Radiation of gravitational waves by a cluster of superdense stars

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An analysis is given of the mechanism whereby collapsed stars in the galactic nucleus will emit gravitational radiation when they pass each other. Estimates are obtained for the annual number of encounters such that a pulse of gravitational radiation with a given spectrum might be observable.

INTRODUCTION

In a 1964 analysis of the evolution of a globular cluster with allowance for relativistic effects, one of the present authors in collaboration with Podurets[1,2] emphasized that gravitational radiation could play a substantial role in the overall balance.

Interest in gravitational radiation rose sharply after Weber's reports that pulses had been observed coming from the galactic nucleus.[3-6] Recent experiments by Braginskii et al.[7] and Tyson[8] have not supported Weber's finding. Even earlier, a detailed theoretical analysis showed that the sensitivity of Weber's equipment would require radiation of up to (10-100) $M_\odot c^2$ in each event, with as many as 10^3 events annually. Both the characteristics of the individual events and the total energy loss during the life of our galaxy appear to conflict with the astronomical data.[9,10]

A natural question arises: What pulses of gravitational radiation may be expected, and what instrumental sensitivity should be achieved for one to have a reasonable chance of recording radiation from these noteworthy processes, which are undoubtedly taking place?

Suppose that at the galactic center there is a cluster consisting of $N = 10^9 n$ superdense stars (neutron stars or black holes), all of the same mass $M = mM_\odot$. Denote the average ("virial") radius of the cluster by $R = 3 \cdot 10^{17}$ r cm = 0.1 r pc. Binary-star formation will take place in this cluster, accompanied by graviational radiation, until the pair of stars merge together. Gravitational radiation will also be emitted when two stars pass each other at a close distance[1)] without forming a bound system.

The cluster parameters should be selected so as to satisfy the condition of a sufficiently long lifetime for the cluster, at least 10^9 yr.

The presence of dense nuclei in many galaxies proves the longevity of the nuclei. An indirect confirmation is provided by the comparatively low activity of the nucleus of our own galaxy in the electromagnetic spectrum. There is no reason to believe that the nucleus is currently in some transient special phase. In principle, the central portions of quasars would presumably have a more rapid evolution ($\approx 10^5$ yr), but the distances of quasars are so great that little of their graviational radiation would reach us.

With the longevity mentioned, the number of pairs that are formed would be no more than n per year. The binary-formation events should yield comparatively strong[2)] but very rare pulses of gravitational radiation. It would hardly be possible or feasible to plan an experiment for detecting events of random occurrence with a frequency of roughly one event per year. Close passages of two stars will occur more often, however, if a larger flyby distance is admitted.

Thus in principle, by appropriately specifying the sensitivity and frequency characteristics of the apparatus, one could adjust matters so as to record events taking place sufficiently often.

We shall give estimates below that can serve as a basis for instrumental design. All these estimates are rough ones, neglecting the radial distribution of stars. As the gravitational radiation varies, there will be a corresponding change in the cluster longevity, but for a given longevity the error should presumably be no more than a factor of 2-3. In a disk-shaped cluster the relative velocities of the stars would be lower, an effect that may partially be compensated by the higher number density of the stars.

The customary methods for attempting to detect gravitational radiation involve a resonance determination of a periodic wave. In the case of nonperiodic pulses, these methods will determine the spectral density A_ν of the radiation, a quantity measured in units of 1 erg $\cdot$ cm^{-2} $\cdot$ Hz^{-1} (see Braginskii, Zel'dovich, and Rudenko,[11] where the introduction of A_ν is substantiated in detail).

For a given pulse or "event," this quantity will be a function of frequency. Note that A_ν is independent of time and does not contain time in its dimensions[3)] ([ergs $\cdot$ cm^{-2} $\cdot$ Hz^{-1}], not [erg $\cdot$ cm^{-2} $\cdot$ Hz^{-1} $\cdot$ sec^{-1}]), as is natural because A_ν refers to a single event.[4)] One condition for using A_ν is that the pulse spectrum should be sufficiently broad, wider than the resonance reception band. For the pulses of interest to us resulting from the flyby of two stars, this condition will always be satisfied.

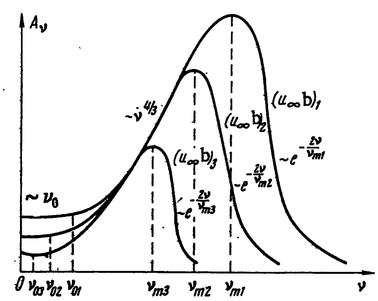

Fig. 1. Schematic behavior of the gravitational-radiation spectrum resulting from close passages of two collapsed stars. A_ν, spectral radiation density; ν, frequency; $\nu_m \approx u_m/r_m$; $\nu_0 \approx c/r_g\beta^3$; $(u_\infty b)_3 > (u_\infty b)_2 > (u_\infty b)_1$.

