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THE RELATIVISTIC ROCHE PROBLEM. I. EQUILIBRIUM THEORY FOR A BODY IN EQUATORIAL, CIRCULAR ORBIT AROUND A KERR BLACK HOLE*

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ABSTRACT

Bodies in orbit around Kerr black holes will experience tidal gravitational forces. A study of the equation of geodesic deviation, referred to reference frames fixed in such bodies, reveals the nature of these forces. We here perform this analysis for all circular, equatorial, geodesic orbits of the Kerr metric whose mass parameter M and angular-momentum parameter Ma satisfy $a \leq M$ (geometrical units). Free-particle motion, which the deviation equation describes, is simple harmonic in the direction perpendicular to the equatorial plane, and we demonstrate the relation between the angular frequency of this simple harmonic motion and the circular orbit frequency.

The equation for free-particle motion leads to a generalized Euler equation that describes fluid flow in the orbiting reference frames. We employ the latter to study the conditions for equilibrium of infinitesimal, incompressible, homogeneous, self-gravitating fluid bodies in circular, equatorial, geodesic motion. Because the potential in the equation depends only quadratically on the local coordinates, ellipsoidal figures of equilibrium are possible. For each possible orbit determined by the Kerr angular momentum Ma and the orbit radius r, a family of ellipsoids parametrized by the fluid density ρ is in equilibrium. A minimum possible density characterizes each family; this is the Roche-limiting density for the orbit.

For fluid bodies in stable circular orbits, the Roche limit is qualitatively like that in Newtonian situations; at the last stable, circular, equatorial orbit of each Kerr metric, where r is a specified function of M and of a, equilibrium bodies must satisfy $[M/(\pi\rho r^3)] \leq 0.0664$. For fluid bodies in the highly energetic, unstable circular orbits near the photon orbits, the fluid density required for the existence of a body in equilibrium is magnified by the square of its energy-at-infinity per unit mass.

Subject headings: binaries — black holes — gravitation — hydrodynamics — relativity

I. INTRODUCTION AND SUMMARY

Since black holes are nonluminous, they can only be detected by their interaction with luminous matter. They may appear as X-ray sources in binary star systems (Tananbaum *et al.* 1972), which are amenable to detailed analysis, and in galactic centers (Lynden-Bell 1969, 1971; Lynden-Bell and Rees 1971), sources of prodigious amounts of energy. Peebles (1972*a*) has recently reviewed the situation.

As a guide to such searches, one needs an understanding of the way in which a black hole would interact with nearby matter and fields. Thus, Peebles (1972*b*) has examined the distribution of stars which might exist outside of a large black hole and has studied star clusters for evidence of such configurations. Test particles in orbit near a black hole radiate gravitationally. The most novel model involves a synchrotron mechanism; particles in astrophysically implausible, unstable, high-energy orbits radiate predominantly in the plane of their orbit (Misner *et al.* 1972; see also Bardeen, Press, and Teukolsky 1972; Breuer *et al.* 1973; Breuer and Vishveshwara 1973; Chitre

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and Price 1972; Chrzanowski and Misner 1973; and Davis *et al.* 1972). Certainly the most tantalizing interaction is the energy extraction process of Penrose (1969; see also Christodoulou 1970; and Christodoulou and Ruffini 1971); here particle decays in the ergosphere of a Kerr (1963) black hole can be arranged to extract energy from the rotation of the black hole. Recent calculations, however, suggest that this process cannot be efficiently employed in realistic breakups of astronomical bodies (Bardeen *et al.* 1972).

In each of these situations, the coupling with the black hole is via a pointlike particle. But the effects of finite size are certainly important for some mechanisms which exploit the exotic environment near the horizon of black holes. The most obvious effect is the tidal deformation of an extended, orbiting body. Though the complete two-body problem is intractable, the influence of a large black hole on a nearby, small fluid body is calculable.

The classical study of such effects was made by Roche (1847–1850), who did the analysis for a fluid body in Newtonian orbit with a rigid sphere. Chandrasekhar (1969 and the references therein) conducted the definitive modern treatment of this situation.

These analyses, and the one given here, all deal with the equilibrium shape of an infinitesimal, homogeneous, incompressible fluid body subject to its own gravity and to the tidal gravitational field of another body. Nduka (1971) recently performed the dynamical Newtonian analysis for the situation of a small satellite which orbits so close to a larger body that no equilibrium solution exists for the shape of the small satellite. He found that catastrophic disruption of the small satellite occurs, with obvious implications for the theory of the origin of the rings of Saturn. Mashhoon (1972) has performed a similar analysis using the gravitational field characterized by the Kerr metric.

We will here calculate the equilibrium configurations appropriate for a body in orbit around a Kerr black hole. As in the Newtonian analysis, a minimum distance of approach exists for an equilibrium body of given density—the Roche limit.

Thus imagine an observer in equatorial, circular, geodesic orbit in a space described by the Kerr metric:

$$\begin{aligned} ds^2 &= -[\omega^{(t)}]^2 + [\omega^{(r)}]^2 + [\omega^{(\theta)}]^2 + [\omega^{(\varphi)}]^2 \\ &\equiv -\left[\frac{\Delta^{1/2}\Sigma^{1/2}}{B^{1/2}}\,dt\right]^2 + \left[\frac{\Sigma^{1/2}}{\Delta^{1/2}}\,dr\right]^2 + [\Sigma^{1/2}d\theta]^2 \\ &\quad + \left[\frac{B^{1/2}\sin\theta}{\Sigma^{1/2}}\left(d\Phi - \frac{2aMr}{B}\,dt\right)\right]^2 . \end{aligned} \tag{I-1}$$

Here $\Delta \equiv r^2 - 2Mr + a^2$, $\Sigma \equiv r^2 + a^2\cos^2\theta$, and $B \equiv (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta$; the parameters M and Ma give the mass and angular momentum of the black hole, and, until § V, we employ the units $G = c = 1$. The coordinates are those of Boyer and Lindquist (1967), and may be considered spherical polar in regions far from the black hole. We describe further the Kerr metric and its geodesics in § II.

