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A NUMERICAL STUDY OF THE STABILITY OF FLATTENED GALAXIES: OR, CAN COLD GALAXIES SURVIVE?*

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ABSTRACT

To study the stability of flattened galaxies, we have followed the evolution of simulated galaxies containing 150 to 500 mass points. Models which begin with characteristics similar to the disk of our Galaxy (except for increased velocity dispersion and thickness to assure local stability) were found to be rapidly and grossly unstable to barlike modes. These modes cause an increase in random kinetic energy, with approximate stability being reached when the ratio of kinetic energy of rotation to total gravitational energy, designated t, is reduced to the value of 0.14 ± 0.02. Parameter studies indicate that the result probably is not due to inadequacies of the numerical N-body simulation method. A survey of the literature shows that a critical value for limiting stability $t \simeq 0.14$ has been found by a variety of methods.

Models with added spherical (halo) component are more stable. It appears that halo-to-disk mass ratios of 1 to $2\frac{1}{2}$, and an initial value of $t \simeq 0.14 \pm 0.03$, are required for stability. If our Galaxy (and other spirals) do not have a substantial unobserved mass in a hot disk component, then apparently the halo (spherical) mass *interior* to the disk must be comparable to the disk mass. Thus normalized, the halo masses of our Galaxy and of other spiral galaxies *exterior* to the observed disks may be extremely large.

Subject headings: galactic structure — stellar dynamics

I. INTRODUCTION

a) *Purpose*

There is some theoretical reason to believe that a highly flattened disk supported mainly by rotation is subject to large-scale (barlike) instabilities, whether the disk is composed of gas or stars. Yet there exist many galaxies where most of the light originates in an apparently flat rotating disk, and where the random stellar motions appear to be small compared to the systematic circular motion; i.e., they are apparently "cold." Our own Galaxy is such a system, and it does not seem to suffer from any large-scale, large-amplitude, short-time-scale instability. Two questions naturally arise. Is a "cold" rotating disk of stars truly unstable? If so, how can we account for the apparent stability of our Galaxy? To supplement previous theoretical and numerical work, we present here some numerical N-body studies designed to test whether or not flattened stellar systems are subject to large-scale instabilities; what, if any, critical value of the random component of the total kinetic energy is needed to prevent these instabilities; and what could be done to add stability to a flat system whose disk would be, by itself, unstable. We then discuss the possible significance of the instability in the light of the present computations, earlier calculations, and the apparent astronomical situation.

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b) *Concept*

i) *Background*

We establish here some notation. An N-body system in equilibrium satisfies the virial theorem in the form

$$2T_{\rm rand} + 2T_{\rm mean} + W = 0\,, \tag{1}$$

where

$$T_{\rm mean} \equiv \frac{1}{2}\int\int \langle \boldsymbol{v}(\boldsymbol{r})\rangle^2 f(\boldsymbol{v}, \boldsymbol{r}) d\boldsymbol{v} d\boldsymbol{r}\,, \tag{2a}$$

$$T_{\rm rand} \equiv \frac{1}{2}\int\int (\boldsymbol{v} - \langle \boldsymbol{v}(\boldsymbol{r})\rangle)^2 f(\boldsymbol{v}, \boldsymbol{r}) d\boldsymbol{v} d\boldsymbol{r}\,. \tag{2b}$$

Here $f(\boldsymbol{v}, \boldsymbol{r})$ is the density of points (atoms, stars) in position and velocity space, $\langle \boldsymbol{v}(\boldsymbol{r})\rangle \equiv \int \boldsymbol{v} f(\boldsymbol{v}, \boldsymbol{r}) d\boldsymbol{v} / \int f d\boldsymbol{v}$ is the streaming velocity, and W is the gravitational energy of the system, $W \equiv \frac{1}{2}\int$ (potential)dm. Consider now the simplest such systems, for which $\langle \boldsymbol{v}(\boldsymbol{r})\rangle$ represents steady rotational motion. Here $T_{\rm mean}$ is the kinetic energy of rotation. Dividing equation (1) by $|W|$ and defining

$$t \equiv T_{\rm mean}/|W|\,; \qquad u \equiv T_{\rm rand}/|W|\,; \tag{3}$$

we have

$$t + u = \tfrac{1}{2}\,, \qquad 0 \leq t \leq \tfrac{1}{2}\,, \tag{4}$$

where the relative magnitudes of t and u represent the relative importance of rotation and "pressure" for maintaining equilibrium.

ii) *Fluid System*

The simplest such system—the uniformly rotating, uniform-density, fluid body—has been studied in great detail. An axisymmetric sequence, the Maclaurin sequence, can be defined in terms of increasing eccentricity or increasing t, the relation between them being

$$t = \tfrac{1}{2}\{(3e^{-2} - 2) - 3(e^{-2} - 1)^{1/2}[\arcsin(e)]^{-1}\}\,.$$

Thus the various known equilibrium and stability properties of the sequence can be rephrased as limits on permissible values of t (cf. Bardeen 1971; Bodenheimer and Ostriker 1973 [BO]; Ostriker and Bodenheimer 1973 [OB]). All equilibrium values of t are attainable along the Maclaurin sequence ($0 \leq t \leq \frac{1}{2}$), but objects with $t > 0.1376$ are secularly unstable and those with $t > 0.2738$ are dynamically unstable to the formation of bars. If axisymmetry is rigorously maintained, the spheroids are secularly unstable to ring formation for $t > 0.3589$, and dynamically unstable to ring formation for $t > 0.4574$. It might be thought that these results are peculiar to the rather contrived and physically improbable Maclaurin spheroids. However, recent studies (cf. OB and references therein) have shown that fluid "stars" constructed with a wide range in degrees of central concentration in density and in angular velocity all become secularly unstable at nearly the same point (when measured by the parameter t). For all the cases considered it was found that

$$t_{\rm crit} = 0.137 \pm 0.002 \quad \text{(calculated fluid limits)}\,. \tag{5}$$

The secular instability originates in the advantage a rapidly rotating object has in maximizing its moment of inertia. Beyond the critical point, barlike equilibria exist having lower total energy but the same angular momentum, mass, and central density.