The radiation spectrum for an individual flyby will have the form shown in Fig. 1. The maximum A_ν will be reached at a characteristic frequency ν_m of the order of the reciprocal of the time spent by the stars at their minimum distance of approach: $\nu_m \approx r_m^{-1} u_m$. In the Newtonian approximation all quantities are readily expressed in terms of the encounter parameters at infinite distance, namely, the velocity u_∞ and the impact parameter b:

$$u_m = \frac{GM}{bu_\infty} + \sqrt{\left(\frac{GM}{bu_\infty}\right)^2 + u_\infty^2},$$

$$r_m = \frac{bu_\infty}{u_m}, \qquad \nu_m \approx u_m \cdot r_m^{-1}.$$

The encounters actually may be divided quite sharply into two categories. For a given velocity u_∞ we shall examine these two types, first for large and then for small b.

1. RADIATION FROM A DISTANT FLYBY

If $b > b_1 = 2GM/u_\infty^2$, the encounter will be weak one, with the trajectory deflected by a small angle. Then $u_m \approx u_\infty$, $r_m \approx b$, and $\nu_m \approx u_\infty/b$, so that the maximum frequency in this situation will be $\approx u_\infty^3/2GM = 10^5\beta^3/m$ sec^{-1}, where $\beta = u_\infty/c$. For $\beta^2 \approx 10^{-3}$ the frequency will be $\nu \approx$ 1-3 sec^{-1}.

A flat spectrum may here be adopted as an approximation. The exact form of the spectrum for this case has been calculated by Ruffini and Wheeler[12] (see also Appendix 1).

In the right-hand member of the equation

$$^1/_2\,\Box h_{ik} = \frac{8\pi G}{c^4}(T_{ik} - {}^1/_2\delta_{ik}T)$$

the tensor T_{ik} consists of two parts: the contribution

$$M_1 u_{1i} u_{1k} \delta(\mathbf{r} - \mathbf{r}_1(t)) + M_2 u_{2i} u_{2k} \delta(\mathbf{r} - \mathbf{r}_2(t))$$

from the two moving particles, and the contribution from the pseudotensor expressing the gravitational force between the particles. If the particles were charged, we would have to form the Maxwellian attraction tensor $E_i E_k - {}^1/_2 E^2 \delta_{ik}$, and representing the field by $\mathbf{E}_1 + \mathbf{E}_2$, discard the terms $E_{1i}E_{2k}$, for the terms E_1^2 and E_2^2 already occur as the energy in the observed mass of a single particle, while as the attraction they would be balanced by the internal forces of the charge.

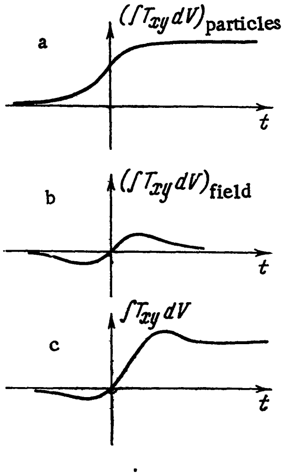

Fig. 2. Time dependence of: a) the contribution of the particles to $\int T_{xy} \cdot dV$; b) the contribution of the field to $\int T_{xy}\, dV$; c) the complete integral $\int T_{xy}\, dV$.

Clearly, then, the field contribution to $\int T_{ik} dV$ will be of the same order as GM^2/r, or at closest approach, as GM^2/b (the electric-field analog is $M\sqrt{G}/r^2$).

In this first case the T_{ik} for the particles will be greater than the field value. But no radiation will be emitted if T_{ik} is constant. Let us consider the difference in T_{ik} for the particles at $t = -\infty$ and $t = +\infty$. If the motion has taken place along the x axis, then $T_{xy} = 0$ at $t = -\infty$, but

$$\int T_{xy}\, dV = 2Mu_\infty \Delta u = 2Mu_\infty \frac{GM}{b^2} \cdot \frac{b}{u_\infty} = \frac{2GM^2}{b}$$

at $t = +\infty$.

The field value of $\int T_{xy}$ will be of the same order, so that the complete curve will have the form shown in Fig. 2c.

To a first approximation such a curve may be represented by the function $\theta = \int \delta(t)dt$, corresponding to a spectrum of the form $h_{ik}(\omega) \approx \text{const}/\omega$. But the energy density of the radiation is of the order of $(c^3/16\pi G)\dot{h}_{ik}^2(\omega)$. Hence a step in h_{ik}, that is, $h_{ik}(\omega) \sim \omega^{-1}$, will yield a flat radiation spectrum $\dot{h}_{ik}(\omega) \sim \omega^0$ as far as the falloff in A_ν for $\nu > \nu_m$ (see above). The quantity $A_\nu \approx G^3M^4/c^5b^2L^2$ [ergs $\cdot$ cm^{-2} $\cdot$ Hz^{-1}] (L is the distance to the galactic center) is independent of the velocity u_∞.

