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ASTRONOMY AND ASTROPHYSICS

Obliquity and Precession for the Last 5000000 Years

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Summary. A new solution is numerically determined for the obliquity of the ecliptic and the longitude of the perihelion measured from the equinox of date. For the eccentricity, the longitude of the perihelion measured from the equinox of epoch, the inclination and the longitude of the node, the solution by Bretagnon has been used. It includes terms to the "second" order to the disturbing masses and to the third degree with respect to the planetary eccentricities and inclinations. For the obliquity and the precession in longitude, Sharaf's analytical expansions are used, which means that terms to the second degree with respect to the Earth's eccentricity have been kept. In this paper their numerical determination takes into account all terms of Bretagnon. Parameters of the expansions are tabulated and a graphical representation extends back to 5000000 years before present and forewards to 1000000 years after present.

Key words: planetary theory—long-term variation—paleoclimatology

1. Introduction

In a recent paper (Berger, 1976), the author has demonstrated the importance of terms generally neglected in the solution giving the long-term variations of the Earth's orbital elements. Table 1 (Berger, 1976) recalls the successive improvements which have been made to calculate these solutions. The results obtained when comparing all these solutions may be summarized in the following way with regard to the influence of different terms on the accuracy of the expansions used. First, further improvements in planetary masses will not have significant influence. Second, for the system in eccentricity e and longitude of the perihelion π measured from the equinox of epoch, terms depending upon the second order as to the disturbing masses are more important than the ones coming from the third degree with respect to the planetary eccentricities and inclinations. Third, for the inclination i of the ecliptic on the plane of reference and the longitude of the ascending node Ω, the latter terms have highly significant influence, whereas additional terms in masses are negiligible. Fourth, because of the nature of their expansion, similar conclusions can be drawn for the time variation of the obliquity ε and the precession Ψ of the equinoxes. However, in this case, the series expansion of these elements must be examined. Analysing the results obtained both from a Laplace's expansion, i.e. including terms depending on the first degree with respect to the Earth's eccentricity and from the Sharaf's expansion (Sharaf et al., 1967), i.e. including terms depending on the second degree with respect to the Earth's eccentricity, the author came to the following conclusions. For the previous and the next millions years, it seems sufficient to consider a Laplace's expansion but where amplitudes, phases and mean rates are calculated from those present in the (e, π) and (i, Ω) Bretagnon solutions, solutions where terms depending on the second power as to the masses and especially the third degree with respect to the planetary e's and i's (Bretagnon, 1974) have been included. However, for more extended time spans, the Sharaf's terms have to be considered, taking into account all the additional Bretagnon's terms. It is the aim of this paper to construct numerically and to represent graphically such a solution.

2. Fundamental Systems

As numerically demonstrated (Berger, 1976), Bretagnon (1974) makes a decisive improvement in the evaluation of the eccentricity, the longitude of the perihelion, the inclination and the longitude of the ascending node. Let us summarize the main caracteristics of that solution. The elements of the Earth's orbit are referred to the mean ecliptic and the mean equinox of 1850.0. Initial elements at this epoch of reference are the most accurate and secular perturbations have been taking out of the observed mean motions, the semi-major axis of the ecliptic being then given by the third law of Kepler. Disturbing planetary masses as proposed by Kovalesky (1971) are used. Constants of integration have been carefully determined in such a way that the solution coincides with the mean elements at the time origin 1850.0, calculation which is not found in a similar work done by Anolik et al. (1969).

In order to show the coherence in the accuracy through the all solution built in this paper, it is also necessary to remember that, in the Bretagnon's expansions themselves, both terms depending on the third degree with respect to the planetary eccentricities and inclinations and terms of the second order to the planetary masses are included. Indeed, Bretagnon takes into account all long period terms of the disturbing function, of fourth order in the planetary e's and i's. In a second approximation he also introduced the short period terms of the disturbing function.
About the influence of these short period terms to the second order as to the masses, terms limited to the third degree in e's and i's which lead to a modification of the long period frequencies by more than 10^{-3}"/year (around 50 for the only Jupiter-Saturn near-resonance) have been kept for all the planets. Among the long period terms generated through these terms, only those which amplitude is greater than 10^{-4} for the inner planets and 10^{-6} for the major ones have been considered. On the other hand, the short period terms include, in particular, those of third order of the great inequality in the motion of Jupiter and Saturn.

In comparison, let us recall that Brouwer (1950, p. 88) considered, alike Hill (1897), the influence to the second order of the disturbing masses of the only Jupiter-Saturn near-resonance and that, from equations for the eccentricities and longitudes of the perihelia of Jupiter and Saturn obtained by Le Verrier (1874). This method led to contributions up to the fifth order in e's and i's of the short period terms and these generated long period terms of which Brouwer kept only the two most important ones. To stress the importance of all these additional terms, the solution by Bretagnon has been written in a new appropriate form (Berger, 1975b, formulae 11-12-13) which allows the direct determination of the amplitudes associated with arguments of the Lagrange's terms.

For the computational requirement, these solutions of the fundamental systems in eccentricity and in inclination can also be expressed in the following way where only the most significant terms have been kept:

$$\begin{matrix} h \\ k \end{matrix} = e \begin{matrix} \sin \\ \cos \end{matrix} \pi = \sum_{j=1}^{19} M_j \begin{matrix} \sin \\ \cos \end{matrix} (g_j t + \beta_j), \qquad (1)$$

$$\begin{matrix} p_{(i/2)} \\ q_{(i/2)} \end{matrix} = \sin\frac{i}{2} \begin{matrix} \sin \\ \cos \end{matrix} \Omega = \sum_{j=1}^{15} \hat{N}_j \begin{matrix} \sin \\ \cos \end{matrix} (\hat{s}_j t + \hat{\delta}_j). \qquad (2)$$

From (2), it is possible to determine the analytical expression for the long-term variations of i and Ω given by $p = \sin i \sin \Omega$ and $q = \sin i \cos \Omega$, with an accuracy of the same order than one used in the determination of the fundamental systems (1) and (2). If $r = 15$, (2) becomes:

$$\sin i \begin{matrix} \sin \\ \cos \end{matrix} \Omega = \sum_{l=1}^{r} B_l \begin{matrix} \sin \\ \cos \end{matrix} \theta_l + \sum_{j}^{r} \sum_{l \neq j}^{r} B_{lj} \begin{matrix} \sin \\ \cos \end{matrix} \theta_{lj} \qquad (3)$$

$$+ \sum_{j}^{r} \sum_{l \neq j}^{r-1} \sum_{\substack{k>l \\ k \neq j}}^{r} B_{ljk} \begin{matrix} \sin \\ \cos \end{matrix} \theta_{ljk} \qquad (3)$$

where

$$B_l = \hat{N}_l \Big(2 - \hat{N}_l^2 - 2 \sum_{j \neq l} \hat{N}_j^2\Big) \qquad \theta_l = \hat{s}_l t + \hat{\delta}_l$$

$$B_{lj} = -\hat{N}_l^2 \hat{N}_j \qquad \theta_{lj} = 2\theta_l - \theta_j$$

$$B_{ljk} = -2 \hat{N}_l \hat{N}_j \hat{N}_k \qquad \theta_{ljk} = \theta_l + \theta_k - \theta_j .$$