The orbiting observer can describe the motion of particles in his vicinity in terms of a local coordinate system x^i, proper time τ, and basis vectors $\boldsymbol{e}_i$; the vectors $\boldsymbol{e}_i$ appear in figure 1. Free particles move along geodesics, so their motion with respect to the observer's geodesic is given by the equation of geodesic deviation (see Misner 1969). In terms of the observer's coordinates, this equation is (with all x^i dependence explicit)

$$\frac{d^2x^i}{d\tau^2} = -\frac{\partial}{\partial x^i}\left\{\frac{M}{2r^3}\left[-\left(\frac{3\Delta}{P}\right)(x^1)^2 + \left(\frac{3\Delta - 2P}{P}\right)(x^3)^2\right]\right\} \pm 2\epsilon_{ij3}\left(\frac{M}{r^3}\right)^{1/2}\frac{dx^j}{d\tau}\,, \tag{I-2}$$

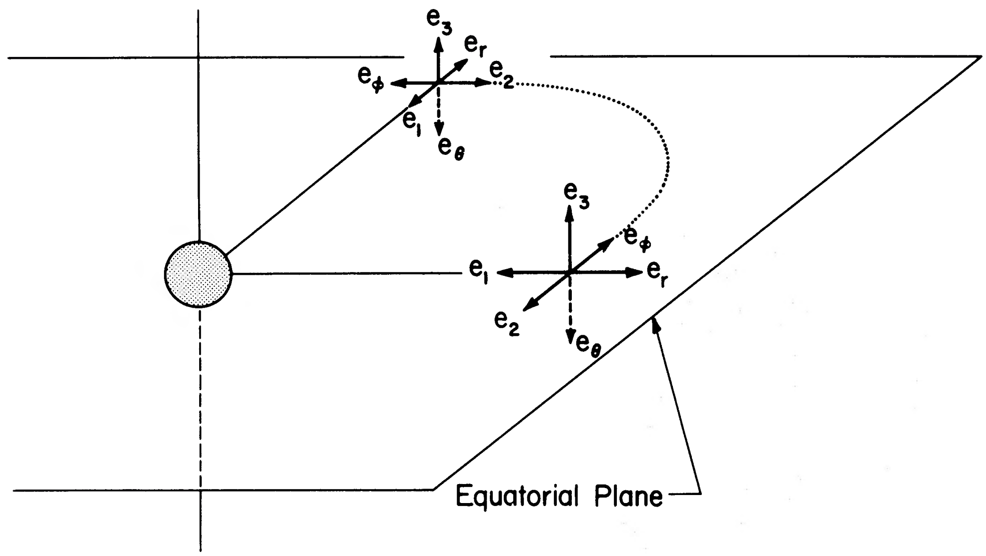

FIG. 1.—The two sets of unit vectors used by an observer in equatorial, circular, geodesic orbit around a Kerr black hole. e_r and e_θ are unit vectors associated with the locally nonrotating frame, while e_ϕ is the unit vector tangent to the direction of the orbital motion. The vectors, e_1, e_2, and e_3 correspond to those used in the Newtonian analysis of Chandrasekhar (1969). With respect to an observer at infinity, the two sets of unit vectors orbit around the black hole at the coordinate angular velocity associated with equatorial, circular geodesics: $\Omega_{\text{coord}} = \pm(Mr)^{1/2}/[r^2 \pm a(Mr)^{1/2}]$.

where ϵ_{ijk} is the totally antisymmetric matrix and the polynomial $P = P(r, a) = r^2 - 3Mr \pm 2a(Mr)^{1/2}$ vanishes at the photon orbit, the innermost circular geodesic. Also, the sign ambiguity refers to prograde or retrograde motion of the observer with respect to the rotation of the black hole. As it must, the equation (I-2) reduces explicitly to the analogous Newtonian equation in the limit of large r. Moreover, the free particle motion it describes is governed by tidal, centrifugal, and Coriolis effects. A derivation of the equation appears in § III.

In a straightforward way, equation (I-2) leads to an equation for the flow of a self-gravitating fluid. This derivation and the limits on the validity of the result are the subject of § IV. Note here that the resulting fluid bodies must be small in mass and in linear extent if the geodesic assumption for their motion is to be valid.

The equilibrium problem of Roche is then the determination of the characteristics of incompressible, homogeneous fluid bodies which are in hydrostatic equilibrium in the coordinates of the orbiting observer. With p the pressure of the fluid, ρ its constant density, and $\Phi_G(x^i)$ its self-gravitational potential, such bodies are in equilibrium if the quantity

$$\frac{p}{\rho} = \left\{\frac{M}{2r^3}\left[-\left(\frac{3\Delta}{P}\right)(x^1)^2 + \left(\frac{3\Delta - 2P}{P}\right)(x^3)^2\right] + \Phi_G(x^i)\right\} \tag{I-3}$$

is constant on their surfaces. Because $\Phi_G(x^i)$ is quadratic in the coordinates x^i (see Chandrasekhar 1969), ellipsoidal figures of equilibrium are possible.

For a given equatorial, circular orbit, i.e., for given r and a, the possible ellipsoids form a one-parameter family of different shapes, *of which one has a minimum density*. The most important orbits are the last stable orbits and the high-energy orbits near the photon orbit; figures 2 and 3 respectively give the minimum densities that equilibrium bodies require. Only in the latter case are the results qualitatively different from

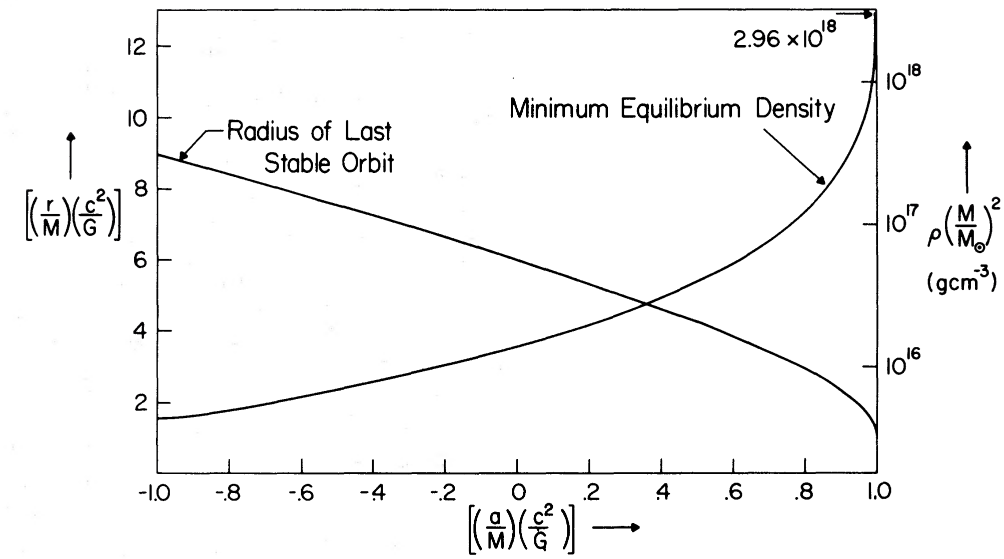

FIG. 2.—Roche limits for infinitesimal, homogeneous, fluid bodies of density ρ in the last stable orbits of the Kerr geometries. Here, $(M/M_{\odot})$ is the black-hole mass in solar units and Mac is the angular momentum of the black hole. For each $a > 0$, two orbits are possible: one corotating with the black hole and one counterrotating. Counterrotating orbits are labeled by $a < 0$, though they occur in the Kerr geometry parametrized by $|a|$. In each orbit, a sequence of ellipsoids with varying density may be in equilibrium. The Roche-limiting configuration is the one with minimum density.

the results of the analysis for Newtonian orbits. Further results of this type appear in § V; some have already been published (Fishbone 1972*a*).

