

Tidal friction and generalized Cassini's laws in the solar system*

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The tidal drift toward a generalized Cassini state of rotation of the spin axis of a planet or satellite in a precessing orbit is described. Generalized Cassini's laws are applied to several solar system objects and the location of their spin axes estimated. Of those considered only the Moon definitely occupies state 2 with the spin axis near to the normal of the invariable plane. Most objects appear to occupy state 1 with the spin axis near to the orbit normal. Iapetus could occupy either state depending on its oblateness. In addition, the resonant rotation of Mercury is found to have little effect on the tidal drift of its spin axis toward state 1.

INTRODUCTION

TO date, the definitive analysis of the history of planetary and satellite obliquities is that of Peale (1974), who treats the tidal evolution of a spinning object orbiting a primary in an inclined uniformly precessing orbit. In the absence of dissipation, coplanar precession of the spin axis $\mathbf{s}$, orbit normal $\mathbf{n}$, and normal to the invariable plane $\mathbf{k}$ is possible for (at most) four positions of the spin axis. The Moon's motion, described by Cassini's laws (e.g., Kaula 1968), is an example of one such position. Colombo (1966) was the first to consider the generalization of Cassini's laws to other objects in the solar system such as Mercury. The four possible coplanar configurations were enumerated and designated Cassini states by Peale (1969). Peale showed that these states represent extremes in the Hamiltonian of the motion and suggested that dissipation by tidal friction would tend to drive the system to one of these states. He has since demonstrated the validity of this hypothesis by developing a detailed account of the tidal evolution of the obliquity of Mercury (Peale 1974). The effect of tidal friction is to drive any initial motion of the spin axis toward one of two states: one lying close to the orbit normal $\mathbf{n}$, but on the opposite side from $\mathbf{k}$ (designated state 1 by Peale); the other lying near $\mathbf{k}$ on the opposite side from $\mathbf{n}$ (designated state 2). Which state is eventually occupied depends primarily on the ratio of the periods for spin axis and orbit precession. The Moon occupies state 2 because of its very rapid orbit precession. Mercury undoubtedly occupies state 1, since for reasonable values of its oblateness, the spin precession period is much shorter than the orbit precession. Similar conclusions were also reached by Ward (1972). The disposition of various other objects in the solar system is considered in Sec. III of this paper where estimates of precession periods and corresponding Cassini positions are made.

Peale's approach is a combination of numerical and analytical techniques and can be employed when there

is a relatively large separation of the spin axis from a Cassini position. However, near a Cassini state the complexity of the problem is greatly reduced. Proximity to a Cassini state permits an approximation of the spin-axis trajectory which greatly facilitates computation. To lowest order, the dynamical equations can be averaged over the precessional time scale analytically. When combined with the particularly simple form of tidal equations used by MacDonald, the nature of the drift toward the Cassini state is clearly illustrated.

TIDAL FRICTION AND CASSINI'S LAWS

The Hamiltonian for the motion of a planet's spin axis expressed in a coordinate frame attached to the precessing orbit is

$$K = \frac{1}{2}C\omega^2 - (C-A)n^2\left[\frac{3}{4}(\mathbf{n} \cdot \mathbf{s})^2 - \frac{1}{4}\right] - C\omega\dot{\Omega}(\mathbf{k} \cdot \mathbf{s}), \quad (1)$$

where K is a constant of the motion, if the system is conservative. The orbital mean motion, the spin angular frequency, and the orbit precession frequency are represented by n , ω , and $\dot{\Omega}$, respectively. The principal moments of inertia are $C > B = A$. There are several techniques for obtaining Eq. 1 (e.g., Colombo 1966; Peale 1969, 1974) and we offer one in the Appendix using a generating function. A rearrangement of Eq. 1 defines a new quantity

$$E = \left\{ K - \frac{1}{2}C\omega^2 - \frac{1}{4}(C-A)n^2 \right\} / C\omega \\ = -(\alpha/2)(\mathbf{n} \cdot \mathbf{s})^2 - \dot{\Omega}(\mathbf{k} \cdot \mathbf{s}), \quad (2)$$

where

$$\alpha = \frac{3}{2}(C-A)n^2/C\omega. \quad (3)$$

In the absence of dissipation, E is also a constant of the motion.