Because of the definition of $\hat{s}_l$ in (2), all arguments in (3) are not different. In fact, ten of the θ_{lj} and θ_{ljk} are identical to some of the θ_l. Consequently, it is more convenient to write this solution back to a form similar to (2) where all mean rates s_l are different:

$$\begin{matrix} p \\ q \end{matrix} = \sin i \begin{matrix} \sin \\ \cos \end{matrix} \Omega = \sum_{k} N_k \begin{matrix} \sin \\ \cos \end{matrix} (s_k t + \delta_k). \qquad (4)$$

Among all terms in (4), respectively 44 and 6 additional terms (terms whose arguments are linear combinations of the first 15 ones) have amplitudes larger than 10^{-7} and 10^{-6}. Even taking only amplitudes larger than 10^{-5}, values of i and Ω differ by less than 1%. So, we would propose the values in Table 1, as being the amplitudes, mean rates and phases for the long-term variations of e, π, i, Ω given by (1) and (4).

3. Obliquity and Precession

The method by Sharaf et al. (1967) has been used to compute the long-term variations of the luni-solar precession in longitude Ψ_f and of the inclination ε_f of the equator on the mean ecliptic of 1850.0. It means that the following expansions include terms of the second degree with respect to the Earth's eccentricity:

$$\varepsilon_f = \bar{\varepsilon} + \sum_l C_l^f \tilde{N}_l \cos(\tilde{f}_l t + \tilde{\delta}_l) \qquad (5)$$

$$\Psi_f = \bar{\Psi} t + \zeta + \sum_l D_l^f \tilde{N}_l \sin(\tilde{f}_l t + \tilde{\delta}_l)$$

$$+ \sum_{j<k} D_{jk}^{f'''} M_j M_k \sin[(g_j - g_k)t + \beta_j - \beta_k]. \qquad (6)$$

In (5) and (6), the mean rates $\tilde{f}$ and phases $\tilde{\delta}$ are linear combinations respectively of

$f_j = \bar{\Psi} + s_j$ and $\delta'_j = \delta_j + \zeta$

and the $\tilde{N}_l$'s are equal to N_j or $N_j N_k (j \leqq k)$.

The mean value of the obliquity is given by $\bar{\varepsilon}$ and $\bar{\Psi}$ represents the mean value of the precession in longitude over one Julian year. The coefficients C^f and D^f in the amplitudes have been obtained through the expansion of the luni-solar potential of the Poisson's equations in terms of the Newcomb precessional constant P_0, of ε_f, Ψ_f, i and Ω.

Their analytical expressions are given in Sharaf (1967) and it can be seen that they are function of $\bar{\varepsilon}$, $\bar{\Psi}$ and f_j, except for $D^{f''}_{jk}$. Indeed, this last depends upon $\bar{\Psi}$, g_j and the ratio $\frac{P_{0s}}{P_0}$ where P_{0s} is the solar contribution to P_0.

The constants of integration $\bar{\varepsilon}$, $\bar{\Psi}$ and ζ can be determined by solving the following set (7) of three equations, the initial conditions being now referred to the mean ecliptic and the mean equinox of 1850.0:

$$(\varepsilon_f)_0 = 23°4588$$
$$(\Psi_f)_0 = 0° \tag{7}$$
$$\left(\frac{d\Psi_f}{dt}\right)_0 = 50''36946\,.$$

If the constants P_0 and P_{0s} are respectively equal to 54″9066 (Woolard and Clemence, 1966) and 17″3919, the solution which takes into account (1) and (4) becomes:

$$\bar{\Psi} = 50''439721 \qquad \bar{\varepsilon} = 23°394\,328 \qquad \zeta = 1°993\,889\,. \tag{8}$$

However, these values are not independent and they verify the following diagnostic equation:

$$\bar{\Psi} = P_0 \cos\bar{\varepsilon}\left\{1 - \sum_k N_k^2\left[\tfrac{3}{2} + \tfrac{3}{4}c_k^2 - \tfrac{5}{2}c_k - \tfrac{1}{2}\,\mathrm{tg}^2\varepsilon c_k(c_k - 1)\right] - \frac{3P_{0s}}{P_0}\sum_{k<j} M_k M_j \cos(\delta_k - \delta_j)\right\} \tag{9}$$

where

$$c_k = \frac{\bar{\Psi}}{f_k}\,. \tag{10}$$

After 15 iterations, the solution of (7) is stable to the seventh decimal and $\bar{\Psi}$ given by (9) (50″437466) differs only by 0″002 from the computed value. In order to detect computer errors, the same computations have been made starting with the numerical values given by Sharaf (1967). Results found are similar to the ones obtained by Sharaf up to the computational rounding, i.e. referring to the mean equinox and mean ecliptic of 1950.0, these constants are:

Sharaf (1967)	$\bar{\Psi} = 50''440\,174$
	$\bar{\varepsilon} = 23°401\,11$
	$\zeta = 1°964\,59$
Berger	$\bar{\Psi} = 50''440\,175$
	$\bar{\varepsilon} = 23°401\,09$
	$\zeta = 1°964\,55$

From (8) and Table 1, amplitudes, mean rates and phases of ε_f and Ψ_f have been determined (Table 2).

As these computations have been carried out essentially for a paleoclimatological problem, the general precession in longitude Ψ and the inclination ε of the ecliptic of date on the equator of date, must be calculated. These two values can be determined by resolving the classical spherical triangle Ω, γ_1, γ where Ω is the ascending node of the ecliptic of date on the ecliptic of epoch, γ_1 is the direction of the intersection of the equator of date with the ecliptic of epoch and γ the mean equinox of date (Fig. 1):

$$\sin^2\varepsilon = \sin^2 i\left[\sin^2(\Psi_f + \Omega) + \cos^2\varepsilon_f \cos^2(\Psi_f + \Omega)\right] + \cos^2 i \sin^2\varepsilon_f + \frac{\sin 2i}{2}\sin 2\varepsilon_f \cos(\Psi_f + \Omega)\,, \tag{11}$$

$$\sin(\Psi_f - \Psi)\sin\varepsilon = \cos\varepsilon_f \sin i \sin(\Psi_f + \Omega) - \sin^2\tfrac{i}{2}\sin\varepsilon_f \sin 2(\Psi_f + \Omega)\,. \tag{12}$$

4. Paleoclimatological Solution

The system of Equations (1), (5), (6), (11) and (12) allows the computation of the eccentricity, of the longitude of the perihelion $\tilde{\omega} = \pi + \Psi$ measured from