In a subsequent paper, we will investigate the stability of the relativistic Roche ellipsoids with respect to small perturbations, as has already been done in the Newtonian limit (Chandrasekhar 1969).

Finally, let us give some additional conventions which we subsequently employ.

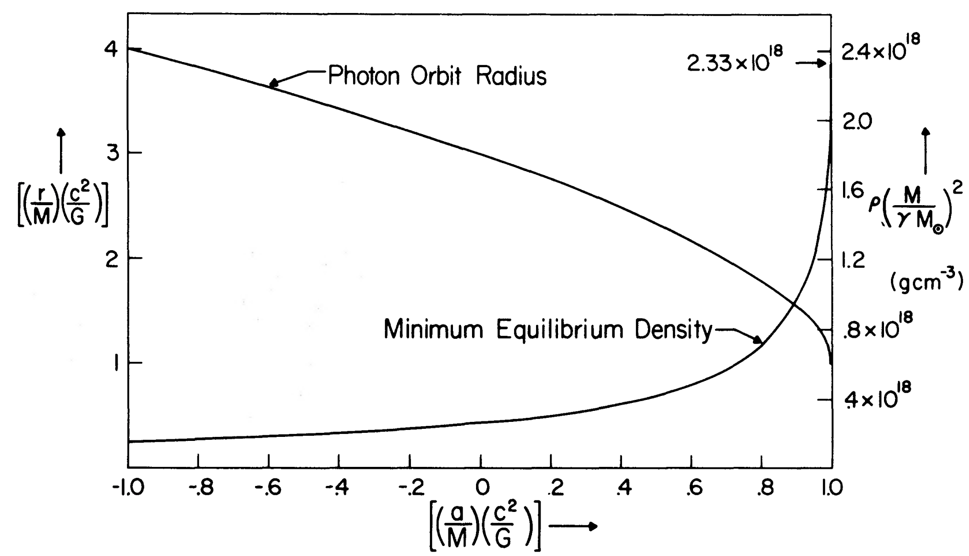

FIG. 3.—Roche limits for infinitesimal, homogeneous, fluid bodies of density ρ near the circular photon orbits of the Kerr geometries. Here γ is the energy-at-infinity per unit mass characteristic of the orbit; all other notation is explained in the caption of fig. 2.

Repeated indices imply a sum, with Latin letters ranging over spatial indices (which may mean r, θ, and Φ or one to three) and Greek indices over spacetime indices. The symbols ω^r and $\omega^{(r)}$ are radial one-forms in different reference frames. We also employ the term "Roche limit" either to denote the minimum density of an equilibrium configuration consistent with a given orbit or simply to characterize that configuration. The meaning will emerge from the context of any given discussion.

II. THE KERR METRIC, ITS GEODESICS, AND FRAMES

In Boyer-Lindquist (1967) coordinates, the Kerr (1963) metric appears (Bardeen 1970*a*) as equation (I-1). The metric is stationary and invariant with respect to translations in the Φ- and t-directions. For the case $a = 0$, the metric describes the static, spherically symmetric Schwarzschild geometry; while if $M = 0$, it describes flat space in peculiar coordinates (Boyer and Lindquist 1967). The constants M and a are parameters subject to $0 \leq a^2 \leq M^2$ in order that causality not be violated (Carter 1968). The former is the mass of the black hole while Ma is its angular momentum (Boyer and Price 1965; Cohen 1968). With no loss of generality, we will take $a \geq 0$; the black hole then rotates in the positive Φ-direction.

As one approaches a rotating black hole from afar, he will be dragged around by its rotation unless he accelerates otherwise. At the so-called static limit, no amount of acceleration can prevent a timelike (physical) observer from being dragged around. Alternatively, inside the static limit, the world lines of constant r, θ, and Φ are not those of any physical particles. This limit occurs at (Vishveshwara 1968) radius $r_e = M + (M^2 - a^2 \cos^2 \theta)^{1/2}$. Between the static limit and the horizon, located at radius

$$r_h = M + (M^2 - a^2)^{1/2}\,, \tag{II-1}$$

is the ergosphere.

Since the world lines of constant r, θ, and Φ are not always those of physical observers and since there is indeed no physical reason for this choice of world lines, a different choice of observers (i.e., frames of reference) is desirable. Bardeen (1970*c*) and Bardeen *et al.* (1972) have shown that the "locally nonrotating frames" satisfy the requirements of physical identifiability and global utility. Their world lines are orthogonal to the constant-t hypersurfaces. They are the frames implicit in the orthonormal description (I-1) of the metric.

Whereas the locally nonrotating frames are accelerated reference frames, the frames of geodesic observers are not. The latter form the other set selected by physical reasons, and it is in these that we wish to understand local physics. Thus we must now review some aspects of geodesic motion.

The Kerr metric leads to four geodesic equations for the motion of test particles (Carter 1968; see also de Felice 1968 and Wilkins 1972). These equations show that the energy-at-infinity per unit rest mass, γ, and the angular momentum per unit rest mass about the symmetry axis, l, are conserved quantities. For orbits in the equatorial plane, $\theta = \pi/2$, the effective potential equation

$$\Sigma^2(r_{,\xi})^2 = [\gamma(r^2 + a^2) - la]^2 - \Delta[r^2 + (l - a\gamma)^2] \tag{II-2}$$

governs the motion of test particles; ξ is an affine path parameter and $r_{,\xi} \equiv dr/d\xi$. For the case of circular orbits in the equatorial plane, one obtains (Bardeen *et al.* 1972)

$$\gamma = \gamma_{\pm} = \frac{r^2 - 2Mr \pm a(Mr)^{1/2}}{r[r^2 - 3Mr \pm 2a(Mr)^{1/2}]^{1/2}}\,. \tag{II-3}$$

Here and subsequently, the upper sign refers to prograde orbits corotating with the black hole, while the lower sign refers to retrograde orbits. (The signs must be chosen consistently.)

The parameter γ becomes infinite when the denominator in equation (II-3) vanishes. This defines the photon orbits, the prograde and retrograde circular orbits respectively closest to the black hole. Thus, at a photon orbit (Bardeen *et al.* 1972),

$$P(r, a) \equiv r^2 - 3Mr \pm 2a(Mr)^{1/2} = 0\,, \tag{II-4}$$

an equation with explicit solutions $r_\gamma = r_{\gamma\pm}(M, a)$. The last radially stable orbit in the equatorial plane occurs when (Bardeen *et al.* 1972)

$$S(r, a) \equiv r^2 - 6Mr \pm 8a(Mr)^{1/2} - 3a^2 = 0\,, \tag{II-5}$$

an equation with explicit solutions $r_s = r_{s\pm}(M, a)$. Note too that the last stable orbit is also the circular orbit of minimum energy, as can be verified by solving the equation $\partial\gamma(r)/\partial r = 0$. The high-energy orbits between the last stable orbit and the photon orbit seem astrophysically implausible, but can in principle be achieved (Boyer and Lindquist 1967).