The instability consists of a developing eccentricity of the equatorial plane and flattening at the poles. Dynamical instability ($t > 0.27$) has been followed in the nonlinear regime by Rossner (1967) and Fujimoto (1968). The very complicated post-instability motions can be roughly characterized as those of an extremely elongated (25 to 1) prolate spheroid tumbling end over end in space. These elongated objects (really ellipsoids, not spheroids) are probably unstable to fragmentation, although this last point has not been proven. Thus the instability which leads to bar formation in fluids appears to be large-scale and irreversible.

iii) *N-Body Systems*

The advantage a triaxial configuration has over an axisymmetric one should carry over to rapidly rotating stellar systems, so it is interesting to see if there is an analog of the instability in an N-body system. Previous numerical calculations by Miller, Prendergast and Quirk (1970 [MPQ]) and Hohl (1971 [H]) appeared to indicate that flat cold systems ($u/t \ll 1$) are subject to large-scale instabilities. While these studies are, in terms of numbers of stars, far superior to the work to be described here, they were not designed to explore this instability. Therefore, we describe next some numerical studies of the instability, and then in § III return to a discussion of its possible significance.

II. COMPUTATION

a) *The N-Body Model*

The model is based on the numerical integration of the equations of motion in three dimensions for N mutually interacting particles. In addition to this flattened (but not flat) system, we suppose that there is a spherical component, which we call the halo, having an assumed mass distribution designed to produce a relatively level rotation curve. One issue of definition is important here: by "halo" we mean "spherically symmetrical component," without prejudice as to whether the correct astronomical term would be "halo," "galactic bulge," "galactic nucleus," or some combination thereof.

The acceleration of the ith particle is

$$\boldsymbol{a}_i = -\frac{(1.1)^2 \boldsymbol{r}_i M_H}{R(r_i + 0.1R)^2} + \sum_{j \neq i} \frac{(\boldsymbol{r}_j - \boldsymbol{r}_i)}{(r_{ij}^2 + c^2)^{3/2}} \,. \tag{6}$$

The first term is the contribution by the "hard" spherically symmetric halo mass M_H. The halo stops at radius R equal to the initial disk radius. Beyond this radius the first term is $-\boldsymbol{r}_i M_H / r_i^3$. The second term is an approximation to the Newtonian gravitational interaction, valid when $r_{ij} \equiv |\boldsymbol{r}_i - \boldsymbol{r}_j|$ is much greater than the cutoff c. Because of the departure from a Newtonian potential, the virial theorem in its usual form is not exactly satisfied for a system in equilibrium. However, in the numerical models the discrepancy is only on the order of 10 percent, causing an uncertainty in our estimate of t of the same order. Units are chosen such that $G = 1$ and the mass of each particle is unity. The cutoff at $r_{ij} \sim c$ greatly simplifies the numerical computation by eliminating the infrequent but very large accelerations at close encounters. Such large fluctuations are in any case only the uninteresting consequence of the very small number of particles in the model compared to a galaxy. We are interested in the question of large-scale stability and the hope is that, if the model radius is much greater than c, the acceleration (6) may give a realistic description of the large-scale dynamic motion. As described below, we can test this assumption by seeing how the results depend on the value of c.

When N is large, the main computational problem is the large number, $\propto N(N - 1)/2$, of operations required to determine the accelerations. We were fortunate that Professor

E. G. Groth designed a program for evaluating the accelerations utilizing the full capabilities of the Princeton computer, saving a factor of about 3 over the more naïve approach. In the computation the integration time step is a fixed number Δt. For each point the position, velocity, and acceleration at t are used with $\boldsymbol{a}(t - \Delta t)$ and $\boldsymbol{a}(t - 2\Delta t)$ to predict $\boldsymbol{r}(t + \Delta t)$ by a fourth-order polynomial. This new position is used to compute the new acceleration, and the new acceleration with the previous five numbers is used to get a corrected position $\boldsymbol{r}(t + \Delta t)$ by a fifth-order polynomial. The time step is chosen so that the integration ought to reliably carry a point past another point at any impact parameter and any reasonable incident speed. Accuracy is checked by comparing the results for different time steps (§ II*d* below). Earlier versions of this general scheme have been used in other applications (Peebles 1969, 1971).

b) *Initial Values in the Standard Model*

A particular model called model 1 is taken as the "standard" to which we compare the results of varying any of the parameters. The initial values and other parameters for this standard model are described here, the variants in § II*e*.

The general parameters are: (i) particle number $N = 300$; (ii) disk radius $R = 1$; (iii) cutoff radius (eq. [1]) $c = 0.05$; (iv) time step $\Delta t = 0.001$; and (v) halo (eq. [6]) $M_H = 0$.

The initial surface density of points $\Sigma(r)$ varies as r^{-1}. This is achieved by distributing the points uniformly in the radius interval $0 < r < R$, as follows. The disk is divided into $N/10$ rings, with width in radius $\Delta r = 10R/N$ each, and into 10 equal pie-shaped radial segments ($\Delta\phi = 36°$). One point is placed in each of the N cells, with the radius r and longitude ϕ randomly chosen over the range of arguments for the cell. The points are placed in a flat disk. The thickness of the disk is established by assigning random velocities normal to the disk. The initial particle distribution in the plane is shown in figure 4*a*.

The first step in assigning initial velocities is to estimate the angular velocity needed to hold each particle in circular orbit. Each point is temporarily rotated by an angle ϕ_i equal to a random fraction of 360°, and $\boldsymbol{r}_i \cdot \boldsymbol{a}_i$ computed. The average value of this quantity yields the desired mean speed if the radial positions of the particles do not vary and if the angular positions are not correlated. The initial velocity of each particle is directed in the plane perpendicular to the radial vector, the magnitude being $\langle -\boldsymbol{r}_i \cdot \boldsymbol{a}_i \rangle^{1/2}$. This initial velocity averaged over the 10 particles in each radial interval is plotted as a circle in figure 1. The particle speed does not vary much with r, as expected from the density distribution, and in rough agreement with the observed rotation law of the Galaxy. When the halo is added, the shape of the rotation curve is very nearly unchanged—a condition that is expected because the surface density distributions have roughly the same forms for the assumed halo and disk components.