Colombo (1966) has pointed out that the trajectory traced out by the spin axis can be represented by the intersection of a parabolic cylinder with a unit sphere. Inspection of Fig. 3 of Colombo's paper indicates that near Cassini positions 1, 2, and 3, the trajectories resemble nearly circular precession about that state. This is especially true if either $\alpha \gg |\dot{\Omega}|$ or $\alpha \ll |\dot{\Omega}|$.

In the rotating frame used in deriving K , vectors $\mathbf{n}$ and $\mathbf{k}$ are fixed. We define a coordinate system with $\mathbf{n}$

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and $\mathbf{k}$ in the z, y plane and the x axis in the direction of the ascending node of the orbit on the invariable plane. The direction of the z axis is chosen so that

$$\alpha \cos \xi \sin \xi \cos \theta = \dot{\Omega} \sin(i - \xi), \quad (4)$$

where $\mathbf{n} \cdot \mathbf{k} = \cos i$, $\mathbf{n} \cdot \mathbf{z} = \cos \xi$, $\mathbf{k} \cdot \mathbf{z} = \cos(i - \xi)$, $\mathbf{s} \cdot \mathbf{z} = \cos \theta$, $\mathbf{s} \cdot \mathbf{x} = \sin \theta \cos \phi$, and $\mathbf{s} \cdot \mathbf{y} = \sin \theta \sin \phi$. For $\theta = 0$, $\mathbf{z}$ is in the direction of the appropriate Cassini state. Equation 4 has up to four solutions. We restrict our attention to states 1 and 2 and positions of the spin axis not far removed from these. Equation 2 becomes

$$E = -(\alpha/2) \cos^2 \xi \cos^2 \theta - \dot{\Omega} \cos(i - \xi) \cos \theta - (\alpha/2) \sin^2 \xi \sin^2 \theta \sin^2 \phi. \quad (5)$$

It is apparent that the variation in θ depends on the last term which is second order in $\sin \xi$. Evaluating E for $\theta = 0$ and then subtracting the result from Eq. 5 enables one to solve for $\sin \theta$ to second order in $\sin \xi$ ($\sin \theta$ is also assumed to be small).

$$\sin \theta \approx \left\{ \frac{2[E - E(0)]}{\alpha + \dot{\Omega} \cos(i - \xi)} \right\}^{1/2} \times \left\{ 1 + \frac{1}{2} \sin^2 \xi \left[\frac{1 + \sin^2 \phi}{1 + (\Omega/\alpha) \cos(i - \xi)} \right] \right\}. \quad (6)$$

To first-order accuracy, θ and ξ remain constant during a cycle of ϕ and the spin axis precesses about the z axis, which is very nearly the Cassini state. [To this level of accuracy, $E(0)$ can be identified as the value of E when the Cassini state is occupied.] From Eq. 6 it is clear that for $\alpha \gg |\dot{\Omega}|$, $E(0)$ is a minimum while for $\alpha \ll |\dot{\Omega}|$, it is a maximum. (Note that $\dot{\Omega} < 0$, i.e., the nodes regress.) Computing the change in $E - E(0)$ produced by the tides is equivalent to knowing the change in the angular separation of $\mathbf{s}$ from the appropriate Cassini state.

Differentiating Eq. 2 with respect to time yields

$$dE/dt = -\alpha(\mathbf{n} \cdot \mathbf{s})(\mathbf{n} \cdot d\mathbf{s}/dt) - \dot{\Omega}(\mathbf{k} \cdot d\mathbf{s}/dt) - \frac{1}{2}(\mathbf{n} \cdot \mathbf{s})^2 d\alpha/dt. \quad (7)$$

In the absence of dissipative motion such as the tides, substituting $d\mathbf{s}/dt$ into Eq. 7 yields zero since E is constant of motion. Only the time rate of tidal change for $\mathbf{s}$ and α need be considered. Differentiating $E(0)$ and employing Eq. 4,

$$dE(0)/dt = -\frac{1}{2} \cos^2 \xi d\alpha/dt. \quad (8)$$

Expressions for $(d\mathbf{s}/dt)_{\text{tidal}}$ and $(d\alpha/dt)$ must be obtained from the tidal model. As an example, we use the simple model of MacDonald, i.e., the equilibrium tidal bulge of the secondary is displaced from the subprimary point in the direction of the rotation by an angle δ given by the relation $\sin 2\delta \sim Q^{-1}$ where Q is the specific dissipation function, assumed inversely proportional to frequency. For this model,

