important than the ones coming from the third degree with respect to the planetary eccentricities and inclinations. Third, for the inclination $i$ of the ecliptic on the plane of reference and the longitude of the ascending node $\Omega$, the latter terms have highly significant influence, whereas additional terms in masses are negligible. Fourth, because of the nature of their expansion, similar conclusions can be drawn for the time variation of the obliquity $\varepsilon$ and the precession $\Psi$ of the equinoxes. However, in this case, the series expansion of these elements must be examined. Analysing the results obtained both from a Laplace's expansion, i.e. including terms depending on the first degree with respect to the Earth's eccentricity and from the Shara's expansion (Shara et al., 1967), i.e. including terms depending on the second degree with respect to the Earth's eccentricity, the author came to the following conclusions. For the previous and the next millions years, it seems sufficient to consider a Laplace's expansion but where amplitudes, phases and mean rates are calculated from those present in the $(e, \pi)$ and $(i, \Omega)$ Bretagnon solutions, solutions where terms depending on the second power as to the masses and especially the third degree with respect to the planetary $e$'s and $i$'s (Bretagnon, 1974) have been included. However, for more extended time spans, the Shara's terms have to be considered, taking into account all the additional Bretagnon's terms. It is the aim of this paper to construct numerically and to represent graphically such a solution.

2. Fundamental Systems

As numerically demonstrated (Berger, 1976), Bretagnon (1974) makes a decisive improvement in the evaluation of the eccentricity, the longitude of the perihelion, the inclination and the longitude of the ascending node. Let us summarize the main characteristics of that solution. The elements of the Earth's orbit are referred to the mean ecliptic and the mean equinox of 1850.0. Initial elements at this epoch of reference are the most accurate and secular perturbations have been taking
out of the observed mean motions, the semi-major axis of the ecliptic being then given by the third law of Kepler. Disturbing planetary masses as proposed by Kovalesky (1971) are used. Constants of integration have been carefully determined in such a way that the solution coincides with the mean elements at the time origin 1850.0, calculation which is not found in a similar work done by Anolik et al. (1969).

In order to show the coherence in the accuracy through the all solution built in this paper, it is also necessary to remember that, in the Bretagnon’s expansions themselves, both terms depending on the third degree with respect to the planetary eccentricities and inclinations and terms of the second order to the planetary masses are included. Indeed, Bretagnon takes into account all long period terms of the disturbing function, of fourth order in the planetary $e^s$ and $i^s$. In a second approximation he also introduced the short period terms of the disturbing function.

About the influence of these short period terms to the second order as to the masses, terms limited to the third degree in $e^s$ and $i^s$ which leads to a modification of the long period frequencies by more than $10^{-3}$/year (around 50 for the only Jupiter-Saturn near-resonance) have been kept for all the planets. Among the long period terms generated through these terms, only those which amplitude is greater than $10^{-4}$ for the inner planets and $10^{-6}$ for the major ones have been considered. On the other hand, the short period terms include, in particular, those of third order of the great inequality in the motion of Jupiter and Saturn.

In comparison, let us recall that Brouwer (1950, p. 88) considered, alike Hill (1897), the influence to the second order of the disturbing masses of the only Jupiter-Saturn near-resonance and that, from equations for the eccentricities and longitudes of the perihelia of Jupiter and Saturn obtained by Le Verrier (1874). This method led to contributions up to the fifth order in $e^s$ and $i^s$ of the short period terms and these generated long period terms of which Brouwer kept only the two most important ones. To stress the importance of all these additional terms, the solution by Bretagnon has been written in a new appropriate form (Berger, 1975b, formulae 11-12-13) which allows the direct determination of the amplitudes associated with arguments of the Lagrange’s terms.