The above procedure introduces a scatter of approximately 20 percent in the circular velocities of particles in the same ring. We increase this scatter by adding a velocity dispersion designed to fit the Toomre (1964) criterion for stability against the development of small-scale irregularities. The added velocity components are drawn from random normal distributions with standard deviations σ_r, σ_θ, σ_z. The standard deviation of the velocity added in the radial direction is the smaller of

$$\sigma_r^T = \sigma N \nu^{-1}, \qquad \sigma_r^S = 0.4\nu, \tag{7}$$

where

$$\nu \equiv v(1 + d \ln v / d \ln r)^{1/2}.$$

Here $v(r)$ is the mean particle speed as a function of position obtained from the first step, and N is the number of mass points in the model. The first equation is based on Toomre's condition (1964) for stability against growth of irregularities of small scale.

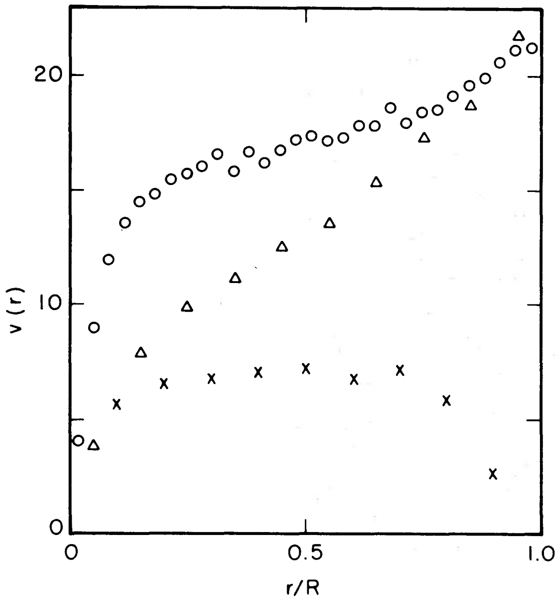

FIG. 1.—Initial velocities in the disk. *Circles*, the initial velocities in the standard, model 1. *Crosses*, the dispersion σ_r in the initial random radial velocities. *Triangles*, the initial rotation curve for model 6.

In our standard integration we choose the constant

$$\sigma = 0.454\,, \tag{8}$$

20 percent larger than the minimum for stability according to Toomre. The quantity σ_r^T gets very large at small r, so the condition $\sigma_r \leq \sigma_r^S$ defined by the second equation is added to assure that the orbits are at least roughly circular.

The derivatives of v in equation (7) are obtained from v averaged in intervals $\Delta r = 0.1$, and σ_r is evaluated for nine zones in radius. The final values for the dispersion are indicated as the crosses in figure 1. In the outer half, $\sigma_r = \sigma_r^T$. For $r \leq 0.5$, σ_r^S falls below σ_r^T, reaching $\sigma_r^S = 0.8\sigma_r^T$ at $r = 0.2$ and $\sigma_r^S = 0.6\sigma_r^T$ at $r = 0.1$. Finally, we reduce σ_r in the outermost zone by a factor of 2 in order to reduce the expansion of the edges of the disk. The σ_r for the radial velocity added to any particle is the value for the zone in which the particle finds itself.

For σ_θ we take the equilibrium expression required for steady epicyclic motions:

$$\sigma_\theta = \frac{\sigma_r}{2^{1/2}}(1 + d\ln v/d\ln r)^{1/2}\,. \tag{9}$$

The dispersion normal to the plane is taken to be

$$\sigma_z = \sigma_\theta\,. \tag{10}$$

As a final step, the vector velocity v_i is multiplied by a factor, separately determined for each of the original rings, such that the ring has the same total kinetic energy as before. The purpose of this last correction is to approximately reestablish the equilibrium between gravitational and centripetal accelerations that existed before the random energies were added.

This calculation differs from previous work (MPQ and H) in several respects. Fewer "stars" are used here. The gravitational interaction is dealt with in a very different way. Both approaches round off the interaction at small separations, but ours does not induce the fourfold symmetry built into some versions of the Fourier technique. In our model, the equilibrium stellar disk has a finite (in fact, substantial) thickness perpendicular to the rotation axis. Thus we avoid, among other difficulties, the rapid two-body relaxation in an infinitely thin disk of mutually interacting Newtonian mass points. For this degenerate case the relaxation time is independent of N and of the same order as the crossing time.

c) Measures of Evolution and Some Results for the Standard Model

A convenient measure of rotation, and a theoretically interesting criterion for instability in the numerical and analytic fluid models, is $t \equiv T_{\text{mean}}/|W|$ (eq. [3]). The systematic rotation energy in the model is defined as

$$T_{\text{mean}} = \frac{1}{2} \sum_{\alpha} n_\alpha v_\alpha^2 , \tag{11}$$

where n_α is the number of points in successive rings of distance perpendicular to the angular-momentum axis $\Delta r = 0.1$, and v_α is the mean value of the component of the velocity in the ϕ-direction (any backward-moving particles counted as negative) for the particles in the ring; this definition agrees with definition (2a) when the streaming velocities are purely rotational but differs from it if a bar forms. Since initial and final states are roughly axisymmetric, only a small error is introduced by use of equation (11). $|W|$ is the magnitude of the sum of the potential energies of interaction of the particles in the field of the halo plus the self-energy of the halo. The evolution of this estimate of t for the standard model is shown as the open circles (model 1) in figure 2.