For the first limiting case there is a single spectrum of unified form with b and u_∞ the only parameters. For a range of b and for specified $\beta = u_\infty/c$, the spectra have the behavior[12] shown in Fig. 3.

At the limit $b \approx b_1$ of the range of impact parameters we have considered, $A_\nu \approx 2.5 \cdot 10^3 \beta^4$ ergs $\cdot$ cm^{-2} $\cdot$ Hz^{-1} for m = 1 and L = 10 kpc ($\nu_{m1} \approx$ 1-3 sec^{-1}, as already mentioned); if $\beta^2 \approx 10^{-3}$, $A_\nu \approx 2.5 \cdot 10^{-3}$ ergs $\cdot$ cm^{-2} $\cdot$ Hz^{-1}.

The occurrence frequency of such events can be estimated from the customary relation

$$P[\text{events/yr}] \approx 3 \cdot 10^7 \frac{N^2}{2V} \pi b_1^2 c\beta \approx 0.15 \frac{n^{1/2} m^{1/2}}{r^{3/2}}.$$

Finally, for radiation of frequency $\nu \lessgtr 3n^{3/2}m^{1/2}r^{-3/2}$ the frequency of events[5)] with given ν is

$$P \approx 1.5 \cdot n^{7/2} r^{-9/2} m^{+3/2} \nu^{-2}.$$

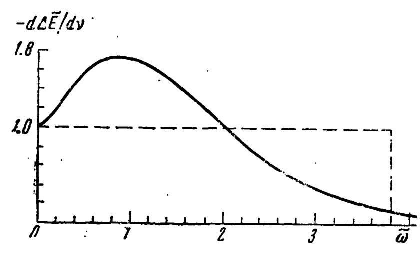

Fig. 3. Gravitational-radiation spectrum for distant passages of two stars:[12] spectral density $d\Delta\tilde{E}/d\nu$ of the radiation in units of $64G^3M^4/5c^5b^2$ as a function of the angular frequency $\tilde{\omega}$ in units of $\beta c/b$.

For given ν, the quantity P depends on A_ν in the following manner, on a logarithmic scale. The number of events[6)] for which the radiation density lies within the range between $A_\nu/\sqrt{e}$ and $A_\nu\sqrt{e}$ is equal to

$$P[\text{events/yr}] \approx 0.15 \frac{n^{1/2}m^{1/2}}{r^{3/2}} \frac{2.5\cdot 10^{-3}n^2m^4/r^2}{A_\nu} \approx 0.4 n^{5/2}m^{9/2}r^{-7/2}\left(\frac{10^{-3}}{A_\nu}\right)$$

for $2.5\cdot 10^{-4}\ m^3 r\nu^2/n < A_\nu < 2.5\cdot 10^{-3}\ m^4n^2/r^2$, and $P = 0$ for $A_\nu < 2.5\cdot 10^{-4}\ m^3r\nu^2/n$.

2. RADIATION FROM A CLOSE FLYBY

Let us turn now to the other limiting case, where the impact parameter b corresponds to a strong deflection, but with Newtonian theory still applicable:

$$b_1 > b > r_g c/u_\infty.$$

If we here consider only flyby encounters and not captures as a lower limit, we will have a critical value b_c, which, when gravitational radiation is taken into account, is given[13] by the expression

$$b_c \approx \beta^{-6/7} r_g.$$

Parabolic motion is approached at the limit; the velocity at infinity may be neglected. The maximum frequency and the radiation at the maximum frequency depend only on the periastron distance r_m, which in turn is determined by the moment $u_\infty b$ but not by b and u_∞ separately. As before, high-frequency radiation falls off exponentially, $\sim\exp(-2\nu/\nu_m)$. To establish the low-frequency radiation one must consider the asymptotic behavior of T_{ik} as $t\to\pm\infty$. It is readily shown that $T_{ik} \sim |t|^{-2/3}$ for parabolic motion, whence we obtain $h_{ik}(\omega) = \int T_{ik}\cos\omega t\,dt \sim \omega^{-1/3}$; accordingly, $h_{ik}(\omega)\sim\omega^{2/3}$ and $A(\omega)\sim\omega^{4/3}$. For parabolic motion this part of the spectrum does not depend on the moment. The total power $\int A(\omega)d\omega = \text{const}\,\omega_m^{7/3}$ does depend on the moment, since $\omega_m \sim (u_\infty b)^{-3}$. Figure 1 illustrates the behavior of the frequency spectrum for several values of the parameter $u_\infty b$. For exact calculations, see Appendix 1.

Thus every parabolic flyby will produce radiation of the same spectral density. The total number of events with a given flux is proportional to the number of collisions with an impact parameter b smaller than some value required for a given frequency.