Table 1. Parameters in series expansions of fundamental ecliptical elements

	ECCENTRICITY AND LONGITUDE OF PERIHELION				INCLINATION AND LONGITUDE OF NODE			
I	AMPLITUDE	MEAN RATE ('' /YEAR)	PHASE (DEGREE)	PERIOD (YEARS)	AMPLITUDE	MEAN RATE ('' /YEAR)	PHASE (DEGREE)	PERIOD (YEARS)
1	0.00333077	5.199079	87.050581	249275.	0.01207583	-5.610937	12.138097	230977.
2	0.01627522	7.346091	193.584714	176420.	0.00508255	-6.771027	305.221442	191404.
3	0.00988829	17.220546	319.721289	75259.	0.02003973	-18.829299	249.033075	68829.
4	-0.01300660	17.857263	307.810989	72576.	0.00760908	-17.818769	277.935044	72732.
5	0.01860798	4.207205	28.503222	308043.	0.02767151	0.0	106.153214	0.
6	0.00140015	26.216758	127.715144	49434.	0.00280772	-26.267070	125.642842	49339.
7	0.00064990	3.065181	114.775439	422814.	-0.00173126	-2.999837	316.293317	432023.
8	0.00001250	0.667863	72.090286	1940518.	-0.00129855	-0.691431	201.287664	1874374.
9	0.00037800	18.493980	295.900689	70077.	-0.00024791	-4.624015	51.755575	280276.
10	0.00059900	16.583829	331.631589	78148.	-0.00080171	-6.602811	313.590739	196280.
11	-0.00033700	6.190953	145.597939	209338.	0.00180535	-7.757949	265.603964	167054.
12	-0.00017400	6.186001	126.668058	209505.	0.00099776	-18.192582	237.122775	71238.
13	0.00100700	6.359169	153.967236	203800.	-0.00238320	-19.466016	260.943375	66578.
14	-0.00012400	18.417441	210.155603	70368.	0.00348076	-18.455486	289.845344	70223.
15	0.00018200	17.425567	151.608244	74373.	-0.00075611	-19.839829	220.131106	65323.
16	-0.00235400	18.231076	348.623258	71087.				
17	0.00027600	18.867793	336.712958	68688.				
18	0.00085700	16.210016	290.819319	79951.				
19	-0.00336700	16.846733	278.909019	76929.				

Table 2. Obliquity and precession relative to mean ecliptic and mean equinox of 1850.0 A.D.

I	OBLIQUITY ('')	PRECESSION ('')	MEAN RATE (''/YEAR)	PHASE (DEGREE)	PERIOD (YEARS)
1	-2802.58	6629.89	44.828784	14.13199	28910.
2	-1210.90	2880.22	43.668694	307.21533	29678.
3	-6595.68	16945.54	31.610422	251.02696	40999.
4	-2426.80	6183.01	32.620952	279.92893	39729.
5	-5707.66	13193.20	50.439721	108.14710	25694.
6	-1208.44	3361.40	24.172651	127.63673	53614.
7	379.68	-888.01	47.439884	318.28721	27319.
8	271.57	-629.36	49.748290	203.28155	26051.
9	56.30	-132.59	45.815706	53.74946	28287.
10	190.27	-452.21	43.836910	315.58463	29564.
11	-440.07	1051.81	42.681772	267.59785	30364.
12	-321.91	822.66	32.247139	239.11666	40190.
13	800.51	-2068.01	30.973705	262.93726	41842.
14	-1132.23	2899.78	31.984235	291.83923	40520.
15	257.08	-666.34	30.599892	222.12499	42353.
16	-22.98	113.51	89.657568	28.26397	14455.
17	-138.14	641.89	63.220844	142.05393	20500.
18	-19.28	90.04	65.241904	199.85787	19865.
19	-91.27	461.42	100.879442	216.29421	12847.
20	-19.81	97.60	88.497478	321.34732	14644.
21	-116.49	553.80	76.439206	265.15895	16955.
22	-42.53	203.00	77.449736	294.06092	16733.
23	-91.73	457.99	95.268505	122.27909	13604.
24	-22.98	105.31	69.001435	141.76872	18782.
25		-30.61	92.268668	332.41919	14046.
26		35.80	87.510556	281.72984	14810.
27		26.96	77.075923	253.24865	16815.
28	14.21	-67.38	75.802489	277.06925	17097.
29	-19.94	94.93	76.813019	305.97122	16872.
30	-50.48	239.65	75.279116	198.24229	17216.
31	-18.43	87.85	76.289646	227.14426	16988.
32	-39.85	198.38	94.108415	55.36243	13771.
33		45.55	67.841345	74.85206	19103.
34		-29.16	74.642399	210.15259	17363.
35		41.08	75.652929	239.05456	17131.
36	-107.78	502.05	64.231374	170.95590	20177.
37	-234.45	1120.78	82.050143	359.17407	15795.
38	-55.78	252.69	55.783073	18.66369	23233.
39	15.69	-74.79	79.050306	209.31417	16395.
40	11.17	-53.36	81.358712	94.30852	15929.
41		-37.65	75.447332	206.61159	17178.
42	-18.39	87.22	74.292194	158.62482	17445.
43	-14.01	65.18	63.857561	130.14363	20295.
44	37.01	-171.68	62.584127	153.96423	20708.
45	-36.49	169.74	63.594657	182.86620	20379.
46	11.29	-52.32	62.210314	113.15196	20833.
47	-85.55	410.72	83.060673	28.07604	15603.
48	-20.45	92.76	56.793603	47.56566	22819.
49		-27.41	80.060836	238.21614	16188.
50		31.97	75.302724	187.52679	17211.
51	-18.05	84.17	64.605187	211.76817	20060.
52	-46.51	213.55	74.612372	235.78383	17370.
53	12.29	-61.72	97.879605	66.43431	13241.
54		-43.96	100.188011	311.42866	12936.
55		-31.16	94.276631	63.73173	13747.
56	-14.55	72.26	93.121493	15.74496	13917.
57	-11.38	54.56	82.686860	347.26377	15674.
58	28.61	-136.39	81.413426	11.08437	15919.
59	-40.12	192.10	82.423956	39.98634	15724.
60		-43.87	81.039613	330.27210	15992.
61		-30.72	55.146356	30.57399	23501.
62		43.34	56.156886	59.47596	23078.
63		-27.90	62.957940	194.77650	20585.
64	20.01	53.23	1.160090	66.91666	1117155.
65	117.55		13.218362	123.10502	98045.
66	42.90		12.207832	94.20305	106161.
67	91.73	33.92	5.610937	94.01512	230977.
68	23.28		20.656133	246.49525	62742.
69		165.90	0.986922	39.61748	1313174.
70	-14.34		13.855079	111.19472	93540.
71	20.12		12.844549	82.29275	100899.
72	51.04		12.058272	56.18837	107478.
73	18.63		11.047742	27.28640	117309.
74	39.85		6.771027	160.93177	191404.
75	10.10		19.496043	179.57860	66475.
76		-45.42	0.168216	8.36930	7704380.
77	90.92	32.96	1.010530	28.90197	1282495.
78	234.45	36.90	18.829299	217.12014	68829.
79	58.72	-62.73	7.437771	123.39023	174246.
80	-15.76		15.829462	67.26024	81873.
81	-11.18		18.137868	312.25459	71453.
82	18.63		11.071350	16.57089	117059.
83		-1973.65	0.636717	348.08970	2035441.
84	57.74	-496.17	0.373813	40.81227	3466974.
85	85.55		17.818769	188.21817	72732.
86	21.45		8.448301	152.29220	153404.
87	-13.85	-42.05	1.647247	16.99167	786767.
88	46.51		26.267070	340.51037	49339.
89	-12.29		2.999837	149.85990	432023.
90	14.55		7.757949	200.54925	167054.
91	11.38		18.192582	229.03044	71238.
92	-28.61		19.466016	205.20984	66578.
93	40.12		18.455486	176.30787	70223.
94	10.05		7.811584	164.20250	165907.
95		-48.31	1.155138	47.98678	1121944.
96		-27.18	1.273434	336.17940	1017721.
97		-133.53	0.262904	307.27743	4929556.
98		61.57	1.384343	69.71424	936184.
ADDITIONAL TERMS FOR PRECESSION					
99		249.62	2.147012	106.53413	603630.
100		27.09	12.021467	232.67071	107807.
101		-33.84	12.658184	220.76041	102384.
102		606.36	C.991874	58.54736	1306618.
103		161.13	9.874455	126.13658	131248.
104		-199.10	10.511172	114.22627	123297.
105		953.88	3.138886	165.08149	412885.
106		-34.80	10.884985	155.03854	119063.
107		-57.02	9.500642	85.32430	136412.
108		139.79	13.013341	291.21807	99590.
109		86.78	0.205021	191.88696	6321304.
110		-175.30	13.650058	279.30777	94945.
111		28.46	0.560178	262.34461	2313550.
112		-54.21	0.431696	156.20274	3002113.
113		104.69	1.142024	273.72778	1134827.
114		-31.25	1.983748	117.09472	653309.
115		86.09	2.151964	125.46401	602241.
116		-30.88	14.023871	320.12004	92414.
117		-49.01	12.639528	250.40580	102535.
118		117.07	0.004952	18.92988	261712439.