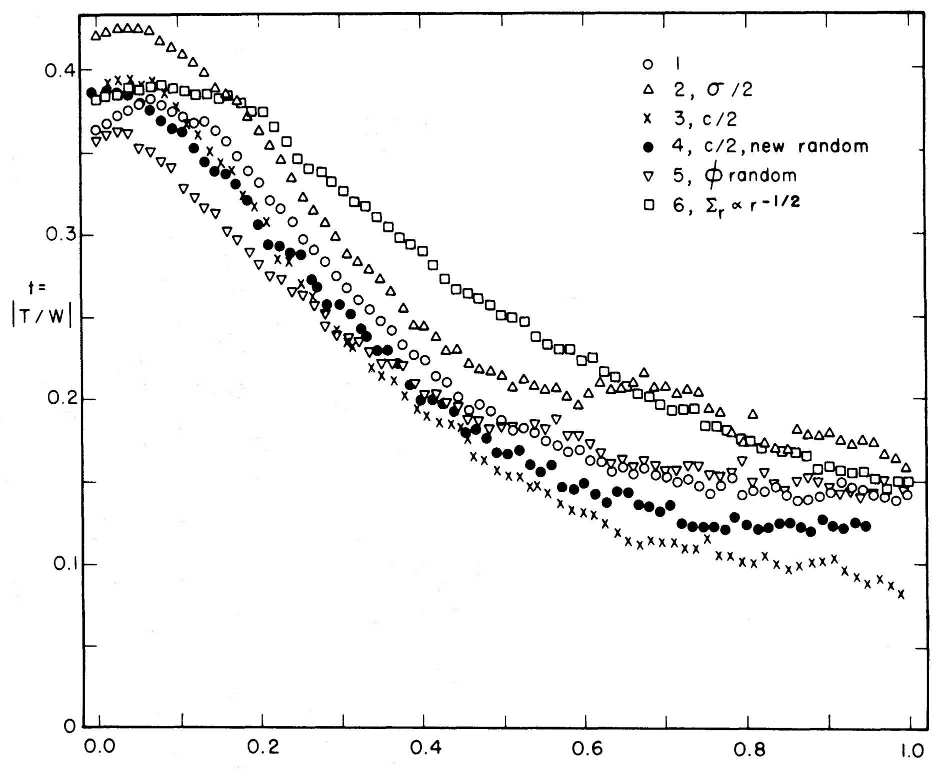

FIG. 2.—Evolution of the model galaxies. Abscissa is time measured in units of the orbit period for the outermost particles in the initial system under the assumption of circular orbits. Ordinate is defined in equation 3.

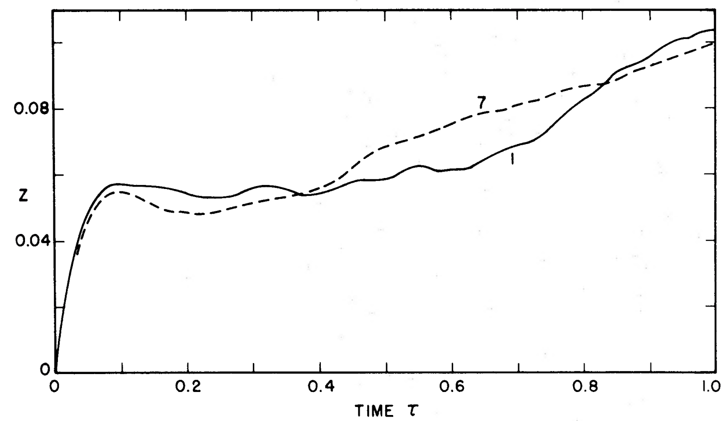

FIG. 3.—Evolution of the model galaxies. Ordinate is the rms thickness of the disk, in units of the initial radius of the disk.

The time τ in this and the following figures is measured in units of the orbit time for the particles in the outermost part of the original disk.

A second interesting measure is the root-mean-square value of the coordinates perpendicular to the disk, z. One might expect the effect of two-body collisions, or of any tendency for the disk to buckle, to show up as a progressive thickening. The evolution of the disk thickness is shown in figure 3. The rapid initial rise comes about because the initial disk has no thickness but a random normal component of velocity. One might expect $|\bar{z}|$ to overshoot, then settle back to a stable value. This seems to happen in model 7 ($N = 500$; cf. § II*e*), but is not so apparent in the model 1. After one rotation period, the disk is approximately twice as thick as its initial equilibrium value.

We tried one other measure of evolution, the transforms $\sum \sin 2\phi_i$, $\sum \cos 2\phi_i$, quantities designed to measure the growth of a barlike shape. They did initially increase but then behaved in a very complicated way, possibly because the disk does not maintain a truly straight bar.

Particle positions in model 1 at selected times are shown in figure 4. After a short period of adjustment to initial conditions, a large-scale bar develops. This bar is quite apparent after only a small fraction of a rotation period (cf. fig. 4*b*) and is most developed at about $\tau = 0.6$ (fig. 4*c*). After the development of a prominent bar, the potential has a significant time dependence, so the energy of an individual star moving in the "smoothed" potential is not constant. Also, since the smoothed potential is not axisymmetric, the angular momentum of an individual particle is not constant either. Finally, since the turning rate for the bar is comparable to the orbital time for the average star, there is very efficient coupling between the large-scale disturbances and individual particle orbits (effective wave-particle interaction). As a consequence of these effects, the particles' initial, nearly circular, orbits are rapidly altered and the velocity distribution becomes more isotropic in the plane (v_ϕ, v_r). Thus the disk heats up rapidly and t falls to approximately half of its original value in only half a rotation period. After t has fallen to the range 0.1–0.2, further changes are slower, and the system appears to be approaching a stationary state (cf. fig. 4*d*).

d) *Numerical Accuracy*

Numerical errors are a traditional problem in N-body models (cf. Lecar 1967), but we have several reasons for thinking that we have avoided or mitigated the problem:

1. We are interested in the global evolution of the system, not in the detailed

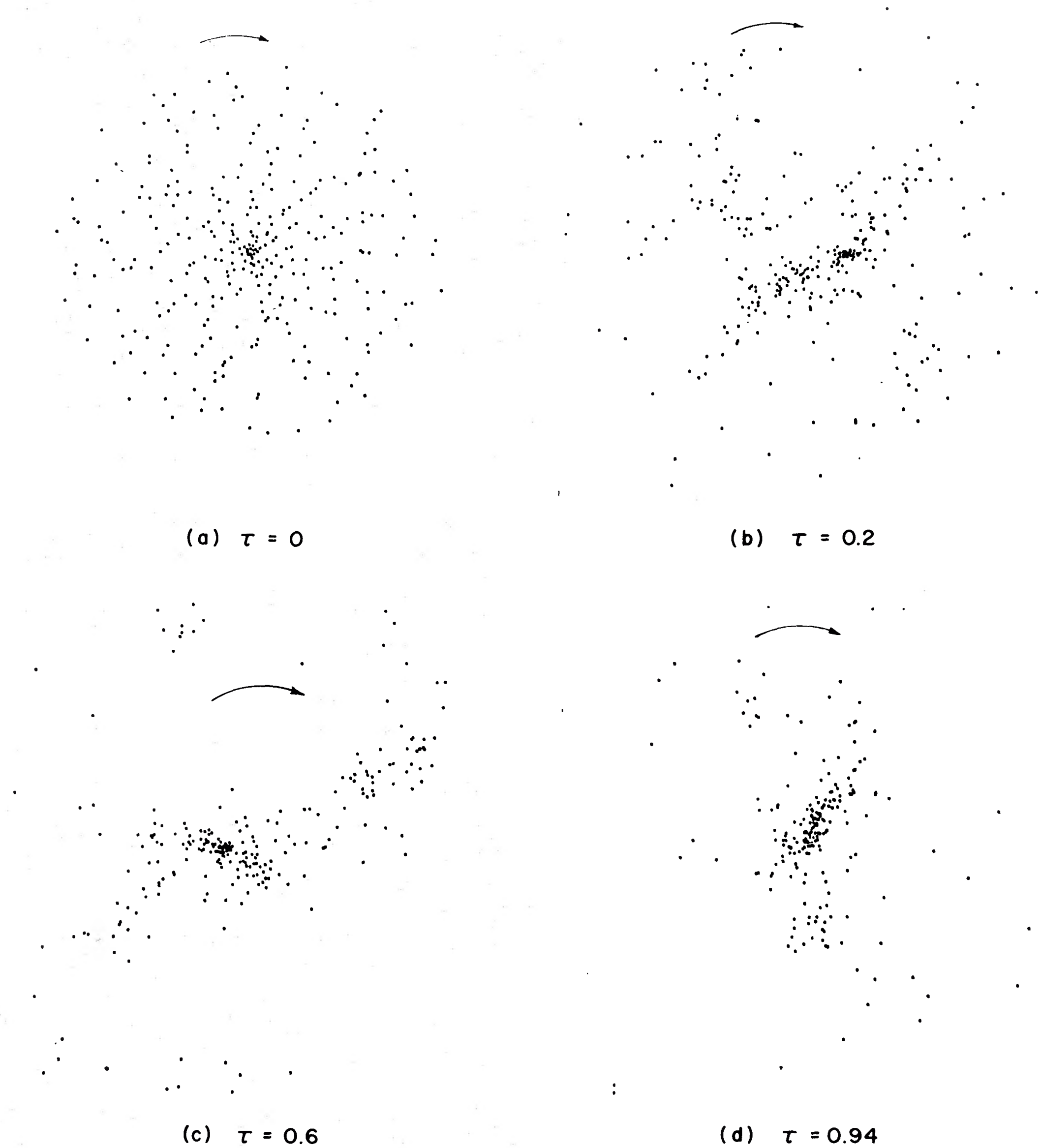

FIG. 4.—Evolution of model 1. The graphs show the positions of the mass points projected onto the plane, at four instants.

relaxation processes for individual particles, whether due to numerical error or to two-body collisions. We may expect, therefore, that a given numerical error may be less serious here than in some other applications of N-body models.

2. We have rounded over the potential at small distance, thereby making the mathematical problem nonsingular and removing the largest fluctuations in acceleration from the computational problem.

3. We judge from trial applications of the integration scheme, where the analytic solution is known, that the time step should be small enough for the integration to carry one particle past another one reliably (but not with great accuracy) whatever the impact parameter for expected particle relative velocities.

4. We find that the results are insensitive to the choice of Δt. We ran a second version of the standard model 1 with Δt doubled. The resulting pattern of distribution of points as a function of τ looks almost identical to the standard case, although some individual orbits do show marked differences. At $\tau = 0.5$ (measured in units of the initial orbit period for the outermost points) the system energy in the standard run has increased by 0.08 percent, and has decreased by 0.6 percent in the new model with Δt doubled. At this time $t = 0.1796$ in the standard case, 0.4 percent larger than this in the new model, the difference being of the same order as the energy error in the second model. The potential energies differ by 1.6 percent. There are comparable discrepancies in the random kinetic energy, the thickness of the disk, and the moment of inertia about the angular-momentum axis. At $\tau = 1$, the energy has increased by 0.22 percent in model 1, decreased by 2.8 percent in the new model, $t = 0.1373$ in the model 1, 4 percent smaller in the new model.

We conclude from this comparison that the interesting aspects of the model, the pattern of the distribution of points and the measure t, are quite insensitive to the choice of time step, hence that it is reasonable to suppose that computation errors are not seriously affecting the results. Other tests are described in the following section.

e) Parameter Studies

In this section we describe the results of varying divers features of the model, usually changing only one parameter at a time.

Model 2.—The introduced velocity dispersion (eqs. [7]–[10]) is reduced to half the value in model 1, with

$$\sigma = 0.227 .$$

The results for $t(\tau)$ are shown as the triangles in figure 2. The system stays somewhat more compact than model 1, the "bar" is less prominent, and one might imagine that the subclusters of points are denser and richer. The system is unstable to local condensations according to Toomre's criterion, so this latter behavior is not unexpected; however, the evolution of t parallels the standard case.