As an estimate for the spectral density in the case of parabolic motion we have

$$A_\nu\ [\text{ergs}\cdot\text{cm}^{-2}\cdot\text{Hz}^{-1}] = \frac{W_\nu}{4\pi L^2} \approx \frac{4G^{7/3}M^{10/3}c^{-5}}{L^2}\nu^{4/3},$$

$$\nu_m \approx 0.5(2GM)^2(u_\infty b)^{-3},$$

$$W[\text{ergs}] \approx \int_0^{\nu_m} W_\nu\,d\nu \approx 20G^{7/3}M^{10/3}c^{-5}\nu_m^{7/3}.$$

It is of interest to compare this spectral density with the value for gradual infall along an orbit:

$$W_\nu^{(\text{orb})} \approx -\frac{dE}{d\omega}2\pi, \quad E = -{}^1/_2\frac{GM^2}{r}, \quad \omega = \left(\frac{GM}{r^3}\right)^{1/2},$$

$$E = -{}^1/_2 G^{2/3}M^{5/3}\omega^{2/3} \qquad W_\nu^{(\text{orb})} \approx G^{2/3}M^{5/3}\nu^{-1/3},$$

$$f = \frac{W_\nu^{(\text{orb})}}{W_\nu} \approx 0.25G^{-5/3}M^{5/3}c^5\nu^{-5/3} \approx 0.8(\nu_{\max}/\nu)^{5/3},$$

where

$$\nu_{\max} \approx c/r_g = 10^5(M/M_\odot)^{-1}.$$

Hence the ratio of the orbital to the flyby spectral density at frequency ν is equal to

$$f \approx 0.8(10^5/\nu)^{5/3}m^{-5/3}.$$

One should recognize that these estimates refer to the spectral density averaged over all angles. Since quadrupole radiation is emitted, the recorded density may vary with the angle by a factor of order 2.

Table 1 gives some numerical estimates, affording a comparison between the properties of radiation in the flyby and orbital modes.

Curiously enough, in the case of a rotating black hole near which the metric is described by the Kerr solution, an amplification of the gravitational waves will be induced by the rotational energy of the black hole.[14,15] As a consequence, highly stable circular orbits could exist, and radiative viscosity would be absent during motion along such orbits.[16] However, such processes would not play a major role in our present problem. All these effects would in fact be possible if the mass of the black hole is significantly greater than the mass of the object in the circular orbit. In such an event, either the mass of the black hole would be large, say $10^8\,M_\odot$ (the radiation would then be emitted at very low frequencies), or the mass of the rotating object would be small (the lowmass body then might be neither a black hole nor a neutron star, and would therefore be disrupted by tidal forces at considerably greater orbital distances than those at which the effects mentioned above occur).

We shall now estimate the frequency of events for the case of parabolic motion, that is, for $3n^{3/2}m^{1/2}r^{-3/2} < \nu < 10^3m^{-4/7}n^{3/7}r^{-3/7}$. The spectral density is given by

$$A_\nu \approx 2\cdot 10^4 m^{10/3}(\nu/10^5)^{4/3}, \quad \nu < \nu_m,$$

$$A_\nu \sim e^{-2\nu/\nu_m}, \quad \nu > \nu_m.$$

TABLE 1

ν	10^2	$5\cdot 10^2$	10^3	$5\cdot 10^3$	10^4
A_ν flyby	2	17	44	$3.8\cdot 10^2$	$9.6\cdot 10^2$
$(u_\infty b)_{max}$	10^{17}	$5.8\cdot 10^{16}$	$4.6\cdot 10^{16}$	$2.6\cdot 10^{16}$	$2.1\cdot 10^{16}$
A_ν orb	$1.6\cdot 10^5$	$9.3\cdot 10^4$	$7.35\cdot 10^4$	—	—
r_{orb}	$7\cdot 10^6$	$2.4\cdot 10^6$	$1.5\cdot 10^6$	—	—

Comparative parameters of gravitational radiation in the orbital and flyby modes. Dimensions: ν[Hz]; A_ν [ergs $\cdot$ cm$^{-2}\cdot$ Hz^{-1}]; $u_\infty b$ [cm^2/sec]; r_{orb} [cm].

The frequency of maximum radiation is

$$\nu_m \approx 5\cdot 10^8 m^{1/2} n^{-3/2} r^{3/2} (b/r_{g\odot})^{-3}.$$

At a given frequency ν, encounters with $b < b_{max}$ contribute, where

$$b_{max} \approx 10^7 m^{1/6} r^{1/2} n^{-1/2} (\nu/10^5)^{-1/3}.$$

For the occurrence frequency of events in the parabolic case we obtain

$$P\,[\text{events/yr}] \approx 10^{-4} n^{3/2} m^{5/6} r^{-5/2} (\nu/10^5)^{-2/3}$$

(per unit logarithmic interval).

Numerical estimates for both of the types of flyby discussed above are displayed in Fig. 4.