If one examines prograde or retrograde orbits only, then $r_s > r_\gamma > r_h$ always holds. The three prograde radii all approach the value M as the parameter a approaches M; a more careful consideration of this limit (Bardeen *et al.* 1972) shows that for $a = M(1 - \epsilon)$ with $\epsilon \ll 1$,

$$r_h \simeq M[1 + (2\epsilon)^{1/2}]\,, \tag{II-6a}$$

$$r_{\gamma+} \simeq M(1 + 2(2\epsilon/3)^{1/2} + \tfrac{2}{3}(2\epsilon/3)]\,, \tag{II-6b}$$

and

$$r_{s+} \simeq M[1 + (4\epsilon)^{1/3}]\,. \tag{II-6c}$$

Another necessary result is the velocity of a circular orbit as a function of its radius. The appropriate velocity is that of a particle in geodesic orbit with respect to the locally nonrotating frame. Thus, from the metric (I-1), for any equatorial, circular orbit, an observer in that frame says

$$\begin{aligned} v &\equiv \frac{d(\text{proper azimuthal distance moved by the particle})}{d(\text{proper time for that movement})} \\ &= \frac{(B \sin^2\theta/\Sigma)^{1/2}[d\Phi - (2aMr/B)dt]}{(\Delta\Sigma/B)^{1/2}dt}\,, \end{aligned} \tag{II-7}$$

which, with the geodesic equations, yields (Bardeen *et al.* 1972)

$$v = \frac{\pm(Mr)^{1/2}[r^2 \mp 2a(Mr)^{1/2} + a^2]}{\Delta^{1/2}[r^2 \pm a(Mr)^{1/2}]}\,. \tag{II-8}$$

A useful relation which follows from this is

$$\begin{aligned} (1 - v^2)^{-1/2} &= \frac{\Delta^{1/2}[r(Mr)^{1/2} \pm Ma]}{M^{1/2}[r^3 + ra^2 + 2Ma^2]^{1/2}[r^2 - 3Mr \pm 2a(Mr)^{1/2}]^{1/2}} \\ &= \gamma\left(\frac{r\Delta^{1/2}[r(Mr)^{1/2} \pm Ma]}{M^{1/2}[r^3 + ra^2 + 2Ma^2]^{1/2}[r^2 - 2Mr \pm a(Mr)^{1/2}]}\right)\,, \end{aligned} \tag{II-9}$$

where the proportionality factor between $(1 - v^2)^{-1/2}$ and γ is finite for all possible circular geodesics. Note that the symbol γ represents $(1 - v^2)^{1/2}$ in the paper of Bardeen *et al.* (1972).

The coordinate angular velocity also follows from the geodesic equations (Bardeen *et al.* 1972):

$$\Omega_{\text{coord}} \equiv \frac{d\Phi}{dt} = \frac{\pm(Mr)^{1/2}}{r^2 \pm a(Mr)^{1/2}} . \tag{II-10}$$

Finally, to an observer in the orbiting, geodesic frame, the proper angular velocity is

$$\Omega_{\text{prop}} \equiv \frac{d\Phi}{d\tau} = \pm\left[\frac{M}{r^3 - 3Mr^2 \pm 2ra(Mr)^{1/2}}\right]^{1/2} , \tag{II-11}$$

where τ is his proper time.

III. EQUATION OF MOTION FOR AN ORBITING OBSERVER

a) Equation of Geodesic Deviation

In order to study the structure of a fluid body in geodesic orbit, we seek the generalization of Newton's second law in a frame of reference fixed in the body. The equation of geodesic deviation (see Misner 1969) is this generalization:

$$\nabla_u{}^2 \boldsymbol{n} + R(\boldsymbol{n}, \boldsymbol{u})\boldsymbol{u} = 0 . \tag{III-1}$$

Here $\boldsymbol{u}$ is the vector tangent to the geodesic orbit, $\boldsymbol{n}$ is a vector from the orbit path to nearby geodesics, ∇_u is the covariant derivative along the orbit, and $R(\boldsymbol{n}, \boldsymbol{u})$ is the curvature tensor. Equation (III-1) is independent of coordinate system.

Consider the term $\nabla_u{}^2\boldsymbol{n}$ in equation (III-1). The deviation vector is $\boldsymbol{n} = n^\alpha \boldsymbol{e}_\alpha$, with $\boldsymbol{e}_\alpha$ a set of basis vectors for the orbiting observer and n^α components. Choose

$$\boldsymbol{u} = \boldsymbol{e}_0 \equiv \left.\frac{d}{d\tau}\right|_{\text{geodesic}} , \tag{III-2}$$

where τ is proper time along the geodesic. The first term in the deviation equation is then $d^2(n^\alpha \boldsymbol{e}_\alpha)/d\tau^2$, which, when expanded, has terms corresponding to derivatives of the components n^α and of the basis vectors $\boldsymbol{e}_\alpha$.

Let us now specify the coordinate system of a circularly orbiting observer. Since $\boldsymbol{e}_0 = \boldsymbol{u}$, $\nabla_u \boldsymbol{e}_0 = \nabla_u \boldsymbol{u} = 0$ by definition of a geodesic. We demand that the basis vectors $\boldsymbol{e}_\alpha$ be orthonormal and retain an unchanging orientation with respect to the radial direction from the center of the black hole (see fig. 1). Thus

$$0 = \frac{d}{d\tau}(\boldsymbol{e}_0 \cdot \boldsymbol{e}_i) = \boldsymbol{e}_0 \cdot \frac{d\boldsymbol{e}_i}{d\tau} + \frac{d\boldsymbol{e}_0}{d\tau} \cdot \boldsymbol{e}_i = \boldsymbol{e}_0 \cdot \frac{d\boldsymbol{e}_i}{d\tau} ,$$

so the spatial basis vectors $\boldsymbol{e}_i$ change according to $d\boldsymbol{e}_i/d\tau = a_i{}^l \boldsymbol{e}_l$, with $a_i{}^l$ some 3×3 transformation matrix. This matrix $a_i{}^l = a_{il}$ is antisymmetric as a consequence of $\boldsymbol{e}_k \cdot \boldsymbol{e}_l = \delta_{kl}$:

$$0 = \frac{d}{d\tau}(\boldsymbol{e}_i \cdot \boldsymbol{e}_j) = \boldsymbol{e}_i \cdot \frac{d\boldsymbol{e}_j}{d\tau} + \frac{d\boldsymbol{e}_i}{d\tau} \cdot \boldsymbol{e}_j$$