Model 3.—The cutoff c in equation (6) is reduced to $c = 0.025$, half the value in model 1. To preserve accuracy we also reduced the time step by a factor of 2. The results for t are shown as crosses in figure 2. This model ended up with the lowest value of t for any of the models described here. Perhaps related to this is the fact that the disk fissioned into two orbiting clusters of roughly equal size, an effect we observed occasionally in our preliminary models. One must bear in mind here that the calculated value of T_{mean} (defined by eq. [11]) is only a nominal estimate of the systematic kinetic energy, both because our estimate of what the systematic streaming velocity is at any position depends on a limited number of particles, and because we are assuming that the systematic motion always is circular about the center of the mass. Clearly the "true" streaming velocity is greater than our estimate if the system has broken into two disks and each rotates about its own center of mass as it revolves about its companion.

Model 4.—To see whether the low values of t in model 3 might be the result of an accidental fission, we ran a model identical to model 3 in all respects save that a different set of random numbers was used to generate the initial values. There is not clear evidence of fission in this model, and the t-values (solid circles in fig. 2) are larger than for model 3, but still somewhat below the other models.

Model 5.—In model 1 we forced a uniform initial distribution of particles by dividing the disk into cells in radius and angle, placing one point at random in each cell. In model 5, the cells in radius remain as before, but the initial longitude of each particle is a random fraction of 360°. This allows greater initial density irregularities. The results look very much like model 1.

Model 6.—Here the radial distances of the points are chosen so that $(r/R)^{3/2}$ is uniformly distributed from 0 to 1, giving surface density $\Sigma(r) \propto r^{-1/2}$. This is less centrally concentrated than model 1 ($\Sigma \propto r^{-1}$). The initial velocities averaged over intervals $\Delta r = 0.1$ are plotted as triangles in figure 1. The rotation curve for this system is rather closer to uniform rotation, but the general features of the instability, including the development of a rough "bar," persist; as before, the velocity distribution rapidly becomes more isotropic in the disk and t falls to the range 0.1–0.2.

Models 7, 8.—Perhaps the most serious concern about our approach is that we may be seeing effects peculiar to systems with limited numbers of particles. This certainly is true for some quantities, like the evolution of the local velocity dispersion, although the fact that the dispersion in z velocities remains small compared to the dispersion of velocities in the disk indicates that two-body relaxation is *not* a primary factor in producing the instability. To further test the assumption that the evolution of t is insensitive to N, we compared models with different numbers of particles. Models 7 and 8 have $N = 500$ and 150, respectively. The initial values are chosen according to the same prescription as for model 1. The sequence of "random" numbers is the same in each model; but of course, because N is different, a given "random" number is applied to different variables in the different models.

The three models differing (in this sense) only in N are compared in figure 5. The closely similar course of evolution, including the appearance of a rough bar, in the three models is strong encouragement for the view that the evolution of t we have found may not be a peculiarity of the number of particles chosen.

f) *Effect of the Halo*

Among the parameters we varied, the only one that markedly changes the course of evolution of t is the halo mass M_H (eq. [6]). The models illustrated in figure 6 all

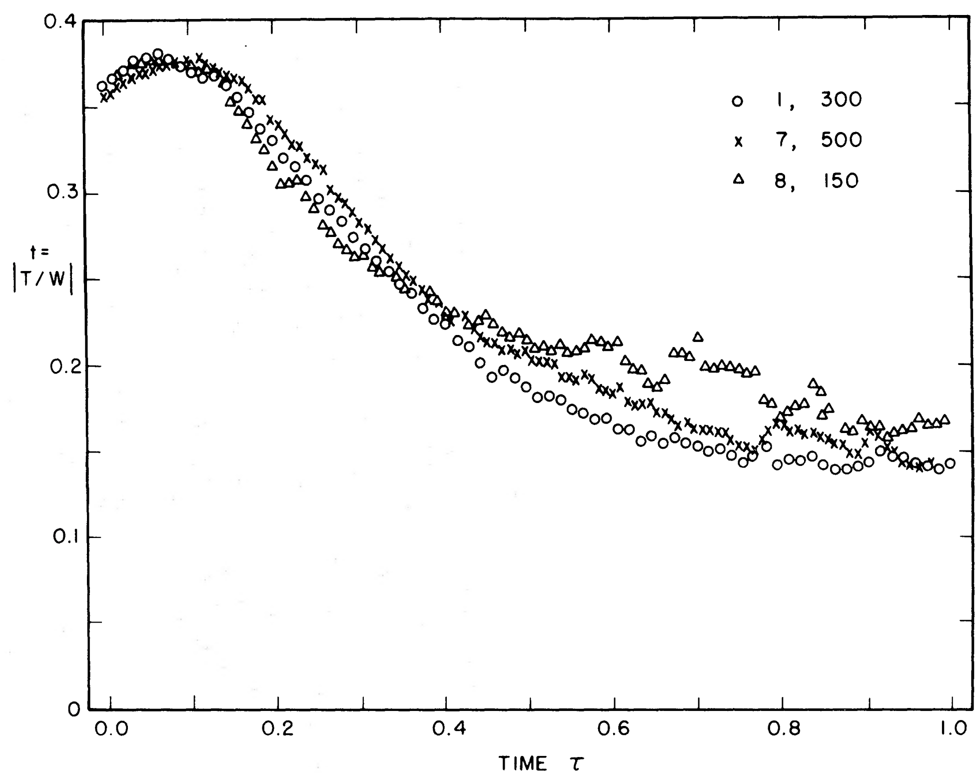

FIG. 5.—Effect of varying the particle number N on the evolution of the model galaxy