3. ESTIMATES FOR A CLUSTER AND THE ROLE OF LOW-MASS BODIES

All the estimates obtained above refer to a globular cluster in which the mean velocity vanishes at every point, and the rms collisional velocity is of the same order as the virial value.

If the cluster evolution time is considered to be sufficiently long, we will thereby be restricting the range of characteristic cluster parameters (see Appendix 2), which in turn will limit the frequency of events.

Let us ascertain whether collisions between neutron stars would be a source of gravitational radiation. For $r_m \approx 10^6$ cm and $u_m \approx c/2$, which corresponds to $b \approx {}^1/_2 r_m \beta^{-1}$, the frequency of direct collisions will be

$$P\,[\text{events/yr}] \simeq 10^{-4} n^{3/2} r^{-5} m^{-1/2}.$$

If an energy of about $0.05 M_\odot c^2 \approx 10^{53}$ ergs is released in each such event in the form of electromagnetic radiation, then the average radiation emitted each second will be

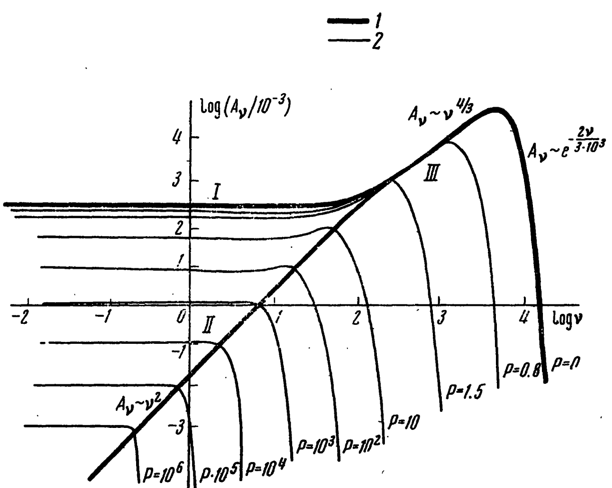

Fig. 4. 1) Spectral density of gravitational radiation from the galactic center in a flyby as a function of the reception frequency (logarithmic scales). I, II) Maximum and minimum possible A_ν, respectively, in a distant flyby; III) A_ν for "parabolic" close encounters. 2) Curves of constant occurrence frequency, P events/yr. The diagram refers to the case r = 0.1, m = 1, n = 1, corresponding to $\tau \approx 2\cdot 10^{10}$ yr.

$$W\,[\text{ergs/sec}] \approx 4\cdot 10^{41} n^{3/2} r^{-5/2} m^{-1/2}.$$

In the case $r \approx 1$, this quantity is nearly equal to the upper estimate on the radiant power at the galactic center ($\lesssim 10^{42}$ ergs/sec); furthermore, the occurrence of highly powerful but very rare flares would be in conflict with observation.

For r = 0.1 there is an even sharper contradiction to the observations: W would be 10^3 times as great as the upper estimate for the power of electromagnetic radiation at the galactic center.

The model would therefore work only for 10^9 n collapsars (black holes, but not neutron stars).

Since the frequency of events will increase with N, the question arises as to the possible role of low-mass objects ($m \ll M_\odot$), which might be extremely numerous without making any appreciable contribution to the total mass of the galactic nucleus, and thus without affecting its lifetime.

However, the graviational radiation of such objects would be very small [$\lesssim (M/M_\odot)^2 M_\odot c^2$] and would be emitted at exceptionally low frequencies.

For objects with a mass much smaller than $M_\odot$, the density $\rho \approx$ 1-10 g/cm^3, and the periastron of the flyby trajectory could not be smaller than the size of the passing body (otherwise only a small portion of the mass of that body would be radiated). Therefore we have

$$\nu < \nu_m \approx \left(\frac{GM}{r_m^3}\right)^{1/2} \leqslant \left(G\rho\frac{M}{m}\right)^{1/2} \approx (10^{-4} \text{ to } 10^{-3}) \left(\frac{M}{m}\right)^{1/2} \text{Hz}.$$

This restriction on the frequency does not hold for bodies that are retained by nongravitational forces and whose mass is very small:

$$m/M_\odot < 10^{-10} \text{ to } 10^{-11};$$

in this case we have

$$A_\nu \leqslant 2\cdot 10^3 \left(\frac{m}{M_\odot}\right)^2 \left(\frac{M}{M_\odot}\right)^{4/3} \left(\frac{G\rho M}{10^{10} m}\right)^{2/3}$$
$$\approx 4\cdot 10^{-7} \left(\frac{m}{M_\odot}\right)^{4/3} \left(\frac{M}{M_\odot}\right)^2$$

As for the radiation of small masses with a gradual infall along a circular orbit,[7] in this case a more stringent limit can be imposed on the frequency of the radiation, on the basis of the condition that the body not be disrupted by tidal forces (otherwise the body would spread out along the orbit and the quadrupole moment would be greatly diminished).