the equinox of date and of the obliquity using the numerical constants given by (8) and in Tables 1 and 2.

For the elements of the Earth's orbit at 1950 (100 Julian years after 1850.0) referred to the mean ecliptic and the mean equinox of 1850.0, comparison between the computed values and the observed ones obtained from Andoyer's formulae (Woolard and Clemence, 1966), shows the accuracy of the solution in restoring the initial conditions:

Bretagnon-Berger	Berger	Andoyer
$e=0.0167239$	$\Psi_f=$ 1°23′57″	1°23′56″
$\pi=100°38'40''$	$\varepsilon_f=$ 23°27′31″	23°27′31.″7
$\Omega=178°21'17''$	$\varepsilon=$ 23°26′42″	23°26′44.″8
$i=$ 0° 0′49″	$\tilde{\omega}=102°$ 2′36″	102° 8′50″

For centuries centered at epoch, it is more appropriate to use power series in time (Laubscher, 1972; Brumberg et al., 1975) to compute all the Earth's orbital elements. Nevertheless, when values must be computed over large periods as being required in paleoclimatology, the trigonometric series method must be used. On Figures 2a–f presented here, $\varepsilon, e, \tilde{\omega}$ have been put together because they are the sensitive ecliptical elements in the astronomical theory of paleoclimates, namely for the determination of the insolation (Berger, 1975a). Moreover, simultaneous occurences can be immediately localized, intervals when $\tilde{\omega}$ is close to 90°, ε is minimum and e is maximum, being of particular interest in order

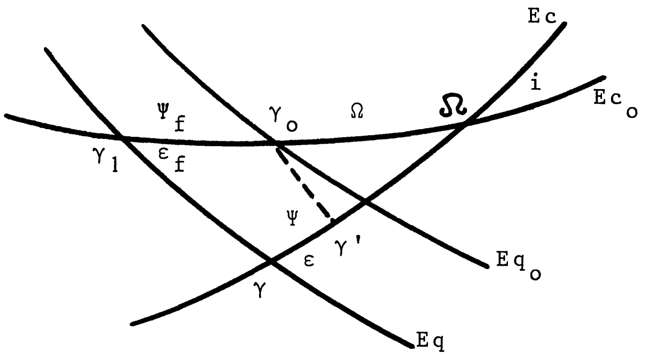
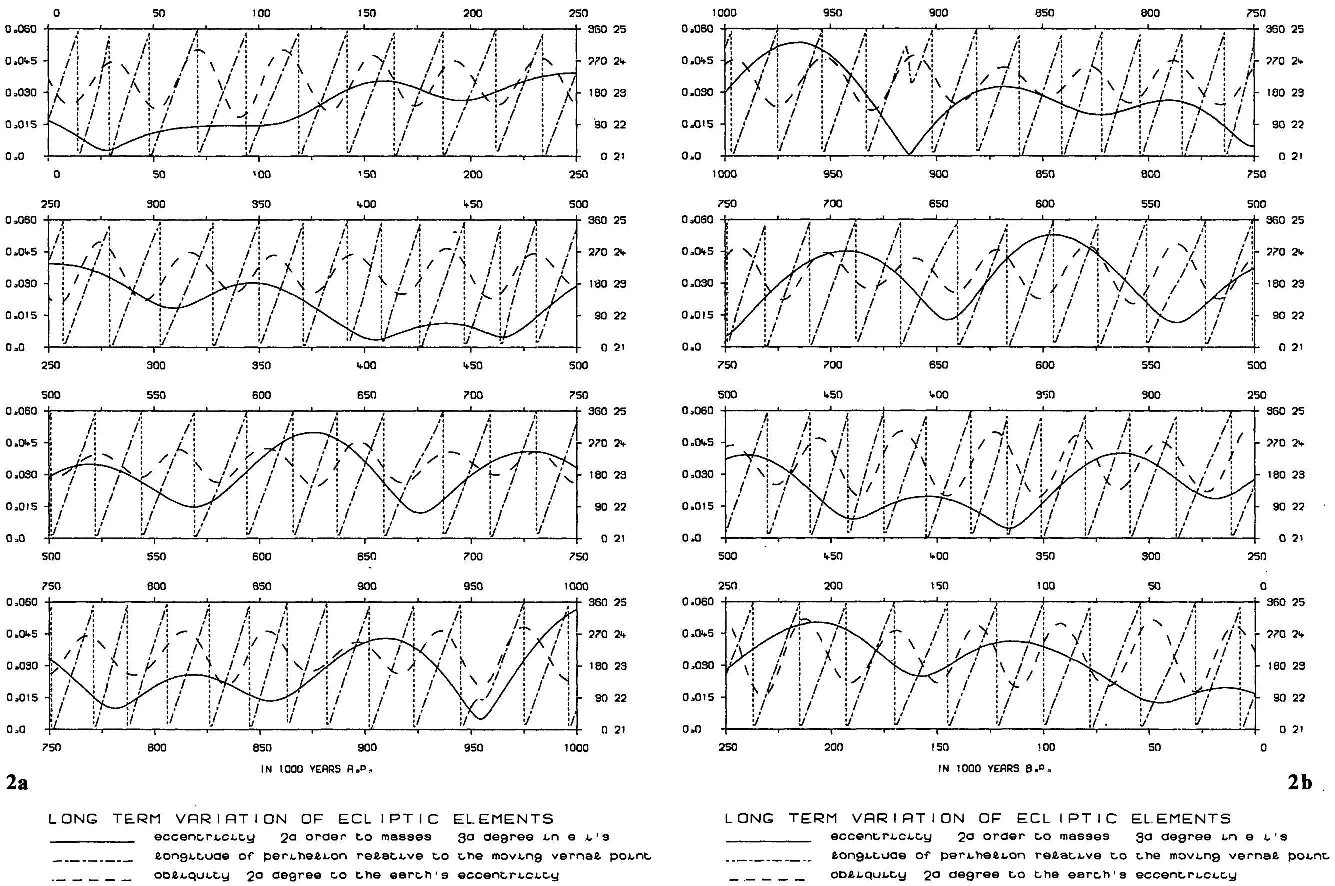