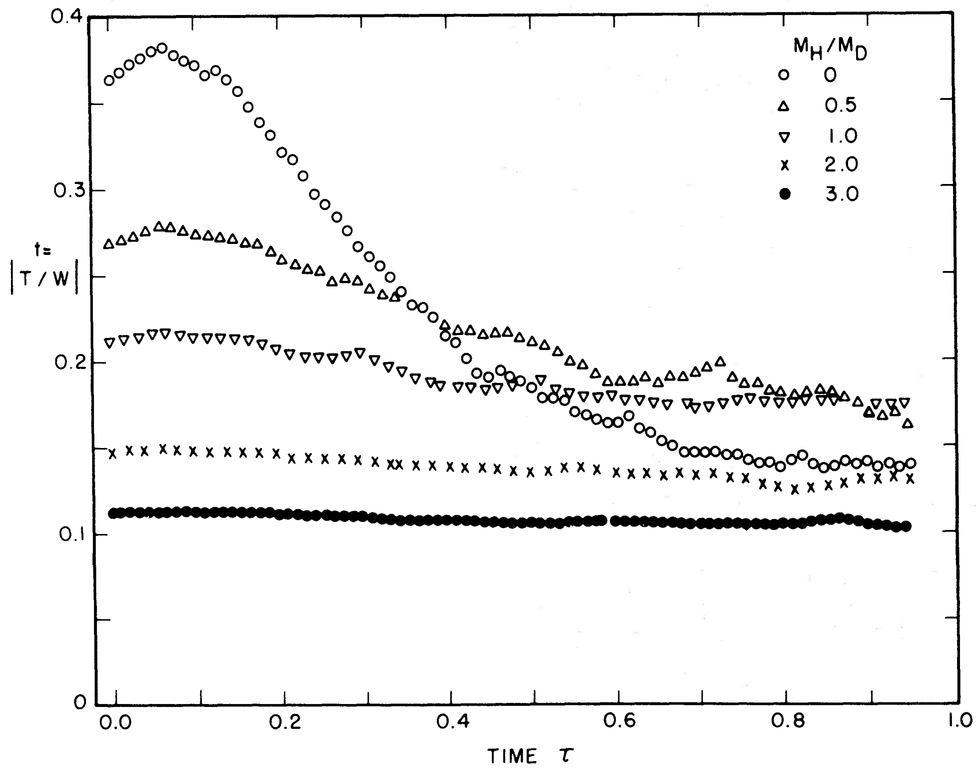

FIG. 6.—Effect of the halo on the evolution of the model galaxy

have the same parameters and commence from the same initial conditions save for the halo mass. In models 9 to 12 the ratios of halo mass to disk mass are 0.5, 1, 2, and 3, respectively. The introduction of the halo increases $|W|$, the total gravitational energy of the system (disk and halo and disk-halo cross-term) and hence lowers the initial value of t, since the halo is assumed to be nonrotating. If t is a direct measure of tendency to instability, and if the instability transforms systematic kinetic energy of rotation into random motions, then, in the models where t exceeds the threshold, the values of this parameter might be expected to asymptotically approach the threshold. Figure 6 shows that model 1 ends up below models 9 and 10. This could be interpreted in several ways, perhaps that in model 1 the instability develops more violently and hence approaches the threshold more rapidly, or perhaps that the instability overshoots, carrying t past the threshold. It is doubtful that we could learn much on this question by carrying the integration of models 9 and 10 further along in time, because the numerical error grows systematically larger. The best we can say is that, by $\tau = 1$, models 9 and 10 do not show clear evidence of the coherent bulk motion so apparent in the initial stages of development of the pure disk models.

In models 11 and 12, t is slowly decreasing even though model 12 starts out with t less than the final value (at $\tau = 1$) in model 1. It is not clear how one should interpret this, for the effect is small and the possibility of relaxation by conventional processes very real. In any case, it appears that models having t initially $\lesssim 0.15$ change slowly and are only weakly unstable if at all.

g) *Summary of Numerical Results*

The cold models without halo exhibit a violent instability that we cannot relate to any peculiarities of the model save the absence of a "hard" component in the potential.

These unstable models appear to approach a stationary state with significantly increased radial motions in the plane. After about one rotation period, the average value of t in the eight models is $t \sim 0.14$, the spread about this value exceeding $\delta t = \pm 0.03$ in only one case, model 3. The behavior of t as a function of time is very similar in the different models. This is a remarkable result in view of the very great differences in parameters among the models.

During the first rotation time period the system of particles goes from a symmetric disk to a highly nonaxisymmetric "barlike" structure, which tends to dissolve and approach rough axial symmetry again. After one orbital period t is roughly comparable to what was indicated as the critical value in analytic studies of fluid models. When a small halo is introduced, this sequence is reproduced in a less pronounced way. When the halo mass is larger, the disk develops random kinetic energy in a manner reminiscent of two-body relaxation processes but does not show a violent instability. For the chosen forms of density distribution in disk and halo components, a halo mass of 1 to $2\frac{1}{2}$ times the disk mass appears to be required to reduce the initial value of t to the stable range $t \simeq 0.14$.

III. DISCUSSION OF NUMERICAL RESULTS AND APPLICATIONS TO REAL SYSTEMS

a) Reality of the Instability

Is the result found in the last section—that cold, axisymmetric, flat galaxies are grossly and irreversibly unstable—true? On the basis of the various checks we have made and the parameter studies described, we believe that the instability is not an artifact of some special errors in these calculations. However, N-body studies are fraught with unexpected difficulties, and it is useful to check any result by independent means. Numerical studies of perfectly flat systems (H, MPQ; Miller 1971 [M]) have been published which, in terms of numerical accuracy and number of stars, are far superior to the present work. Examination of these studies indicates that the same instabilities are found, that bars develop, and that finally "a stable axisymmetric disk with a velocity dispersion much larger than that given by Toomre's criterion is generated" (H). The values of t in the final, apparently stable, systems were not published, but have been kindly calculated by the authors for use in the present paper; these values are given in table 1. They are in good agreement with the results for our model. It is still possible that the result is due to an oddity of N-body calculations, and that the coincidence between the critical value of t and that found in studies of fluid systems is, in fact, coincidental. Perhaps there is a critical value for N-body systems; but it may be nearer 0.2738, the limit for a zero-viscosity Maclaurin spheroid, than