If the body is self-gravitating, then

$$\nu \lesssim (G\rho)^{1/2} \approx 10^{-4} \text{ to } 10^{-3} \text{ Hz}.$$

But if the body can withstand fragmentation by its own elasticity, then

$$\nu \leqslant 10^{-11} \sigma^{1/2} (M_\odot/m)^{1/2} \rho^{-1/6},$$

where $\sigma < 10^4$ kg/cm^2 is the tensile strength of the solid body, so that

$$\nu \leqslant 10^{-7}(M_\odot/m)^{1/2}$$

The directivity of the radiation should also be taken into account. For this purpose we may use an approximate expression for the radiation per unit solid angle emitted at an angle θ relative to the normal to the plane of motion: $A_\nu(\theta) \approx 2.5 A_\nu f(\theta)$, where for a distant encounter $f(\theta) \approx 1 - \sin^2\theta + {}^1/_8\sin^4\theta$, while for a close encounter $f(\theta) \approx 1 - \sin^2\theta + {}^1/_4\sin^4\theta$ (an average has here been taken over angles in the plane of motion).

Evidently in the distant-encounter case the radiation in the plane of motion is nearly an order of magnitude smaller than in the perpendicular direction. In the second, "parabolic," case, the angular dependence is less strong (a behavior consistent with the attractive forces associated with the gravitational interaction of two masses).

But even if the galactic nucleus comprises a highly flattened disk, nevertheless in order for the motion both before and after an encounter to remain in the galactic plane the impact parameter must be far smaller than the thickness of the disk. This situation would correspond to extremely low gravitational-radiation frequencies, much lower than 1 Hz.

Accordingly, the event occurrence frequencies that we have given ought to be "smeared out," and a whole range of spectral densities $A_\nu(\theta)$ should be observed, since the planes of motion of star pairs will form a random angle to the line of sight.

4. NONRESONANCE RECEPTION OF GRAVITATIONAL WAVES BY MEANS OF FREE BODIES

As we have been interested in the spectral properties of the radiation, a resonance system for detecting gravitational waves has been considered. However, another, nonresonance, type of detector is possible, consisting of two noninteracting bodies (such as satellites).

Figure 2 shows that the value of h_{ik} after the encounter of two objects differs from the value before the encounter. As a result the distance between a pair of free bodies should change, and in principle this effect might possibly serve as a nonresonance detector.

For a flyby with an impact parameter b, the relative change in the separation of a pair of free bodies will be

$$\Delta l/l \approx b/L \frac{\beta^4}{1+\beta^4(b/r_g)^2} \Psi(\theta, \varphi, \theta', \varphi'),$$

where $\Psi(\theta, \varphi, \theta', \varphi')$ is a function of order unity allowing for the mutual orientation of the planes of motion of the radiating masses and the detector.

For the distant-encounter case, we have

$$\frac{\Delta l}{l} \approx \frac{r_g^2}{bL}\Psi.$$

In the second, "parabolic," case, we have

$$\Delta l/l \approx b/L\beta^4\Psi.$$

The maximum value of $\Delta l/l$ is in fact reached for $b \approx b_1 = r_g/\beta^2$, and is equal to

$$\left(\frac{\Delta l}{l}\right)_{\max} \approx {}^1/_2\frac{r_g}{L}\beta^2\Psi \approx 5\cdot 10^{-21}\frac{nm^2}{r}\Psi.$$

One should note that although the distance between the free bodies will change, their relative velocity will actually become vanishingly small as the flyby event concludes. Indeed, $\Delta l \approx hl$ while $\Delta v \approx \dot{h}l$, but after the encounter $\dot{h} = 0$, since a gravitational-wave energy flux proportional to $\dot{h}^2$ will then be absent. Thus in particular it would not be possible to have a stochastic acceleration process for the free bodies under the action of random gravitational waves radiated when various masses pass by each other. Rees[17] has suggested such a process for long-wave gravitational radiation of cosmological origin.

The authors are grateful to V. B. Braginskii for formulating the problem of small bodies and for his continuing interest in the investigation.

APPENDIX 1

GRAVITATIONAL-RADIATION SPECTRUM FOR A SINGLE FLYBY

Consider a pair of attracting masses in relative hyperbolic motion. We shall confine attention to the nonrelativistic case, but no restrictions will be placed on the mass ratio of the two bodies.

In the system of the center of mass, the motion is parametrically described[18,19] by

$$r = a(\varepsilon \operatorname{ch}\zeta - 1), \qquad \omega_0 t = (\varepsilon \operatorname{sh}\zeta - \zeta),$$

where

$$a = \frac{G(M_1+M_2)}{u_\infty^2} \approx \frac{r_g}{\beta^2}, \quad \varepsilon = \sqrt{1+\beta^4\left(\frac{b}{r_g}\right)^2}$$

is the eccentricity of the hyperbola, and $\omega_0 = c/r_g\beta^3$.