$$= a_j{}^i + a_i{}^j = a_{ji} + a_{ij} .$$

The basis vectors thus undergo a pure spatial rotation (see Goldstein 1950); we write $a_{ik} = \epsilon_{ijk}\Omega_{cj}$ and

$$\frac{d\boldsymbol{e}_i}{d\tau} = \epsilon_{ijk}\Omega_{cj}\boldsymbol{e}_k , \tag{III-3}$$

where ϵ_{ijk} is the totally antisymmetric matrix and Ω_{cj} is the yet to be determined, constant, angular velocity vector of magnitude Ω_c. The subscript c means "centrifugal." We finally compute:

$$\nabla_u{}^2 \boldsymbol{n} = \nabla_u\left[\frac{dn^\beta}{d\tau}\, \boldsymbol{e}_\beta + n^i \epsilon_{ijk}\Omega_{cj}\boldsymbol{e}_k\right]$$

$$= \left[\frac{d^2n^i}{d\tau^2} + 2\,\frac{dn^j}{d\tau}\,\epsilon_{jki}\Omega_{ck} + n^m\Omega_{cj}\Omega_{cl}\epsilon_{mjk}\epsilon_{kli}\right]\boldsymbol{e}_i + \frac{d^2n^0}{d\tau^2}\,\boldsymbol{e}_0\,. \tag{III-4}$$

The second and third terms on the right side of equation (III-4) correspond to Coriolis and centrifugal forces, respectively.

b) Lorentz Transformation to the Orbiting Frame

We must specify the basis of the orbiting frame. From equation (I-1), an orthonormal basis of one-forms for the locally nonrotating frame in the Kerr geometry is the set

$$\omega^{(t)} = \frac{\Delta^{1/2}\Sigma^{1/2}}{B^{1/2}}\, dt\,; \qquad \omega^{(r)} = \frac{\Sigma^{1/2}}{\Delta^{1/2}}\, dr\,;$$

$$\omega^{(\theta)} = \Sigma^{1/2} d\theta\,; \qquad \omega^{(\varphi)} = \frac{B^{1/2}}{\Sigma^{1/2}} \sin\theta\left(d\Phi - \frac{2aMr}{B}\, dt\right)\cdot \tag{III-5}$$

The basis for an observer in circular orbit is obtained from the basis forms (III-5) by a Lorentz transformation $\Lambda_{(\beta)}{}^\alpha$. Then the ω^α basis, also orthonormal, follows from $\omega^\alpha = \Lambda_{(\beta)}{}^\alpha\omega^{(\beta)}$ and is

$$\omega^\tau = (1 - v^2)^{-1/2}\omega^{(t)} - v(1 - v^2)^{-1/2}\omega^{(\varphi)}\,;$$

$$\omega^r = \omega^{(r)}\,; \qquad \omega^\theta = \omega^{(\theta)}\,;$$

$$\omega^\phi = -v(1 - v^2)^{-1/2}\omega^{(t)} + (1 - v^2)^{-1/2}\omega^{(\varphi)}\,. \tag{III-6}$$

The ordinary velocity v is that of the orbiter, circularly moving at constant θ, as measured by an observer fixed in the locally nonrotating frame. For the case of interest here, the orbit is an equatorial geodesic, so v is given by equation (II-8).

Note too that we have given meaning to this Lorentz transformation only at $\theta = \pi/2$. If the transformation is to bear physical meaning off the equator also, then the velocity v would have to assume appropriate values there that would match smoothly onto the equatorial values. Here, however, we are interested only in the equatorial values of tensor quantities and hence need not worry about nonequatorial values for the transformation velocity.[1]

c) Derivation of the Angular Velocity

In the frame of basis vectors $\boldsymbol{e}_\alpha$ (dual to the basis forms ω^α), we will evaluate equation (III-4). These basis vectors are those associated with the reference frame in the orbiting body.

The relation between the orbiting observer's basis forms ω^α and connection forms $\omega^\alpha{}_\beta$ contains the information needed to specify Ω_{cj}. This follows because the basis

[1] Questions by Alfred Schild at a University of Texas Seminar caused the author to think more carefully about this point.

vectors $\boldsymbol{e}_\beta$ are related by (see Misner 1963, 1969; also Flanders 1963)

$$\boldsymbol{\nabla}_\alpha \boldsymbol{e}_\mu = \boldsymbol{e}_\nu \Gamma^\nu{}_{\mu\alpha}\,, \tag{III-7}$$

with $\Gamma^\nu{}_{\mu\alpha}$ the connection coefficients referred to the orthonormal frame. The connection forms in turn follow from exterior differentiation of the basis forms by

$$d\omega^\nu = -\omega^\nu{}_\mu \wedge \omega^\mu = -\Gamma^\nu{}_{\mu\alpha}\omega^\alpha \wedge \omega^\mu\,; \tag{III-8}$$

the wedge $\wedge$ is the symbol for an antisymmetrized product. Thus the manner of change of the basis vectors will be apparent after we perform this last calculation in the frame of the orbiting body. Alternatively, one can obtain equations (III-8) in the locally nonrotating frame and then transform the results to the frame of the orbiting body; we will proceed in this latter fashion.

The basis forms ω^α and basis vectors $\boldsymbol{e}_\beta$ transform via a Lorentz transformation. For the connection forms $\omega^{(\alpha)}{}_{(\beta)}$, equation (III-8) and the inverse Lorentz transformation ($\Lambda_{(\mu)}{}^\nu \Lambda_\nu{}^{(\alpha)} = \delta_\mu{}^\alpha$) give

$$\omega^\beta{}_\mu = \Lambda_{(\nu)}{}^\beta \Lambda_\mu{}^{(\alpha)} \omega^{(\nu)}{}_{(\alpha)} + \Lambda_{(\nu)}{}^\beta d\Lambda_\mu{}^{(\nu)}\,. \tag{III-9}$$

Equatorial, circular geodesics do not cross, so $d\Lambda_\mu{}^{(\nu)}$ is well defined. The procedure is thus to calculate $\omega^{(\mu)}{}_{(\nu)}$ in the locally nonrotating frame, calculate $d\Lambda_\mu{}^{(\nu)}$ and express the result in terms of the same basis forms (III-5), apply the transformation (III-9), and finally, reexpress the basis forms (III-5) in terms of those of the orbiting observer (III-6).