For the computational requirement, these solutions of the fundamental systems in eccentricity and in inclination can also be expressed in the following way where only the most significant terms have been kept:

$$h \sin \pi = e \pi = \sum_{j=1}^{19} M_j (g_j t + \beta_j),$$

$$k \cos \sin p_{(1/2)} \sin \sin = \sin \frac{\pi}{2} \Omega = \sum_{j=1}^{15} \hat{N}_j (\delta_j t + \delta_j).$$

From (2), it is possible to determine the analytical expression for the long-term variations of $i$ and $\Omega$ given by $p = \sin i \sin \Omega$ and $q = \sin i \cos \Omega$, with an accuracy of the same order than one used in the determination of the fundamental systems (1) and (2). If $r = 15$, (2) becomes:

$$\begin{align*}
\sin i \quad \sin \quad \sin \\
\cos \quad \cos \quad \cos
\end{align*}
$$

$$\begin{align*}
\sin i \Omega = \sum_{j=1}^{r} B_{l, j} \quad \theta_{ij} + \sum_{j=1}^{r} \sum_{l=1}^{r} B_{l, j} \quad \theta_{ij} \\
+ \sum_{j=1}^{r} \sum_{l=1}^{r} \sum_{k=1}^{r} B_{l, j, k} \quad \theta_{ij, k}
\end{align*}
$$

where

$$\begin{align*}
B_{l, j} = \hat{N}_j (2 - \hat{N}_j^2 - 2 \hat{N}_j^2) \quad \theta_{ij} = \delta_j t + \delta_j, \\
B_{l, j} = -\hat{N}_j^2 \hat{N}_j \quad \theta_{ij} = 2 \theta_j - \theta_j, \\
B_{l, j, k} = -2 \hat{N}_k \hat{N}_j \hat{N}_k \quad \theta_{ij, k} = \theta_j + \theta_k - \theta_j.
\end{align*}
$$

Because of the definition of $\theta_j$ in (2), all arguments in (3) are not different. In fact, ten of the $\theta_{ij}$ and $\theta_{ij, k}$ are identical to some of the $\theta_{ij}$. Consequently, it is more convenient to write this solution back to a form similar to (2) where all mean rates $s_j$ are different:

$$\begin{align*}
\sin i \quad \sin \quad \sin \\
\cos \quad \cos \quad \cos
\end{align*}
$$

$$\begin{align*}
\sin i \Omega = \sum_{k=1}^{15} \hat{N}_k (s_k t + \delta_k).
\end{align*}
$$

Among all terms in (4), respectively 44 and 6 additional terms (terms whose arguments are linear combinations of the first 15 ones) have amplitudes larger than $10^{-7}$ and $10^{-6}$. Even taking only amplitudes larger than $10^{-5}$, values of $i$ and $\Omega$ differ by less than 1%. So, we would propose the values in Table 1, as being the amplitudes, mean rates and phases for the long-term variations of $e, \pi, i, \Omega$ given by (1) and (4).

3. Obliquity and Precession

The method by Sharaf et al. (1967) has been used to compute the long-term variations of the luni-solar precession in longitude $\Psi_f$ and of the inclination $\varepsilon_f$ of the equator on the mean ecliptic of 1850.0. It means that the following expansions include terms of the second degree with respect to the Earth’s eccentricity:

$$\varepsilon_f = \varepsilon + \sum \Delta f \sin (f_j t + \delta_j)$$

and

$$\Psi_f = \Psi t + \zeta + \sum \Delta f \sin (f_j t + \delta_j)$$

In (5) and (6), the mean rates $f_j$ and phases $\delta_j$ are linear combinations respectively of

$$f_j = \Psi + s_j$$

and $\delta_j = \delta_j + \zeta$

and the $\hat{N}_j$'s are equal to $N_j$ or $N_j \hat{N}_j (j \leq k)$. 

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The mean value of the obliquity is given by $\overline{\varpi}$ and $\overline{\varpi}$ represents the mean value of the precession in longitude over one Julian year. The coefficients $C'$ and $D'$ in the amplitudes have been obtained through the expansion of the luni-solar potential of the Poisson's equations in terms of the Newcomb precessional constant $P_0$, of $\varepsilon$, $\varpi$, and $\omega$. Their analytical expressions are given in Sharaf (1967) and it can be seen that they are function of $\overline{\varpi}$ and $f$, except for $D'_n$. Indeed, this last depends upon $\overline{\varpi}$, $g$, and the ratio $\frac{P_{os}}{P_0}$ where $P_{os}$ is the solar contribution to $P_0$.