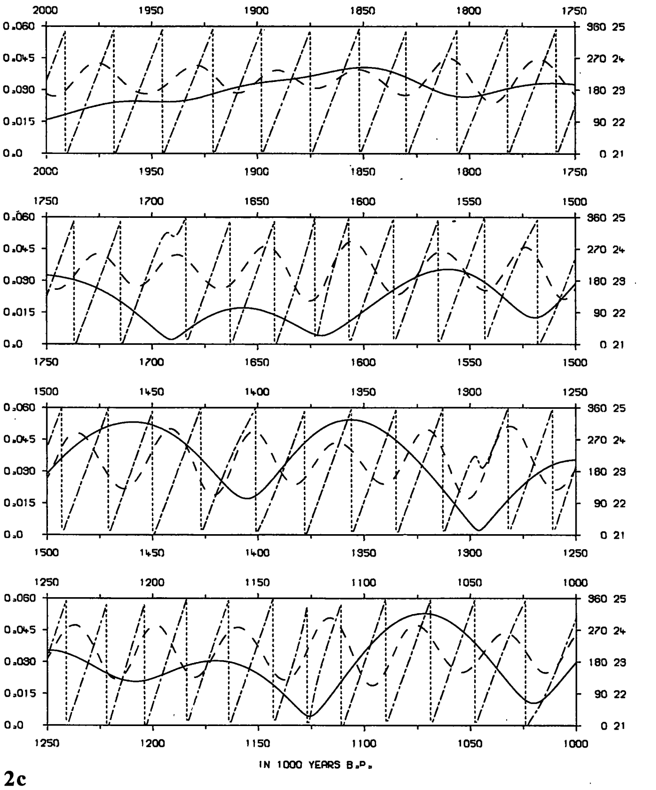
Fig. 1. Obliquity and precession in longitude for the Earth's orbit. Ec is the ecliptic of date, Ec_0 the ecliptic of epoch, Eq the equator of date, Eq_0 the equator of epoch, γ the mean vernal equinox of date, γ_0 the mean vernal equinox of epoch, $\gamma_1\gamma_0=\Psi_f$ the luni-solar precession in longitude, $\gamma'\gamma=\Psi$ the general precession in longitude, $\gamma_0\Omega=\Omega$ the longitude of the node, i the inclination of the Earth's orbit of date on the Earth's orbit of epoch, ε_f and ε the inclination of the equator of date respectively on the ecliptic of epoch and on the ecliptic of date (Woolard and Clemence, 1966)

to explain the glacial periods of the Quaternary Ice Age, according to Milankovitch (1941). The time step is one thousand years and as it would be interesting to look to the future in comparison with the past, the corresponding values for the next million years will be given in parentheses. Because the main features of this next million years are not too different from those of the previous one million years, it will also show how much older periods can depart from our present one. Years before 1950 will be noted B.P. and after 1950, A.P.

During the previous 5 millions years, the eccentricity is seen to vary between 0.0004829 at 4609000 B.P. (0.0026875 at 27000 A.P.) and 0.0607908 at 3072000 B.P. (0.0565598 at 1000000 A.P.). The theoretical minimum and maximum values of e are respectively 0 and 0.072826. The period is extremely regular, the average being 95800 (91400) years. The analysis of such an extended large time-span clearly shows a secondary main period of around 400000 years. The steady high value experienced from 1720000 to 2000000 B.P. is remarkable and will certainly be important for the past climates. We are now at the beginning of a 101000 years period which extends from the 14000 B.P. maximum to a weak maximum, nearly a plateau centered at 75000 A.P., and during which e will remain small. Correlatively, the line of apsides made 39 (9) revolutions in a fixed reference system with a period going from 19900 (11600) to 312000 (224000) years, the dispersion around the mean 125200 (103300) years being very large. In fact, when e becomes zero, π is no more defined and there is a jumb in the π value. Such important jumbs are localized at 913, 1691, 2515, 3375 and 4610 thousands years ago, dates where there is also an abrupt change in the variation of $\tilde{\omega}$. Moreover, when at the same time π is close to 360°, a very short "artifical" period is generated and, as a consequence, the perihelion is maintained close to the mean equinox of 1850.0 during a longer time. It is the case at 1296000 B.P., 3763000 B.P. and 954000 A.P. where e is respectively equal to 0.002, 0.001 and 0.005. If we except these events, the minimum more realistic period becomes 58600 (62600) years. These artifical periods have of course no physical meaning in celestial mechanics, but the numerical results related to them have to be interpreted carefully