Rewrite (III-5) as

$$\omega^{(t)} = e^{\nu-\sigma-\lambda}dt\,, \qquad \omega^{(r)} = e^\lambda dr\,,$$
$$\omega^{(\theta)} = e^\nu d\theta\,, \qquad \omega^{(\varphi)} = e^\sigma \sin\theta (d\Phi - \Omega dt)\,,$$

where λ, ν, σ, and Ω each depend on r and θ. The notation will be that $\nu_{,r} \equiv (\partial\nu/\partial r)$ and $\nu_{,\theta} \equiv (\partial\nu/\partial\theta)$. Then, the known solution to equation (III-8) (Bardeen *et al.* 1972) leads to the results

$$\begin{aligned}\omega^\tau{}_r = {} & \omega^\tau\{(1-v^2)^{-1}[(\nu_{,r} - \lambda_{,r} - \sigma_{,r} - v^2\sigma_{,r})\, e^{-\lambda} + v\Omega_{,r} \sin\theta\, e^{2\sigma-\nu}]\} \\ & + \omega^\phi\{(1-v^2)^{-1}[v(\nu_{,r} - \lambda_{,r} - 2\sigma_{,r})\, e^{-\lambda} + (1+v^2)\tfrac{1}{2}\Omega_{,r} \sin\theta\, e^{2\sigma-\nu}]\}\,,\end{aligned} \tag{III-10a}$$

$$\begin{aligned}\omega^\tau{}_\theta = {} & \omega^\tau\{(1-v^2)^{-1}[(\nu_{,\theta} - \lambda_{,\theta} - \sigma_{,\theta})\, e^{-\nu} - v^2(\sigma_{,\theta} + \cot\theta)\, e^{-\nu} \\ & \qquad + v\Omega_{,\theta} \sin\theta\, e^{2\sigma+\lambda-2\nu}]\} \\ & + \omega^\phi\{(1-v^2)^{-1}[v(\nu_{,\theta} - \lambda_{,\theta} - 2\sigma_{,\theta} - \cot\theta)\, e^{-\nu} \\ & \qquad + (1+v^2)\tfrac{1}{2}\Omega_{,\theta} \sin\theta\, e^{2\sigma+\lambda-2\nu}]\}\,,\end{aligned} \tag{III-10b}$$

$$\omega^\tau{}_\phi = \omega^r\{\tfrac{1}{2}\Omega_{,r} \sin\theta\, e^{2\sigma-\nu}\} + \omega^\theta\{\tfrac{1}{2}\Omega_{,\theta} \sin\theta\, e^{2\sigma-2\nu+\lambda}\} + (1-v^2)^{-1}dv\,, \tag{III-10c}$$

$$\omega^r{}_\theta = \omega^r\{\lambda_{,\theta}\, e^{-\nu}\} + \omega^\theta\{-\nu_{,r}\, e^{-\lambda}\}\,, \tag{III-10d}$$

$$\begin{aligned}\omega^r{}_\phi = {} & \omega^\tau\{(1-v^2)^{-1}[(1+v^2)\tfrac{1}{2}\Omega_{,r} \sin\theta\, e^{2\sigma-\nu} + v(\nu_{,r} - \lambda_{,r} - 2\sigma_{,r})\, e^{-\lambda}]\} \\ & + \omega^\phi\{(1-v^2)^{-1}[v\Omega_{,r} \sin\theta\, e^{2\sigma-\nu} - \sigma_{,r}\, e^{-\lambda} + v^2(\nu_{,r} - \lambda_{,r} - \sigma_{,r})\, e^{-\lambda}]\}\,,\end{aligned} \tag{III-10e}$$

and

$$\begin{aligned}\omega^\theta{}_\phi = {} & \omega^\tau\{(1-v^2)^{-1}[(1+v^2)\tfrac{1}{2}\Omega_{,\theta} \sin\theta\, e^{2\sigma-2\nu+\lambda} + v(\nu_{,\theta} - \lambda_{,\theta} - 2\sigma_{,\theta} - \cot\theta)\, e^{-\nu}]\} \\ & + \omega^\phi\{(1-v^2)^{-1}[v\Omega_{,\theta} \sin\theta\, e^{2\sigma-2\nu+\lambda} - (\sigma_{,\theta} + \cot\theta)\, e^{-\nu} \\ & \qquad + v^2(\nu_{,\theta} - \lambda_{,\theta} - \sigma_{,\theta})\, e^{-\nu}]\}\,.\end{aligned} \tag{III-10f}$$

In our particular case, where the velocity v is given by the geodesic formula (II-8), the third term in the expansion of $\omega^{\tau}{}_{\phi}$ becomes $\omega^{r}[(1 - v^2)^{-1}(\partial v/\partial r)\, e^{-\lambda}\}$. In Appendix A is the complete list of $\omega^{\alpha}{}_{\beta}$ as functions of ω^{γ} for the case of a frame in a geodesic, circular orbit around a Schwarzschild black hole at arbitrary radius r; in this list, all coefficients appear explicitly as functions of r.

By equations (III-2), (III-3), and (III-7), we need only those connection coefficients of the form $\Gamma^{i}{}_{j0}$; we can hence limit our attention to the connection forms $\omega^{r}{}_{\theta}$, $\omega^{r}{}_{\phi}$, and $\omega^{\theta}{}_{\phi}$. But the relation (III-8) and the result (III-10d) specify $\Gamma^{r}{}_{\theta\tau} = 0$, so we only need to examine $\omega^{r}{}_{\phi}$ and $\omega^{\theta}{}_{\phi}$, from which we extract $\Gamma^{r}{}_{\phi\tau}$ and $\Gamma^{\theta}{}_{\phi\tau}$. In the latter, each derivative with respect to θ introduces a factor of $\cos\theta$, so for equatorial orbits, $\Gamma^{\theta}{}_{\phi\tau}$ will not contribute.

Using equation (II-8), we find after much algebra that

$$\Gamma^{r}{}_{\phi\tau}|_{\pi/2} = \mp (M/r^3)^{1/2}\,, \tag{III-11}$$

independent of the parameter a! The angular velocity vector in equation (III-3) is thus

$$\Omega_{cj} = (0,\ \pm [M/r^3]^{1/2}, 0)\,. \tag{III-12}$$

The cross product $\boldsymbol{e}_r \times \boldsymbol{e}_\theta = \boldsymbol{e}_\phi$ defines a right-handed coordinate system.

d) Transformation of the Curvature

The curvature term in equation (III-1) is (see Misner 1969)

$$\begin{aligned} R(\boldsymbol{n}, \boldsymbol{u})\boldsymbol{u} &= n^{\alpha} R^{\beta}{}_{0\alpha 0} \boldsymbol{e}_{\beta} \\ &= n^{j} R^{i}{}_{0j0} \boldsymbol{e}_{i}\,. \end{aligned} \tag{III-13}$$

The components of the Riemann curvature in the orbiting frame follow from a fourfold Lorentz transformation of those in the locally nonrotating frame: $R_{\alpha\beta\gamma\delta} = \Lambda_{\alpha}{}^{(\alpha)}\Lambda_{\beta}{}^{(\beta)}\Lambda_{\gamma}{}^{(\gamma)}\Lambda_{\delta}{}^{(\delta)}R_{(\alpha)(\beta)(\gamma)(\delta)}$. The $R_{(\alpha)(\beta)(\gamma)(\delta)}$ curvature components have been computed via the Newman-Penrose formalism (Newman and Penrose 1962; see also Kinnersley 1969*a*, *b*). Stewart and Walker (1973) display a general formula for computing the tidal acceleration (curvature) in that tetrad formalism, while Bardeen *et al.* (1972) display a list of the curvature components in the locally nonrotating frame. The details of the calculation of these components appear in the author's thesis (Fishbone 1972*b*).