The constants of integration $\overline{\varpi}$, $\overline{\varpi}$, and $\zeta$ can be determined by solving the following set (7) of three equations, the initial conditions being now referred to the mean ecliptic and the mean equinox of 1850.0:

\[
\begin{align*}
\langle \varepsilon \rangle_0 &= 23^\circ 4588 \\
\langle \varpi \rangle_0 &= 0^\circ \\
\frac{d\langle \varpi \rangle}{dt}_0 &= 50^\circ 36946. 
\end{align*}
\]

If the constants $P_0$ and $P_{os}$ are respectively equal to 54°966 (Woolard and Clemence, 1966) and 17°3919, the solution which takes into account (1) and (4) becomes:

\[
\varpi = 50^\circ 439721 \quad \overline{\varpi} = 23^\circ 394328 \quad \zeta = 1^\circ 993889. 
\]

However, these values are not independent and they verify the following diagnostic equation:

\[
\varpi = P_0 \cos \zeta \left\{ 1 - \sum_k N_k^2 \left[ \frac{3}{2} c_k^2 - \frac{1}{2} c_k - \frac{1}{2} \tg^2 \varepsilon_c (c_k - 1) \right] - \frac{3P_{os}}{P_0} \sum_{k < j} M_k M_j \cos (\delta_k - \delta_j) \right\} 
\]

where

\[
\overline{\varpi} = \frac{\varpi}{c_k}
\]

After 15 iterations, the solution of (7) is stable to the seventh decimal and $\overline{\varpi}$ given by (9) (50°437466) differs only by 0'002 from the computed value. In order to detect computer errors, the same computations have been made starting with the numerical values given by Sharaf (1967). Results found are similar to the ones obtained by Sharaf up to the computational rounding, i.e. referring to the mean equino and mean ecliptic of 1950.0, these constants are:

Sharaf (1967) $\overline{\varpi} = 50^\circ 440174$

\[
\overline{\varpi} = 23^\circ 40111 \quad \zeta = 1^\circ 96459
\]

Berger $\overline{\varpi} = 50^\circ 440175$

\[
\overline{\varpi} = 23^\circ 40109 \quad \zeta = 1^\circ 96455
\]

From (8) and Table 1, mean rates and phases of $\varepsilon$, and $\varpi$ have been determined (Table 2).

As these computations have been carried out essentially for a paleoclimatological problem, the general precession in longitude $P_0$ and the inclination $\varepsilon$ of the ecliptic of date on the equator of date, must be calculated. These two values can be determined by resolving the classical spherical triangle $\Omega, \gamma_1, \gamma$ where $\Omega$ is the ascending node of the ecliptic of date on the ecliptic of epoch, $\gamma_1$ is the direction of the intersection of the equator of date with the ecliptic of epoch and $\gamma$ the mean equinox of date (Fig. 1):

\[
\sin^2 \varepsilon = \sin^2 \gamma_1 \left[ \sin^2 (\varpi_0 + \Omega) + \cos^2 \varepsilon \cos^2 (\varpi_0 + \Omega) \right] + \cos^2 \gamma_1 \sin^2 \varepsilon + \frac{\sin 2\gamma_1}{2} \sin 2\varepsilon \cos (\varpi_0 + \Omega), \quad (11)
\]

\[
\sin (\varpi_0 - \varpi) \sin \varepsilon = \cos \gamma_1 \sin \gamma \sin (\varpi_0 + \Omega) - \sin^2 \gamma_1 \sin \varepsilon \sin 2(\varpi_0 + \Omega). \quad (12)
\]