$$(d\mathbf{s}/dt)_{\text{tidal}} = -\frac{1}{\tau} \left[\frac{n}{\omega} - \frac{1}{2}(\mathbf{s} \cdot \mathbf{n}) \right] [\mathbf{n} - (\mathbf{s} \cdot \mathbf{n})\mathbf{s}] \quad (9)$$

and

$$d\omega/dt = -\frac{1}{2\tau} [\omega + \omega(\mathbf{s} \cdot \mathbf{n})^2 - 2n(\mathbf{s} \cdot \mathbf{n})], \quad (10)$$

where Eqs. 9 and 10 have been averaged over an orbit. The characteristic decay time τ is given by

$$1/\tau \sim \frac{3}{2} \frac{k_T}{Q_n} \frac{Gm_p R_s^5}{C n r^6}, \quad (11)$$

where k_T , R_s , and C are the tidal Love number, radius, and moment of inertia of the secondary, respectively; the value of Q for a frequency equal to the mean motion n is designated Q_n ; the orbital radius is r ; and m_p is the mass of the primary. Equations 3 and 10 are then used to calculate $d\alpha/dt = (d\alpha/d\omega)(d\omega/dt)$. The form and sign of $d\alpha/dt$ will depend on whether $(C - A)/C$ is due to intrinsic oblateness or to hydrostatic flattening. In the latter case there is an additional dependence on ω :

$$C = I[1 + (2k_s R_s^5 / 9GI)\omega^2], \quad (12)$$

$$A = I[1 - (k_s R_s^5 / 9GI)\omega^2],$$

where k_s is the secular Love number, and I is the moment of inertia of the equivalent sphere (e.g., Munk and MacDonald 1960).

Before continuing we should first say a word about the motivation behind our approach. Goldreich and Peale (1970) have looked at the tidal decay of a planet's obliquity for the case of a fixed orbit plane and found a rate of change

$$d\theta/dt \sim -\frac{1}{\tau} \sin \theta \left(\frac{n}{\omega} - \frac{1}{2} \cos \theta \right), \quad (13)$$

where θ is here equal to the obliquity. Practically speaking we are interested in cases where the Cassini state is near either $\mathbf{n}$ or $\mathbf{k}$ and the orbital inclination is small. Hence, the motion of the spin vector will differ little from the analysis for a fixed orbit until θ is of order or less than ξ . Accordingly, we will content ourselves with a solution second order in combinations of ξ and θ .

Since the time scale for tidal decay is much longer than a precession period, $d[E - E(0)]/dt$ can be averaged over the latter (Ward 1972).

$$\left\langle \frac{d}{dt} [E - E(0)] \right\rangle \approx -\frac{1}{\tau} \left(\frac{n}{\omega} - \frac{1}{2} \right) (\alpha + \dot{\Omega} \cos i) \sin^2 \theta + \frac{\omega}{2\tau} \left(\frac{d\alpha}{d\omega} \right) \left(\frac{n}{\omega} - 1 \right) \sin^2 \theta - \frac{1}{\tau} \left(\frac{n}{\omega} - 1 \right) [\alpha \cos \xi \sin^2 \xi \cos \theta - \dot{\Omega} \sin \xi \sin(i - \xi)]. \quad (14)$$

The last term, which is not proportional to $\sin^2\theta$, vanishes by virtue of Eq. 4 (we have retained higher-order terms in the expression to assist identification). The decay of θ resembles Eq. 13. The interpretation is somewhat different, however. Equation 14 describes the drift toward the Cassini state, not toward zero obliquity. When the Cassini state is occupied, the obliquity is still nonzero. In the terminal stage of tidal despinning, as the object approaches synchronous rotation, and the position of the Cassini state does not drift significantly, the first term becomes dominant. If $\alpha \gg |\dot{\Omega}|$, $\langle d[E - E(0)]/dt \rangle < 0$. Since $E(0)$ is a minimum, E is approaching $E(0)$. If $\alpha \ll |\dot{\Omega}|$, $\langle d[E - E(0)]/dt \rangle > 0$. Since $E(0)$ is a maximum, E is again approaching $E(0)$.