Thus, the curvature components in the orbiting frame are

$$R_{\tau r\tau r} = (1 - v^2)^{-1}\{R_{(t)(r)(t)(r)} + v^2 R_{(\varphi)(r)(\varphi)(r)} + 2vR_{(t)(r)(\varphi)(r)}\}\,, \tag{III-14a}$$

$$R_{\tau\theta\tau\theta} = (1 - v^2)^{-1}\{R_{(t)(\theta)(t)(\theta)} + v^2 R_{(\varphi)(\theta)(\varphi)(\theta)} + 2vR_{(t)(\theta)(\varphi)(\theta)}\}\,, \tag{III-14b}$$

$$R_{\tau\phi\tau\phi} = R_{(t)(\varphi)(t)(\varphi)}\,, \tag{III-14c}$$

and

$$R_{r\tau\theta\tau} = (1 - v^2)^{-1}\{(1 + v^2)R_{(r)(t)(\theta)(t)} + v(R_{(r)(t)(\theta)(\varphi)} + R_{(r)(\varphi)(\theta)(t)})\}\,. \tag{III-14d}$$

We are here interested only in these components evaluated on the equatorial plane, where $R_{r\tau\theta\tau}$ vanishes. A further simplification occurs if we substitute the formulae (II-8) and (II-9) for geodesic orbits; of course, equations (III-14) apply for any velocity in the Φ direction at any value of colatitude θ. The final forms for equatorial, circular geodesics are, using the expressions of equations (II-3) and (II-4),

$$R_{\tau r \tau r} = \frac{-M}{r^3}\left\{\frac{3\Delta - P(r, a)}{P(r, a)}\right\} \tag{III-15a}$$

$$= \frac{-M\gamma^2}{r^3}\left\{\frac{r^2[3\Delta - P(r, a)]}{[r^2 - 2Mr \pm a(Mr)^{1/2}]^2}\right\}, \tag{III-15b}$$

$$R_{\tau\theta\tau\theta} = \frac{M}{r^3}\left\{\frac{3\Delta - 2P(r, a)}{P(r, a)}\right\} \tag{III-15c}$$

$$= \frac{M\gamma^2}{r^3}\left\{\frac{r^2[3\Delta - 2P(r, a)]}{[r^2 - 2Mr \pm a(Mr)^{1/2}]^2}\right\}, \tag{III-15d}$$

and

$$R_{\tau\phi\tau\phi} = M/r^3 \,. \tag{III-15e}$$

For more details of the algebra, see the author's thesis (Fishbone 1972*b*). In the forms (III-15b, d), any divergence in the curvature occurs purely in the γ^2 factors, which would diverge for photon orbits. The bracketed factors in those expressions are finite for all circular orbits, even when $a = M$.[2]

e) Equation of Motion for an Orbiting Observer

We can now put the pieces (III-12), and (III-15) into the equation of geodesic deviation (III-1) via the forms (III-4) and (III-13) to obtain

$$\begin{aligned} 0 = \nabla_u{}^2 \boldsymbol{n} + R(\boldsymbol{n}, \boldsymbol{u})\boldsymbol{u} &= \frac{d^2 n^0}{d\tau^2}\,\boldsymbol{e}_0 \\ &+ \left\{\frac{d^2 n^i}{d\tau^2} \pm 2\left(\frac{M}{r^3}\right)^{1/2}\frac{dn^j}{d\tau}\,\epsilon_{j\theta i} + \left(\frac{M}{r^3}\right) n^m \epsilon_{m\theta k}\epsilon_{k\theta i}\right\}\boldsymbol{e}_i \\ &+ \frac{M}{r^3}\left\{-\left[\frac{3\Delta - P}{P}\right] n^r \boldsymbol{e}_r + \left[\frac{3\Delta - 2P}{P}\right] n^\theta \boldsymbol{e}_\theta + n^\phi \boldsymbol{e}_\phi\right\}. \end{aligned} \tag{III-16}$$

We set $n^0 = 0$ to coordinate the clocks on the various geodesics.

For easier comparison with existing results (Chandrasekhar 1969), we change the coordinate labels to (1, 2, 3) by $x^1 = -n^r$, $x^2 = -n^\phi$, and $x^3 = -n^\theta$. Figure 1 depicts the relation of these coordinate systems. Incorporating these changes, we rewrite the vector equation (III-16) as three scalar equations:

$$\frac{d^2x^1}{d\tau^2} \mp 2\left(\frac{M}{r^3}\right)^{1/2}\frac{dx^2}{d\tau} - \left(\frac{M}{r^3}\right)\left(\frac{3\Delta}{P}\right)x^1 = 0\,; \tag{III-17a}$$

$$\frac{d^2x^2}{d\tau^2} \pm 2\left(\frac{M}{r^3}\right)^{1/2}\frac{dx^1}{d\tau} = 0\,; \tag{III-17b}$$

$$\frac{d^2x^3}{d\tau^2} + \left(\frac{M}{r^3}\right)\left(\frac{3\Delta - 2P}{P}\right)x^3 = 0\,. \tag{III-17c}$$

These are the equations governing force-free motion from the viewpoint of the orbiting observer. Note that the centifugal and tidal effects cancel identically in the x^2

[2] In this connection, the two "curvature factors" in figure 2 of an earlier *Letter* (Fishbone 1972*a*) have the finite value 3 for the case of prograde orbits when $a = M$.

direction. Also, the motion in the x^3 direction is simple harmonic. We postpone for a moment a discussion of the latter. Gowdy (1972) noted the significance of the former. Imagine a swarm of particles moving along the same circular orbit but distributed throughout it. Because the angular velocities of the particles would be equal, the interparticle distances would remain constant. Thus, from the viewpoint of an observer moving with any such particle, the component of force along the orbit direction must vanish identically. But this is just the meaning of equation (III-17b), where no term proportional to x^2 appears.