### 4. Paleo-climatological Solution

The system of Equations (1), (5), (6), (11) and (12) allows the computation of the eccentricity, of the longitude of the perihelion $\omega = \pi + \varpi$ measured from
the equinox of date and of the obliquity using the numerical constants given by (8) and in Tables 1 and 2. For the elements of the Earth's orbit at 1950 (100 Julian years after 1850) referred to the mean ecliptic and the mean equinox of 1850, comparison between the computed values and the observed ones obtained from Andoyer's formulae (Woodall and Clemence, 1966), shows the accuracy of the solution in restoring the initial conditions:

Bretagnon-Berger

Andoyer
e = 0.0167239
\Psi = 1^{°}23'57''
1^{°}23'56''
\pi = 100^{\circ}38'40''
\epsilon = 23^{°}27'31''
23^{°}27'31.7''
\Omega = 178^{°}21'17''
i = 0^{°}04'99''
\varpi = 102^{°}23'36''
102^{°}8'50''

For centuries centered at epoch, it is more appropriate to use power series in time (Laubscher, 1972; Brumberg et al., 1975) to compute all the Earth's orbital elements. Nevertheless, when values must be computed over large periods as being required in paleoclimatology, the trigonometric series method must be used. On Figures 2a–f presented here, \( \epsilon, \varpi, \bar{\varpi} \) have been put together because they are the sensitive ecliptical elements in the astronomical theory of paleoclimates, namely for the determination of the insolation (Berger, 1975a). Moreover, simultaneous occurrences can be immediately localized, intervals when \( \bar{\varpi} \) is close to 90°, \( \epsilon \) is minimum and \( \epsilon \) is maximum, being of particular interest in order

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to explain the glacial periods of the Quaternary Ice Age, according to Milankovitch (1941). The time step is one thousand years and as it would be interesting to look to the future in comparison with the past, the corresponding values for the next million years will be given in parentheses. Because the main features of this next million years are not too different from those of the previous one million years, it will also show how much older periods can depart from our present one. Years before 1950 will be noted B.P. and after 1950, A.P.

During the previous 5 millions years, the eccentricity is seen to vary between 0.0004829 at 4609000 B.P. (0.0026875 at 270000 A.P.) and 0.0607908 at 3072000 B.P. (0.0565598 at 1000000 A.P.). The theoretical minimum and maximum values of \( e \) are respectively 0 and 0.072826. The period is extremely regular, the average being 95800 (91400) years. The analysis of such an extended large time-span clearly shows a secondary main period of around 400000 years. The steady high value experienced from 1722000 to 2000000 B.P. is remarkable and will certainly be important for the past climates. We are now at the beginning of a 101000 years period which extends from the 14000 B.P. maximum to a weak maximum, nearly a plateau centered at 75000 A.P., and during which \( e \) will remain small. Correlatively, the line of apsides made 39 (9) revolutions in a fixed reference system with a period going from 19900 (11600) to 312000 (224000) years, the dispersion around the mean 125200 (103300) years being very large. In fact, when \( e \) becomes zero, \( \pi \) no more defined and there is a jump in the \( \pi \) value. Such important jumps are localized at 913, 1691, 2515, 3375 and 4610 thousands years ago, dates where there is also an abrupt change in the variation of \( \dot{\omega} \). Moreover, when at the same time \( \pi \) is close to 360°, a very short "artificial" period is generated and, as a consequence, the perihelion is maintained close to the mean equinox of 1850.0 during a longer time. It is the case at 1296000 B.P., 3763000 B.P. and 954000 A.P. where \( e \) is respectively equal to 0.002, 0.001 and 0.005. If we except these events, the minimum more realistic period becomes 86600 (62600) years. These artificial periods have of course no physical meaning in celestial mechanics, but the numerical results related to them have to be interpreted carefully.