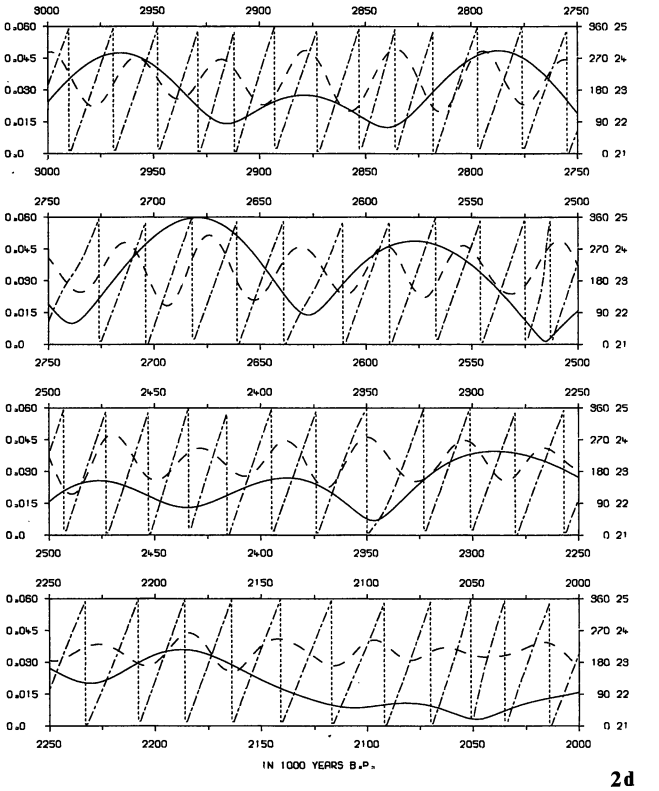


Figs. 2a–f. The variation of the eccentricity e, of the longitude of the perihelion $\tilde{\omega}$ and of the obliquity ε of the Earth's orbit as a function of time for the period from the present time (1950.0) to 1000000 years after present **(a)**, for the period 1000000 years before the present time (B.P.) to the present time **(b)**, for the period 2000000 years B.P. to 1000000 years B.P. **(c)**, for the period 3000000 years B.P. to 2000000 years B.P. **(d)**, for the period 4000000 years B.P. to 3000000 years B.P. **(e)**, and for the period 5000000 years B.P. to 4000000 years B.P. **(f)**. The left-hand scale is related to the eccentricity, the right-hand scales are in degrees of arc and concern respectively $\tilde{\omega}$ and ε



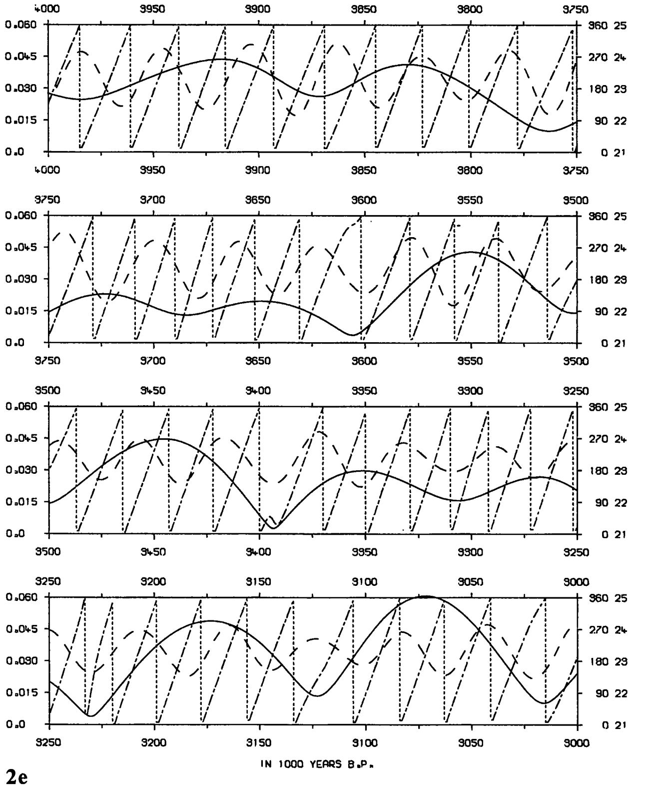
2c

LONG TERM VARIATION OF ECLIPTIC ELEMENTS
eccentricity 2d order to masses 3d degree in e i's
longitude of perihelion relative to the moving vernal point
obliquity 2d degree to the earth's eccentricity



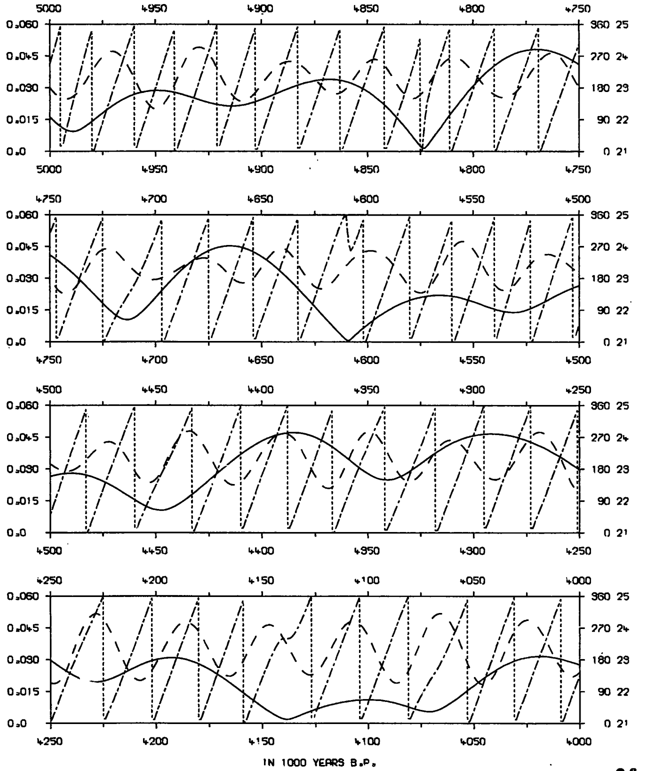
2d

LONG TERM VARIATION OF ECLIPTIC ELEMENTS
eccentricity 2d order to masses 3d degree in e i's
longitude of perihelion relative to the moving vernal point
obliquity 2d degree to the earth's eccentricity



2e

LONG TERM VARIATION OF ECLIPTIC ELEMENTS
eccentricity 3d degree in e i's
longitude of perihelion relative to the moving vernal point
obliquity 2d degree to the earth's eccentricity



2f

LONG TERM VARIATION OF ECLIPTIC ELEMENTS
eccentricity 3d degree in e i's
longitude of perihelion relative to the moving vernal point
obliquity 2d degree to the earth's eccentricity