Figure 1 is a schematic representation of the spin-axis trajectory near Cassini state 1. The angles involved are assumed small enough that the region is essentially flat. The direction of the tidal torque is given by $\mathbf{z} = \mathbf{k} - \mathbf{s}(\mathbf{n} \cdot \mathbf{s})$, i.e., the component of $\mathbf{n}$ perpendicular to $\mathbf{s}$, so that if acting alone, the tides would drive the spin axis coincident with the orbit normal. The precession of $\mathbf{s}$ about the Cassini position results in the decay of the spin axis toward $\mathbf{z}$ instead. At a typical point O , the vector $\mathbf{z}$ can be resolved into a component tangential to the path and a component directed toward $\mathbf{z}$. The tangential component has little effect on the precession motion and averages out over a cycle. It is the latter component labeled $\mathbf{z}^*$ that is responsible for the decay of θ . This component vanishes at $\sin\varphi_c = \tan\theta \cot\xi$, where $\mathbf{z}$ becomes tangent to the trajectory. During the passage from B to D , the tidal torque tends to increase θ by pulling $\mathbf{s}$ toward $\mathbf{n}$, while from D to B the continued rotation of $\mathbf{s}$ toward $\mathbf{n}$ causes a decrease in θ . Not only is the latter path longer but the magnitude of the torque is somewhat longer at A than C . The result is a net decrease in θ for each cycle of the precessional motion.

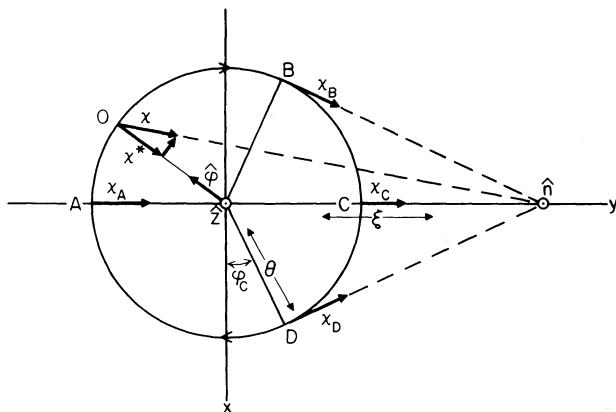


FIG. 1. Schematic decay of spin vector toward state 1. Circular path is spin-axis trajectory. The tidal torque $\mathbf{z}$ is always directed toward orbit normal $\mathbf{n}$. Over a cycle of the motion, the net effect of $\mathbf{z}$ is to decrease θ ; the radius of the cone of precession.

APPLICATIONS

Interest in the literature has been primarily concentrated on the Moon and Mercury as examples of motion of the generalized Cassini type. In addition to these objects, several other interesting applications shall be discussed.

The Moon. The lunar orbit undergoes precession due to solar gravity with a period of 18.6 yr while maintaining an inclination of $5^\circ 8'$ to the ecliptic. The oblateness $(C - A)/C$ has a value of 6.2×10^{-4} . This is roughly two orders of magnitude greater than the hydrostatic value produced by rotation. The Earth's torque on the lunar figure would be sufficient to precess the lunar spin axis about a stationary orbit normal in only 78.5 yr. Nevertheless, the motion of the orbit dominates and the Moon is in state 2 with the spin axis lying $1^\circ 35'$ from the normal to the ecliptic. There is good reason to believe, however, that the Moon occupied state 1 during the early history of the Earth-Moon system and that a transition occurred when the semimajor axis of the lunar orbit was somewhere near ~ 35 Earth radii (Ward 1974).

Triton. Unlike Mercury and the Moon, the orbit of Triton experiences precession almost exclusively due to the torque arising from the equatorial bulge of its primary, Neptune. The orbit itself is retrograde and inclined by $20^\circ 1'$ to the equatorial plane. If tidal friction has driven the satellite's spin to synchronous rotation, the spin should be retrograde also. Observational evidence indicates that Triton has an orbit precession period on the order of 600 yr (Brouwer and Clemence 1961). A lower limit for the oblateness of Triton is that due to hydrostatic flattening with the length of the day set equal to the length of the month, $(C - A)/C \gtrsim 4.1 \times 10^{-4} \rho^{-1} (k/3K')$, where ρ is the density of the satellite. Lewis (1971) has suggested that large outer satellites such as Triton may have liquid water-ammonia mantles as a consequence of radioactive heating. If so, the hydrostatic value is probably appropriate. For $k \sim 1.5$, $K' \sim 0.4$, $\rho \sim 2$ g/cm³, Triton occupies state 1 with an obliquity of $\sim 1^\circ 5'$. We have used K' for the square of the ratio of gyration to the satellite radius.