Figs. 2a-f. The variation of the eccentricity \( e \), of the longitude of the perihelion \( \dot{\omega} \) and of the obliquity \( \epsilon \) of the Earth's orbit as a function of time for the period from the present time (1950.0) to 1000000 years after present (a), for the period 1000000 years before the present time (B.P.) to the present time (b), for the period 2000000 years B.P. to 1000000 years B.P. (c), for the period 3000000 years B.P. to 2000000 years B.P. (d), for the period 4000000 years B.P. to 3000000 years B.P. (e), and for the period 5000000 years B.P. to 4000000 years B.P. (f). The left-hand scale is related to the eccentricity, the right-hand scales are in degrees of arc and concern respectively \( \dot{\omega} \) and \( \epsilon \).
in paleoclimatology where $e \sin \omega$ is an important factor in most of the insolation parameters.

Very much the same behaviour is found for the inclination and the longitude of the node (Figs. 3a–f). However, for that system, the one outstanding term will make the average period around 68800 years for the inclination on the mean ecliptic of 1850.0. This inclination varies between 0° and 4°13′48″ (3°38′24″), the maximum theoretical value being 5°5′38″. This absolute maximum is reduced to 3°30′44″ if the invariable plane is chosen as the plane of reference.

Another important quantity relevant to paleoclimatology, as well as to geophysics and astronomy, is the position of the rotation pole of the Earth with respect to the moving ecliptic.

The period covering a revolution of the mean equinox relative to the moving perihelion is 21740 (21350) years on an average, and goes from 13900 (15390) to 31300 (29700) years. Simultaneously, the inclination of the Earth's equator on the moving ecliptic has varied between 22°2′33″ (22°13′44″) and 24°30′16″ (24°20′50″) with a mean period of 41000 (41040) years, combination between the precessional motion and the period characteristic of the large term in the motion of the ecliptic. The precessional motion has a very steady mean period of 25700 (25700) years and the inclination of the Earth's equator on the fixed mean ecliptic of 1850.0 varies from 18°16′44″ (18°59′56″) to 28°26′23″ (27°38′23″). The maximum deviation between the luni-solar and the general precession in longitude is close to 10° (8°).

The digits given here for each value of the Earth's orbital elements are raw results of computation and some of them are probably not significant. However, it seems that a very high degree of accuracy has been reached now. For the last million years, the difference with the solution proposed by Berger (1976), is generally small and the two solutions are perfectly in phase, indicating a good convergence to the ideal solution. Nevertheless, this difference may reach occasionally some half a degree for the obliquity and values found here are systematically larger. For the longitude of the perihelion, the deviation amounts only 5° as far as one million years ago.

Figs. 3a–f. The variation of the inclination $i$ of the Earth's orbit on the ecliptic of 1850.0, of the longitude of the node $\Omega$ and of the longitude of the perihelion $\kappa$ as a function of time for the period from the present time (1950.0) to 10000000 years after present (a), for the period 1000000 years before the present time (B.P.) to the present time (b), for the period 2000000 years B.P. to 1000000 years B.P. (c), for the period 3000000 years B.P. to 2000000 years B.P. (d), for the period 4000000 years B.P. to 3000000 years B.P. (e), and for the period 5000000 years B.P. to 4000000 years B.P. (f). The left-hand scale is related to both $\kappa$ and $\Omega$. The right-hand scale concerns the inclination $i$. Both scales are in degrees of arc.

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Some improvements are now expected to test the accuracy of this solution for the long-term variations of the Earth's orbital elements. These improvements may occur in the field of theoretical researches (Brumberg and Chapront, 1973; Chapront et al., 1975), in the techniques used to determine the planetary motion (Cohen et al., 1973) or even in the number of terms kept in the expansions used, just like it has been done for the secular perturbations (Brumberg et al., 1975) and already stated by Bretagnon (1974): although the neglected terms would not introduce large modifications of the constants of integration, the calculation should be repeated including long period terms of fifth order and also short period terms of higher order.

At the end, it is also advisable to be careful as regards to the absolute accuracy of the results, inaccuracies in the frequencies producing an effect the importance of which becomes larger and larger as the time increases.

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References

Milankovitch, M.: 1941, Kanon der Erdbestrahlung, Kunglische Serbische Akademie, Beograd