By straightforward calculation one obtains the following expressions for the Fourier components of the first derivatives with respect to time of the quadrupole-moment tensor $D_{\alpha\beta} = \mu(3x_\alpha x_\beta - \delta_{\alpha\beta}r^2)$ (here μ is the reduced mass):

$$\dot{D}_{xx}(\omega) = \frac{2\mu a^2\pi\omega_0}{\omega}\frac{\varepsilon^2-3}{\varepsilon^2}\left[H^{(1)}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu}) - \frac{3(\varepsilon^2-1)}{\varepsilon^2-3}i\tilde{\nu}\varepsilon H^{(1)\prime}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu})\right],$$

$$\dot{D}_{yy}(\omega) = -\frac{2\mu a^2\pi\omega_0}{\omega}\frac{2\varepsilon^2-3}{\varepsilon^2}\left[H^{(1)}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu}) - \frac{3(\varepsilon^2-1)}{2\varepsilon^2-3}i\tilde{\nu}\varepsilon H^{(1)\prime}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu})\right],$$

$$\dot{D}_{zz}(\omega) = \frac{2\mu a^2\pi\omega_0}{\omega}H^{(1)}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu}),$$

$$\dot{D}_{xy}(\omega) = \frac{6\mu a^2\pi\omega_0}{\omega}\sqrt{\frac{\varepsilon^2-1}{\varepsilon^2}}\left[H^{(1)\prime}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu}) - \frac{\varepsilon^2-1}{\varepsilon^2}i\tilde{\nu}\varepsilon H^{(1)}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu})\right];$$

$\tilde{\nu} = \omega/\omega_0$, and the $H^{(1)}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu})$ are the Hankel functions of the first kind.

Case 1 ($\varepsilon \gg 1$). This case corresponds to a hyperbola differing insignificantly from a straight line, that is, to the distant-encounter mode.

The asymptotic behavior of the Hankel function in this case follows at once from its integral representation:

$$i\pi H^{(1)}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu}) = \int_{-\infty}^{\infty} e^{i\tilde{\nu}(\xi-\varepsilon\,\mathrm{sh}\,\xi)}\,d\xi\ ; \qquad \text{(A1)}$$

for $\varepsilon \gg 1$, $\varepsilon \sinh\xi \gg \xi$, and

$$i\pi H^{(1)}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu}) = \int_{-\infty}^{\infty} e^{-i\tilde{\nu}\varepsilon\,\mathrm{sh}\,\xi}\,d\xi = i\pi H^{(1)}_0(i\nu').$$

Similarly, $H^{(1)\prime}_{i\tilde{\nu}}(i\varepsilon\tilde{\nu}) \approx H^{(1)\prime}_0(i\nu')$, with

$$\nu' = \omega/\omega_0\varepsilon \approx \frac{\omega_b}{u_\infty}.$$

For $\nu' \ll 1$ we shall use the standard asymptotic expression[19] for the zeroth Hankel function:

$$iH_0^{(1)}(i\nu') \approx \frac{2}{\pi}\ln\frac{2i}{\gamma\nu'},$$

where the natural logarithm of $\gamma = 1.781\ldots$ is Euler's constant; then

$$iH_0^{(1)\prime}(i\nu') \approx -\frac{2i}{\pi\nu'}.$$

In this case $\dot{D}_{xx}(\omega) \approx \dot{D}_{yy}(\omega) \approx \dot{D}_{zz}(\omega) \approx \nu' \ln(2/\gamma\nu')$ approaches zero as $\nu' \to 0$, whereas

$$D_{xy}(\omega) \to -\frac{12\mu a^2 i\omega_0^2}{\varepsilon\omega^2} = -12i\mu\left(\frac{r_g}{b}\right)c^2,$$

$$A_\nu = 2\pi A_\omega = \frac{4G}{45c^5}\,\frac{|\dddot{D}_{xy}(\omega)|^2}{4\pi L^2} \approx 2.5\cdot10^3 m^4\left(\frac{r_{g\odot}}{b}\right)^2.$$

In the other limiting case, $\nu' \gg 1$, we have

$$H_0^{(1)}(i\nu') \sim e^{-\nu'}$$

so that $A_\nu \sim e^{-2\nu'}$.

Case 2 ($\varepsilon \approx 1$). Here the motion is almost parabolic.

For $\nu \ll 1$ the main contribution to the integral (A1) comes from large ξ, and we may set $H^{(1)}_{i\tilde{\nu}}(i\tilde{\nu}\varepsilon) \approx H^{(1)}_0(i\tilde{\nu}\varepsilon)$, so that for $\tilde{\nu} \ll 1$ the preceding asymptotic expression, leading to a constant, is valid.