Iapetus. The near equality of the solar torque to the combined torques of Saturn and Titan makes it impossible for the orbit of Iapetus to maintain a constant inclination to either the orbital or equator plane of Saturn. These planes are tilted with respect to each other by $26^\circ 7'$. As a compromise, the orbit describes a circular path about a mean pole which has an angular separation of $\sim 14^\circ 7'$ from the spin axis of Saturn, but which is coplanar with this axis and Saturn's orbit normal (Tisserand 1896). The orbit normal of Iapetus remains inclined to the mean pole by 8° during its ~ 3000 -yr precession period. Iapetus is known to have achieved synchronous rotation from its photometric phase curve. The hydrostatic value of $(C - A)/C$ is $\sim 3 \times 10^{-6} \rho^{-1} (k/3K')$. If this value is appropriate for

Iapetus, then the spin axis lies close to the mean pole in state 2. On the other hand, if a lunar value is more appropriate, Iapetus occupies state 1 with an obliquity of $\sim 0^\circ.65$.

Equatorial Satellites. A somewhat less obvious application of generalized Cassini's laws are the equatorial satellites. The orbit planes of equatorial satellites experience precession with the equator plane of the primary (Goldreich 1966). To compute primary's precession period, it is not sufficient to merely compute the solar torque on the primary alone. In general, an accompanying entourage of satellites can affect this period in two ways: Those satellites which tend to partake of the precession add their contribution of angular momentum to that of the system. In addition, the solar torque on their orbits is transmitted to the primary and must be added to the solar torque acting on the equatorial bulge. For this situation, we have estimated the precession period by the expression,

$$T = P \left(\frac{P}{D} \right) \left\{ \frac{K + (\omega_c/\omega) \sum_i (m_i/M_p) (a_i/R_p)^{1/2}}{J + (3/4) \sum_i (m_i/M_p) (a_i/R_p)^2} \right\} \sec \theta_p, \quad (15)$$

where $(C-A)/C = 2J/3K'$, M_p , R_p , θ_p , ω , P , and D are the oblateness, mass, radius, obliquity, spin frequency, orbit period, and spin period of the primary, respectively, $\omega_c^2 = GM_p/R_p^3$; while m_i and a_i are the mass and semimajor axis of the i th satellite. On the other hand, if a satellite is not locked to the planet (such as the Moon), it assists the Sun in exerting a torque on the equatorial bulge. In this case the period is given by,

$$T = P \left(\frac{P}{D} \right) \frac{K}{J} [1 + \sum_i (m_i/M_\odot) (r/a_i)^3]^{-1} \sec \theta_p, \quad (16)$$

where $M_\odot$ is the solar mass and r the Sun-planet distance. Table I lists the approximate precession periods for the planet-satellite systems.

An upper limit for the spin-precession period of the satellites is obtained by assuming hydrostatic flattening and synchronous rotation. The precession period is approximately,

$$P_s = 2(3\pi)^{1/2} (K_s/k_s) (\rho_s/\rho_p) (G\rho_p)^{-1/2} (a_s/R_p)^{9/2} \sec \theta, \quad (17)$$

where the subscripts s and p denote satellite and primary values, respectively. Taking $K'_s \sim 2/5$, $k_s \sim 1.5$, and $\theta \sim 0$; $P_s = 2 \times 10^{-4} \rho_s \rho_p^{-1/2} (a_s/R_p)^{9/2}$ yr. This period is a strongly, monotonically increasing function of the satellite's orbital semimajor axis. Hence, the larger the orbit, the more difficult it is for the spin axis to follow the moving orbit normal. For the outer most equatorial satellite of Jupiter, Saturn, and Uranus: P_s (Callisto) $\sim 320\rho_s$ yr, P_s (Hyperion) $\sim 670\rho_s$ yr, and P_s (Oberon) $\sim 190\rho_s$ yr, respectively. Comparison of these values with Table I reveals that for each, the satellite's obliquity will remain extremely small. It follows that

TABLE I. Spin precession periods of the planets.

Planet	Obliquity	K'^2	J^3	Precession period	
				Without satellites	With satellites
Mercury ¹	...			2.4×10^3	...
Venus	...	0.34	$4.1 \times 10^{-5.4}$	4.8×10^3	...
Earth	$23^\circ 27'$	0.335	$1.64 \times 10^{-3.2}$	8.1×10^4	2.6×10^4
Mars	$25^\circ 06'$	0.359	2.95×10^{-3}	1.7×10^5	1.7×10^5
Jupiter	$3^\circ 05'$	0.25	0.0296	1.1×10^6	4.5×10^5
Saturn	$26^\circ 44'$	0.22	0.027	6.8×10^6	1.8×10^6
Uranus	$97^\circ 55'$	0.13	0.013 ⁵	7.3×10^8	1.2×10^8
Neptune	$28^\circ 48'$	0.29	0.007 ⁶	7.8×10^8	2.4×10^7

¹ Computed for $(C-A)/C \sim 10^{-4}$.