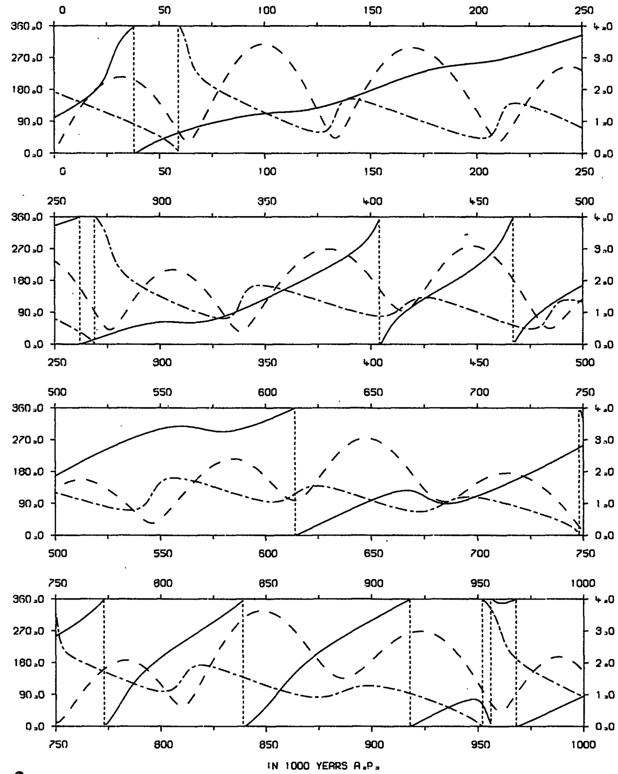
in paleoclimatology where $e \sin \tilde{\omega}$ is an important factor in most of the insolation parameters.

Very much the same behaviour is found for the inclination and the longitude of the node (Figs. 3a–f). However, for that system, the one outstanding term will make the average period around 68800 years for the inclination on the mean ecliptic of 1850.0. This inclination varies between 0° and 4°13′48″ (3°38′24″), the maximum theoretical value being 5°5′38″. This absolute maximum is reduced to 3°30′44″ if the invariable plane is chosen as the plane of reference.

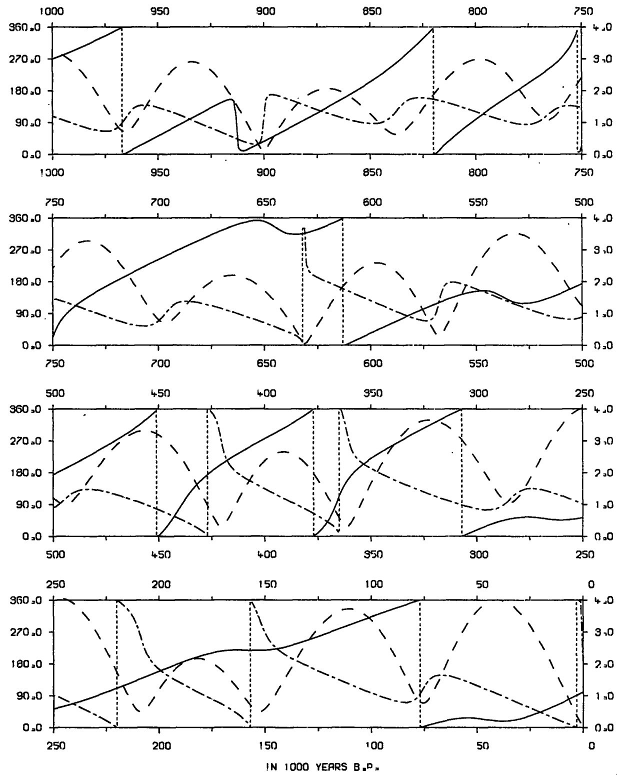
Another important quantity relevant to paleoclimatology, as well as to geophysics and astronomy, is the position of the rotation pole of the Earth with respect to the moving ecliptic.

The period covering a revolution of the mean equinox relative to the moving perihelion is 21740 (21350) years on an average, and goes from 13900 (15390) to 31300 (29700) years. Simultaneously, the inclination of the Earth's equator on the moving ecliptic has varied between 22°2′33″ (22°13′44″) and 24°30′16″ (24°20′50″) with a mean period of 41000 (41040) years, combination between the precessional motion and the period characteristic of the large term in the motion of the ecliptic. The precessional motion has a very steady mean period of 25700 (25700) years and the inclination of the Earth's equator on the fixed mean ecliptic of 1850.0 varies from 18°16′44″ (18°59′56″) to 28°26′23″ (27°38′23″). The maximum deviation between the luni-solar and the general precession in longitude is close to 10° (8°).

The digits given here for each value of the Earth's orbital elements are raw results of computation and some of them are probably not significant. However, it seems that a very high degree of accuracy has been reached now. For the last million years, the difference with the solution proposed by Berger (1976), is generally small and the two solutions are perfectly in phase, indicating a good convergence to the ideal solution. Nevertheless, this difference may reach occasionally some half a degree for the obliquity and values found here are systematically larger. For the longitude of the perihelion, the deviation amounts only 5° as far as one million years ago.



3a

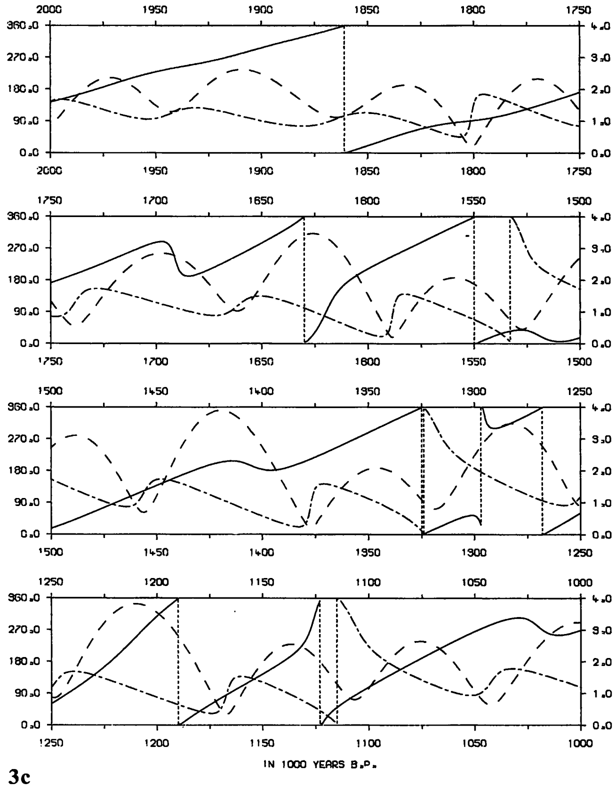


3b

LONG TERM VARIATION OF ECLIPTIC ELEMENTS
——— longitude of perihelion relative to the fixed vernal point
–·–·– longitude of the ascending node
– – – inclination

LONG TERM VARIATION OF ECLIPTIC ELEMENTS
——— longitude of perihelion relative to the fixed vernal point
–·–·– longitude of the ascending node
– – – inclination