The frequency range $\nu \gg 1$ is, however, the most interesting one for the parabolic-motion case. We express the Hankel functions in terms of MacDonald functions:

$$H^{(1)}_{i\tilde{\nu}}(i\tilde{\nu}\varepsilon) \approx -\frac{2i}{\pi\sqrt{3}}(\varepsilon^2-1)^{1/2}K_{1/3}\left[\frac{\tilde{\nu}}{3}(\varepsilon^2-1)^{3/2}\right],$$

$$H^{(1)\prime}_{i\tilde{\nu}}(i\tilde{\nu}\varepsilon) \approx -\frac{2i}{\pi\sqrt{3}}(\varepsilon^2-1)\,K_{2/3}\left[\frac{\tilde{\nu}}{3}(\varepsilon^2-1)^{3/2}\right].$$

Then using the asymptotic expression for the MacDonald functions,

$$K_\mu(y) \approx \frac{2^{\mu-1}\Gamma(\mu)}{y^\mu}, \qquad y \ll 1,$$

$$K_\mu(y) \approx \sqrt{\frac{\pi}{2y}}\,e^{-y}, \qquad y \gg 1,$$

we readily find that $\tilde{\nu}(\varepsilon^2-1)^{3/2} \ll 1$ is equivalent to $\nu \ll u_m/r_m$. The main contribution comes from the diagonal components of $D_{\alpha\beta}$, and we have

$$|\dddot{D}_{\alpha\beta}(\omega)|^2 \approx 8\cdot6^{2/3}\Gamma^2(1/3)\,\mu^2 a^4\omega_0^4\tilde{\nu}^{4/3}$$

and

$$A_\nu \approx 2\cdot10^3 m^{10/3}(\nu/10^5)^{4/3}.$$

APPENDIX 2

RELATION BETWEEN PARAMETERS OF THE GALACTIC NUCLEUS AND ITS LIFETIME

If the virial velocities are sufficiently high and the nucleus is composed of superdense stars, the lifetime will be determined by gravitational captures.[2] With gravitational radiation neglected, a necessary condition for this situation is

$$\left(\frac{u_\infty}{c}\right)^2 \gtrsim 1.6\cdot10^{-3}.$$

Let us now compare the characteristic evolution times τ_c, τ_e due to captures and to evaporation from the nucleus, after correction for gravitational radiation:[13]

$$1/\tau_c = \frac{N}{\frac{4}{3}\pi R^3}\left(\frac{u_\infty}{c}\right)c\sigma_c,$$

$$\sigma_c \approx 2^{-2/7}\pi m^2 r_{g\odot}^2(u_\infty/c)^{-18/7},$$

$$\frac{1}{\tau_c} \approx 0.6\cdot10^{-22}(u_\infty/c)^{-11/7}\frac{m^2 n}{r^3};$$

$$\frac{1}{\tau_e} \approx 0.45\cdot10^{-24}(u_\infty/c)^{-3}\frac{m^2 n}{r^3}.$$

Thus $1/\tau_e < 1/\tau_c$ for $(u_\infty/c)^2 \gtrsim 10^{-3}$, that is, for $mn/r \gtrsim 1$. In particular, we have

$$\tau_c \approx 7.5\cdot10^{19}\,m^{-17/14}n^{-3/14}r^{31/14}\ \text{sec},$$

$$\tau_e \approx 7.5\cdot10^{19}\,m^{-1/2}n^{1/2}r^{3/2}\ \text{sec};$$

For $\tau \approx \tau_c > 3\cdot10^{16}$ sec, $r > 3\cdot10^{-2}m^{17/31}n^{3/31}$.

If $m = n = r = 1$, $\tau \approx 2.5\cdot10^{12}$ yr; if $m \approx n \approx 1$ and $r \approx 0.1$, $\tau \approx 17\cdot10^9$ yr; while if $m \approx n \approx 1$ and $r \approx 0.03$, $\tau \approx 1\cdot10^9$ yr.

Evidently, then, in the cases of greatest interest, those with small r (where events occur with considerable frequency), the lifetime will be determined by captures.

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1)At IAU Symposium No. 64 (Warsaw, 1973), such an encounter was called a "flyby."

2)These events would be more powerful than flyby encounters but nevertheless weak compared to the events considered by Weber.

3)Misner introduces the special term "gravitational pulse unit" GPU for 10^7 ergs$\cdot$cm$^{-2}\cdot$Hz^{-1}.

4)If f events occur per unit time (dimension $[f] = \text{sec}^{-1}$), then a quantity describing the corresponding time-averaged noise will be $B_\nu = fA_\nu$ (dimensions $[B_\nu] = \text{erg}\cdot\text{cm}^{-2}\cdot\text{Hz}^{-1}\cdot\text{sec}^{-1}$).

5)In principle, A_ν will be very small but finite for even more distant encounters, but in determining $P(\nu)$ we shall neglect events with exponentially small $A\nu$.

6)The quantity P may be interpreted approximately as the number of events with a spectral density exceeding some lower limit; see the preceding note.

7)For gradual infall along a circular orbit the gravitational radiation will be proportional to m/M.

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