TABLE 1

CRITICAL VALUE OF $t \equiv T_{\text{mean}}/|W|$

Study	t_{crit}
Maclaurin spheroid (fluid, exact)	$= 0.1376$
Generalized polytropes (OB) (fluid, approximate)	$\simeq 0.137 \pm 0.002$
n-body, flat (H) ($n = 10^5$, approximate)	$\simeq 0.141$
n-body, flat (M) ($n = 1.25 \times 10^5$, approximate)	$\simeq 0.130 - 0.135$
n-body, 3-D (present work)*	
(n = 150–500, approximate)	$\simeq 0.14 \pm 0.02$
n-body, flat (K) ($n \to \infty$, exact)	$= 0.125 < t_{\text{crit}} < 0.173$

* Average and standard deviation of t at $\tau = 1$ for the 12 models discussed in this paper.

0.14, the limit for secular stability of fluid systems (for a careful discussion of the equilibrium and stability of self-gravitating figures see Chandrasekhar 1969), and numerical errors and small values of N combine to give so much relaxation that the computed N-body systems simulate a high-viscosity fluid. Thus it is extremely important that there exists one exact study of the stability of a disk of stars to nonaxisymmetric modes, that of Kalnajs (1972, [K]). Kalnajs was able to construct stable composite models for sufficiently small values of t but not for larger values (the values of t were provided by Toomre 1972). Kalnajs's results are consistent with the previously described conclusions (cf. table 1); but, since he investigated only a very small sample of quite special models, it is possible that further exact studies will produce counterexamples.

In the absence of such counterexamples, it appears that $t \simeq 0.14$ *represents approximately the maximum rotational energy an axisymmetric stellar system can contain and remain stable to the formation of a bar.*

b) *Ways to Construct Stable Systems*

The first way to construct a stable system is that followed in the course of evolution of many of the N-body models—the disk heats up until t is approximately 0.14. This state, a hot flat disk, appears to be a quite satisfactory stable equilibrium, but it does not correspond to the observed stellar motions in the apparently flat spiral galaxies. Second, the system might evolve into an equilibrium rotating bar which is cold, in the sense that streaming motions dominate over random motions, but which is stable against further deformations; existing calculations shed very little light on this point. In any case, this is not a satisfactory model for an ordinary—not barred—spiral galaxy.

Finally, one can add another hot component and thus stabilize the total system. Adding a hot disk component reduces to alternative (1) and would require an unseen disk component with large mass and largely radial orbits. Adding an extended component corresponding to the "halo" described in § II apparently will stabilize the system if the halo mass is equal to or somewhat greater than the disk mass. A similar conclusion was reached by Kalnajs (1972) from an independent consideration of possible stabilizing influences.

Of these three alternatives, the last—the massive halo—seems the most likely solution for our own Galaxy. Though we have not exhausted the possibilities of constructing ingenious models having hot components interior to the Sun but most of the total mass in a flat cold component (a variant of alternative [1]), we have not found a way to produce a stable model by this means that does not do violence to the observed rotation curve. Further work assessing this alternative would be quite useful.

c) *Astronomical Plausibility of a Massive Halo*

This is a lengthy question, and we only outline a few essential points here. The direct evidence from star counts is inconclusive. According to Oort (1965): "Some 5 percent of the total mass of the galaxy may be estimated to consist of these (K, M) dwarfs. There is no way for estimating how much more mass there may be in the form of intrinsically still fainter stars. The real mass of the halo remains entirely unknown. It is quite possible that there might be enough halo stars to make the halo an important contributor, or even *the* most important contributor to the mass of the Galactic System. The uncertainty concerning the relative contributions of the halo and disc to the total mass is the greatest obstacle in the way of constructing a model of the mass distribution in the Galaxy" (Oort's italics).

There is some information available from dynamical studies, and several authors (see, for example, Belton and Brandt 1963; Vandervoort 1970) have concluded that the halo mass must be large; this finding was based on a comparison of the force laws perpendicular to the plane and perpendicular to the rotation axis. Although the standard models for the mass distribution in the Galaxy place most of the mass in the disk (e.g., Schmidt 1965), one of us (J. P. O.) has constructed simple dynamical models of the Galaxy from mass distributions which have halo-to-disk mass ratios of $\gtrsim 2/1$ but which satisfy all the usual dynamical constraints.

Finally, it is interesting to consider the potential applicability of these results to other galaxies. Presumably even Sc and other relatively "pure" spirals must have some means of remaining stable, and the possibility exists that these systems also have very large, low-luminosity halos. The picture developed here agrees very well with the fact, noted by several authors (see, for example, Brandt, Kalinowski, and Roosen 1972; Rogstad and Shostak 1972), that the mass-to-light ratio increases rapidly with distance from the center in these systems; the increase may be due to the growing dominance of the high mass-to-light ratio halo ($\sim 10^2$) over the low mass-to-light ratio disk ($\sim 10^1$). It also suggests that the total mass of such systems has been severely underestimated. In particular, the finding of Roberts and Rots (1973) that the rotation curves of several nearby spirals become flat at large distances from the nucleus may indicate the presence of very extended halos having masses that diverge rapidly $[M(r) \propto r]$ with distance. The inferred mass-radius law is that within the outer parts of an isothermal sphere.

The plausibility of these and other speculative implications of the present work can be tested by at least two routes. Further dynamical studies can be made to ascertain the generality of the bar-making instability in a stellar system; and direct observational searches can be pursued to see if numerous very faint high-velocity stars exist in the solar neighborhood.

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Note added in proof.—The authors wish to reemphasize a point noted in the paper but stressed by a referee. The criterion $t \lesssim 0.14$ is necessary but not sufficient for stability; systems with $t < 0.14$ may be constructed which are unstable to various models.