² Allen (1963).

³ JPL Report 32-1306, except where otherwise specified.

⁴ Anderson (1973).

⁵ Dunham (1971).

⁶ Brouwer and Clemence (1961).

this will also be the situation for the inner satellites.

Stevenson (1950) has reported the apparent brightness of Titania and Oberon to be variable when the plane of the satellite orbits is virtually normal to the line of sight, indicating a large obliquity for these objects. On the other hand, the tidal time scale for the satellites of Uranus is quite short, i.e., of order $\sim 10^6$ yr for reasonable values of Q . Yet it is clear that once the obliquity is driven essentially to zero, it cannot be subsequently increased by the motion of the orbit plane. Indeed, Andersson (private communication, 1974) has recently reexamined this system and found no such brightness variations.

Mercury. Calculations of the mutual secular perturbations of the planets indicate an orbit precession period of 2.5×10^5 yr for Mercury (Brouwer and Clemence 1961). The length of the day is known to be $\frac{2}{3}$ of the 88-day orbital period (Pettengill and Dyce 1965; Goldreich and Peale 1966). The hydrostatic value of the oblateness can be estimated as a few times 10^{-6} . However, since both Mars (Lorell *et al.* 1972) and the Earth are known to process nonhydrostatic contributions of a few times 10^{-5} , whereas the lunar value is 6.2×10^{-4} , a more reasonable value for Mercury is probably near $\sim 10^{-4}$, which, together with the $\sim 7^\circ$ orbital inclination, puts Mercury in state 1 with an obliquity of ~ 4.1 arcmin (Colombo 1966).

The gravitational torque exerted by a primary on a secondary with three unequal moments of inertia can be written,

$$\mathbf{T} = 3 \frac{\mu}{r^3} [(C-B)(\mathbf{r} \cdot \mathbf{J})(\mathbf{r} \cdot \mathbf{K})\mathbf{I} + (A-C)(\mathbf{r} \cdot \mathbf{K})(\mathbf{r} \cdot \mathbf{I})\mathbf{J} + (B-A)(\mathbf{r} \cdot \mathbf{J})(\mathbf{r} \cdot \mathbf{I})\mathbf{K}], \quad (18)$$

where μ is the product of the primary's mass and the gravitational constant, $A < B < C$ are the three principal moments of inertia of the secondary, $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ are unit vectors along the principal axes of its inertial

ellipsoid, $\mathbf{r}$ is a unit vector locating the primary from the center of the secondary, and r is the separation of the bodies. For noncommensurable motion, terms associated with the difference $(B-A)$ vanish upon long-term averaging. In the case of the $\frac{3}{2}$ resonance of Mercury there is a net contribution. Indeed, it is just this contribution that is thought to account for the stability of the commensurable rate. It can be shown that for reasonable starting values Mercury probably had a sizeable obliquity when it entered resonance (Fig. 2). To inquire as to whether the torque on the planet's intrinsic figure could have a stabilizing influence on the obliquity similar to that on the spin rate, the time rates of change of the spin rate, ω , and the component of the spin vector in the direction of the orbit normal, $\omega \cos \theta$, are derived by taking the dot product of Eq. 18 with $\mathbf{K}$ and $\mathbf{n}$, respectively. (For this calculation $\mathbf{n}$ is considered fixed in space and θ is the planet's obliquity.)

$$Cd\omega/dt = \mathbf{K} \cdot \mathbf{T}, \quad Cd(\omega \cos \theta)/dt = \mathbf{n} \cdot \mathbf{T}. \quad (19)$$