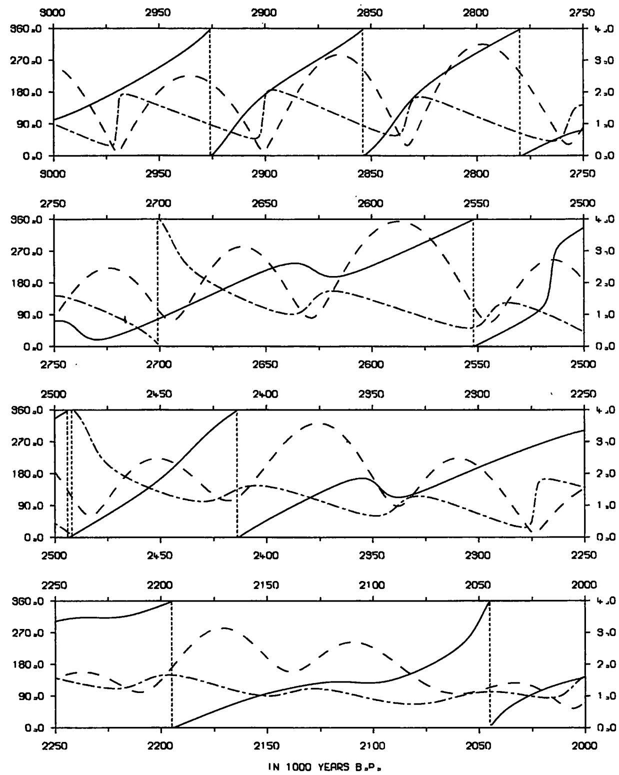
Figs. 3a–f. The variation of the inclination i of the Earth's orbit on the ecliptic of 1850.0, of the longitude of the node Ω and of the longitude of the perihelion π as a function of time for the period from the present time (1950.0) to 1000000 years after present **(a)**, for the period 1000000 years before the present time (B.P.) to the present time **(b)**, for the period 2000000 years B.P. to 1000000 years B.P. **(c)**, for the period 3000000 years B.P. to 2000000 years B.P. **(d)**, for the period 4000000 years B.P. to 3000000 years B.P. **(e)**, and for the period 5000000 years B.P. to 4000000 years B.P. **(f)**. The left-hand scale is related to both π and Ω. The right-hand scale concerns the inclination i. Both scales are in degrees of arc



3c

LONG TERM VARIATION OF ECLIPTIC ELEMENTS

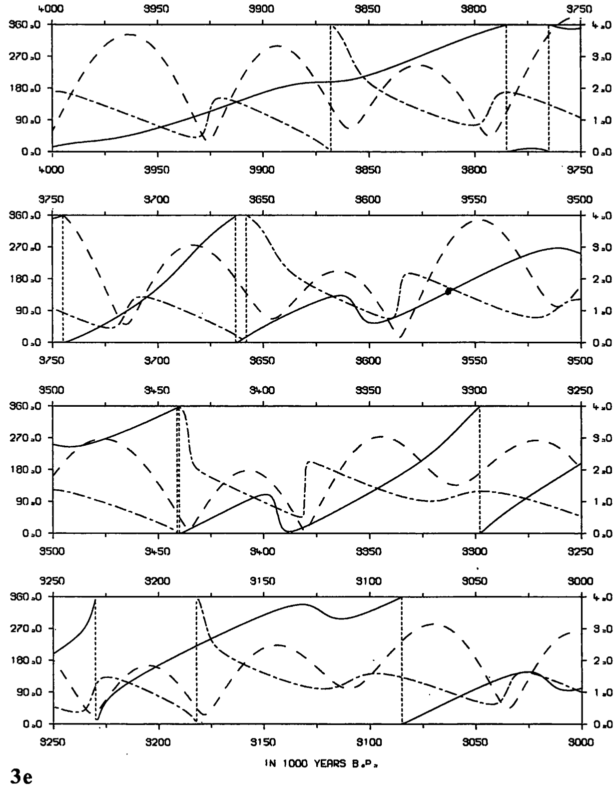
- longitude of perihelion relative to the fixed vernal point (solid line)
- longitude of the ascending node (dash-dot line)
- inclination (dashed line)



3d

LONG TERM VARIATION OF ECLIPTIC ELEMENTS

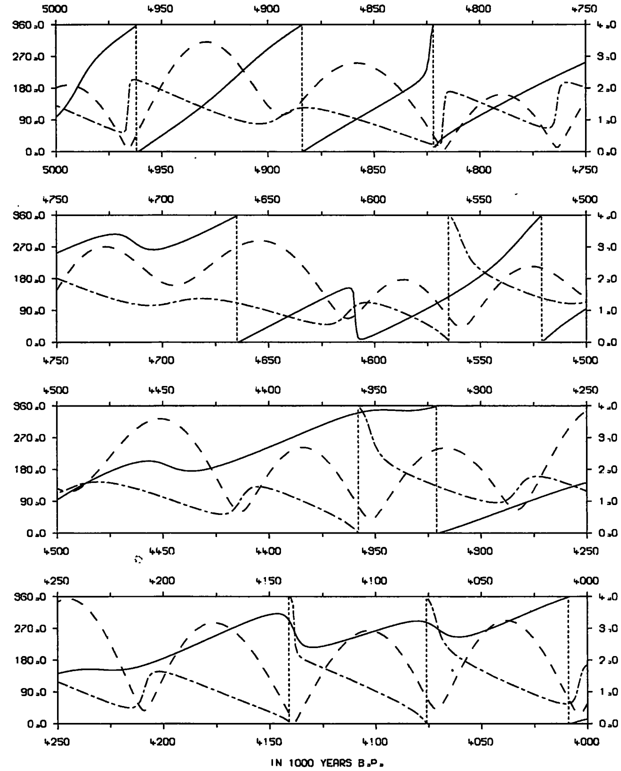
- longitude of perihelion relative to the fixed vernal point (solid line)
- longitude of the ascending node (dash-dot line)
- inclination (dashed line)



3e

LONG TERM VARIATION OF ECLIPTIC ELEMENTS

- longitude of perihelion relative to the fixed vernal point (solid line)
- longitude of the ascending node (dash-dot line)
- inclination (dashed line)



3f

LONG TERM VARIATION OF ECLIPTIC ELEMENTS

- longitude of perihelion relative to the fixed vernal point (solid line)
- longitude of the ascending node (dash-dot line)
- inclination (dashed line)

Some improvements are now expected to test the accuracy of this solution for the long-term variations of the Earth's orbital elements. These improvements may occur in the field of theoretical researches (Brumberg and Chapront, 1973; Chapront et al., 1975), in the techniques used to determine the planetary motion (Cohen et al, 1973) or even in the number of terms kept in the expansions used, just like it has been done for the secular perturbations (Brumberg et al., 1975) and already stated by Bretagnon (1974): although the neglected terms would not introduce large modifications of the constants of integration, the calculation should be repeated including long period terms of fifth order and also short period terms of higher order.

At the end, it is also advisable to be careful as regards to the absolute accuracy of the results, inaccuracies in the frequencies producing an effect the importance of which becomes larger and larger as the time increases.

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