We have assumed internal dissipation constrains the planet to rotate about its maximum principal moment of inertia C . Both equations can be averaged over orbital and precessional motions. Although straightforward, this task is lengthy (Ward 1972). The twice-averaged expressions are

$$\langle d\omega/dt \rangle = \langle d(\omega \cos \theta)/dt \rangle$$

$$= \frac{21}{16} en^2 \left(\frac{B-A}{C} \right) (1 + \cos \theta)^2 \sin 2\Phi, \quad (20)$$

where e is the orbital eccentricity and Φ is the angle between the line of apsides and the principal axis $\mathbf{I}$ when the planet is at perihelion. Since these rates are equal, it is apparent that they cannot both act to cancel out their (unequal) tidal counterparts. Clearly, if the resonance state is to survive,

$$\langle d\omega/dt \rangle_T + \langle d\omega/dt \rangle_P = 0, \quad (21)$$

where the subscripts T and P denote tidal and permanent asymmetry torques. The total rate of change of the obliquity can now be obtained from

$$d(\cos \theta)/dt = [\langle d(\omega \cos \theta)/dt \rangle_T + \langle d(\omega \cos \theta)/dt \rangle_P] / \omega. \quad (22)$$

Equations 20 and 21 can be used to eliminate the second term in Eq. 22 in favor of $\langle d\omega/dt \rangle_T$. For the MacDonald frequency dependent torque Eq. 22 becomes,

$$d(\cos \theta)/dt = -\frac{1}{\tau} (1 - \cos \theta) \left[(1/6)(7 - 3 \cos \theta) - (e^2/4)(51 - 11 \cos \theta) \right]. \quad (23)$$

This expression is positive for all values of θ so that the obliquity always decays. Integrating Eq. 23 with $a = 7/6 + 51e^2/4$ and $b = 1/2 + 11e^2/4$,

$$\cos \theta = (1 - ac)/(1 - bc), \quad (24)$$

where

$$c = \left(\frac{1 - \cos \theta_0}{a - b \cos \theta_0} \right) \exp[-(a - b)t/\tau]. \quad (25)$$

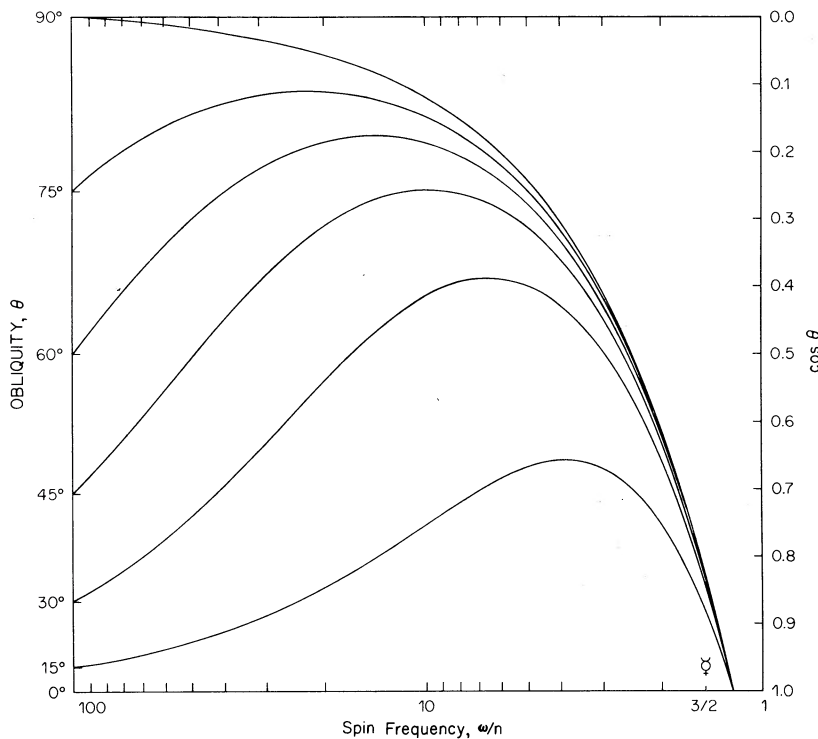


FIG. 2. Obliquity as a function of spin frequency for an object being despun by tidal friction. The asymptotic spin rate of $1.23n$ is an artifact of the MacDonald model employed and the eccentricity of the orbit. The $\frac{3}{2}$ -resonance position is marked. All evolutionary tracts considered have a sizeable obliquity at this point.

This decay rate turns out to be essentially the same as that for nonresonance. We conclude from the inability of the asymmetry torque to stabilize the obliquity, that the rotation axis of Mercury has been driven to state 1 which, for likely values of $(C-A)/C$, implies an essentially zero obliquity for this planet.

Venus. Goldreich and Peale (1970) have pointed out that Venus's retrograde rotation is unstable in the presence of solar tides and must be maintained by an opposing torque perhaps due to core-mantle viscous coupling. Even so, the object may still be in Cassini state 3. The orbit plane of Venus undergoes precession in $\sim 7 \times 10^4$ yr (Brouwer and Von Woerkom 1950). The best estimate of J_2 for Venus based on Mariner 5 data is $2.7 \pm 0.9 \times 10^{-5}$ (Anderson 1973). From Eq. 4 combined with the $3^\circ 4'$ orbital inclination, one obtains an obliquity of $179^\circ 8'$ for Venus if state 3 is occupied.

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APPENDIX

The Hamiltonian describing the motion of the spin axis of an oblate object with two equal moments of inertia is

$$H = \frac{1}{2}(\mathbf{S} \cdot \mathbf{L}) + n^2(C-A)\left[\frac{3}{4}(\mathbf{s} \cdot \mathbf{n})^2 - \frac{1}{4}\right], \quad (\text{A1})$$

where only the lowest-order terms in the potential energy have been retained. The spin angular momentum is represented by $\mathbf{L} = \mathbf{I} \cdot \mathbf{S}$, where $\mathbf{S} = \omega \mathbf{s}$ is the spin velocity vector. Assuming principal axis rotation $\mathbf{L} = C\mathbf{s}$. This Hamiltonian can be identified as the energy but is not a constant of motion if the orbit is precessing because $\mathbf{n}$ is time dependent and $dH/dt = \partial H / \partial t \neq 0$. A time independent Hamiltonian can be obtained by performing a canonical transformation to a new system rotating with the orbit plane.

Let $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$ be unit vectors in the direction of the principal axes fixed in the planet and located by Euler angles with respect to interial space vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ chosen such that the z axis is in the direction of the normal to the invariable plane $\mathbf{k}$ about which the orbit normal $\mathbf{n}$ precesses (Fig. 3). If we transfer to a new

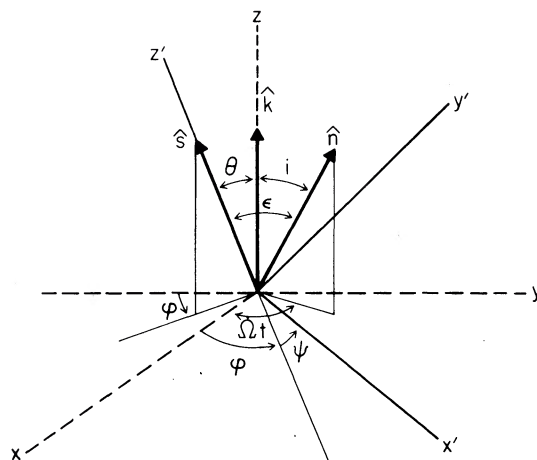


FIG. 3. The orientation of the principal axes of the secondary ($\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$) with respect to the inertial space vectors ($\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$). Angles θ , φ , and ψ are Euler angles. Unit vectors $\mathbf{s}$ and $\mathbf{n}$ locate the spin axis and the orbit normal of the secondary, respectively. The vector $\mathbf{k}$ is normal to the invariable plane.

coordinate system rotating about $\mathbf{k}$ in which $\mathbf{n}$ is fixed, the new Euler angles are

$$\varphi' = \varphi - \dot{\Omega}t, \quad \theta' = \theta, \quad \psi' = \psi. \quad (\text{A2})$$

This is a simple point transformation and is, therefore, canonical. The new Hamiltonian K is given in terms of the old one by

$$K = H + \partial F / \partial t, \quad (\text{A3})$$

where F is the generating function:

$$F = \theta P_\theta' + \psi P_\psi' + (\varphi - \dot{\Omega}t) P_\varphi' \quad (\text{A4})$$

(e.g., Goldstein 1950). Substituting Eq. A4 in Eq. A3 and identifying $P_\varphi' = (\mathbf{L} \cdot \mathbf{k})$, i.e., the component of the planet's spin angular momentum in the direction of the normal to the invariable plane one obtains,

$$K = \frac{1}{2}(\mathbf{S} \cdot \mathbf{L}) - n^2(C-A)\left\{\frac{3}{4}(\mathbf{s} \cdot \mathbf{n})^2 - \frac{1}{4}\right\} - \dot{\Omega}(\mathbf{L} \cdot \mathbf{k}). \quad (\text{A5})$$

This Hamiltonian is a constant since $\mathbf{n}$ is now fixed. However, since the transformation Eq. A2 contains time K can no longer be identified as the energy.

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