Equations (III-17) can also be combined into the useful form (I-2), where the expression.

$$\Phi_{t+c} \equiv \frac{M}{2r^3}\left[-\left(\frac{3\Delta}{P}\right)(x^1)^2 + \left(\frac{3\Delta - 2P}{P}\right)(x^3)^2\right] \tag{III-18}$$

is the potential for (tidal and centrifugal) noninertial effects; it plays a key role subsequently.

f) Force-free Motion

With the constraint of small motions, equation (III-17c) governs simple harmonic motion perpendicular to the equatorial plane. Thus, slightly nonplanar orbits (Wilkins 1972) can be built from equatorial circular orbits, upon which equations (III-17) are based, and this motion perpendicular to the plane. The angular frequency characteristic of the latter is $\Omega^2{}_{\text{perp}} = (M/r^3)([3\Delta - 2P]/P)$. We compare this to the proper angular velocity for the equatorial circular orbit:

$$\frac{\Omega_{\text{prop}}}{\Omega_{\text{perp}}} = \left[\frac{r^2}{r^2 + 3a^2 \mp 4a(Mr)^{1/2}}\right]^{1/2}. \tag{III-19}$$

Note immediately that in the limit $r \to \infty$, or for the case $a = 0$ at any r, the ratio is unity. This means that the nonequatorial orbits are in fact planar—just what we demand in the Schwarzschild and Newtonian limits. When the ratio is an integer, the motion is periodic. For a/M very near unity, such nonplanar orbits, built from either the last stable or photon orbit, look like tightly wound helices around the $\theta = 0$ axis.

Observe also that this simple harmonic motion guarantees the stability of all equatorial, circular geodesics with respect to perturbations out of the equatorial plane (Bardeen 1970*b*), a necessity if beaming mechanisms associated with the preferred equatorial plane (Misner 1972) are to be plausible.

IV. THE EULER EQUATION AND ITS LIMITATIONS

We wish to study the behavior of a fluid body in orbit, so we must find the generalized Euler equation which will govern fluid flow in the frame of an orbiting observer. Such an equation arises from the equation of geodesic deviation (I-2) by the weak equivalence principle, i.e., the fundamental principle that freely falling particles move along geodesics of a metric (see Thorne, Will, and Ni 1971).

First let us discuss the limits of validity of the assumption of circular geodesics, the orbits we assume for the fluid body. This assumption is true only if the energy of the body in orbit is so small that the body does not significantly perturb the background metric and that it does not radiate sufficiently to change orbits quickly. Both of these conditions require $\gamma m \ll M$, where m is the mass of the orbiting fluid body.

Because the body will be fixed in a noninertial reference system, it will have a nonzero spin angular momentum. Such bodies obey more complicated equations of motion than the simple geodesic laws (Papapetrou 1951), but, to a good approximation, geodesic motion will hold if the linear size of the body is much less than the radius of curvature of the Kerr gravitational field (Mashhoon 1972). This requirement also limits the validity of the equation of geodesic deviation itself, so we shall discuss it in that context.

The maximum extent that a spatial deviation vector $n^i e_i$ can have and still obey equation (III-1) well is one much less than that length associated with spatial changes in the curvature. The latter is the inverse square root of the curvature, i.e., the radius of curvature. With respect to temporal limits on particle motions $n^i(\tau)$, the deviation equation will hold for as long a time as the particles remain in the spatial domain of validity. For a situation of hydrostatic equilibrium, as will concern us, this requirement is met forever or not at all.

Thus let L be the linear "size" of a fluid body. From equations (III-15), the spatial requirement becomes

$$L \ll \left[\frac{M}{r^3}\left(\frac{3\Delta - P}{P}\right)\right]^{-1/2} . \tag{IV-1}$$

This condition guarantees that no part of the fluid body extends through the horizon or to a place where its velocity exceeds the local light velocity. We demonstrate this explicitly for high-energy orbits in the Schwarzschild geometry ($a = 0$). With the orbit at $r = (3 + \delta)M$ and $\delta \ll 1$, limit (IV-1) becomes $L \ll 3M\delta^{1/2}$. The proper radial distance Δs from the horizon at $r = 2M$ to the orbit radius satisfies

$$\Delta s = \int_{2M}^{(3+\delta)M} \frac{dr}{(1 - 2M/r)^{1/2}} > (2M)^{1/2}\int_0^{(1+\delta)M} \frac{dx}{x^{1/2}} \simeq 2 \times 2^{1/2}M \gg 3M\delta^{1/2} . \tag{IV-2}$$

Thus, the restriction (IV-1) on the maximum linear size L of the body does not allow it to extend through the horizon. Second, the coordinate angular velocity from equation (II-10) is $(M/r^3)^{1/2} = [M^2(3 + \delta)^3]^{-1/2}$. To find the radius of the null (photon), *nongeodesic* orbit with the same angular velocity, let $ds^2 = 0$ in the metric (I-1). Then $(d\Phi/dt)_\gamma = \pm[(r - 2M)/r^3]^{1/2}$. If we use the ansatz $r = (3 + \alpha)M$, then a second-order expansion of

$$\left(\frac{M}{r^3}\right)^{1/2} = \frac{1}{M(3 + \delta)^{3/2}} = \frac{(1 + \alpha)^{1/2}}{(3 + \alpha)^{3/2}M} = \left(\frac{d\Phi}{dt}\right)_\gamma \tag{IV-3}$$

yields $\alpha = (3\delta)^{1/2}$. Thus, the circular, geodesic, *timelike* orbit at $r = (3 + \delta)M$ has the same coordinate angular velocity as the circular, nongeodesic, *photon* orbit at $r = [3 + (3\delta)^{1/2}]M$. The proper radial distance Δs between these two orbits satisfies

$$\Delta s = \int_{(3+\delta)M}^{[3+(3\delta)^{1/2}]M} \frac{dr}{(1 - 2M/r)^{1/2}} = \int_{(1+\delta)M}^{[1+(3\delta)^{1/2}]M} \frac{(x + 2M)^{1/2}dx}{x^{1/2}} \simeq 3M\delta^{1/2} . \tag{IV-4}$$

Hence, the limitation $L \ll 3M\delta^{1/2}$ precludes any extension of the body to a place where its velocity equals the local light velocity. In short, equation (IV-1) summarizes all length restrictions on the validity of the equation of geodesic deviation for situations of interest here.

To explain better the transition from the free-particle equation (I-2) to an equation for fluid flow, we split the potential Φ_{t+c} into a tidal part Φ_t and a centrifugal part Φ_c:

$$\Phi_t = -\frac{M}{r^3}\left[\left(\frac{3\Delta - P}{2P}\right)(x^1)^2 - \tfrac{1}{2}(x^2)^2 - \left(\frac{3\Delta - 2P}{2P}\right)(x^3)^2\right] \tag{IV-5a}$$

and

$$\Phi_c = -\tfrac{1}{2}\Omega_c{}^2[(x^1)^2 + (x^2)^2] ; \tag{IV-5b}$$

here $\Omega_c = \pm(M/r^3)^{1/2}$, and a particular choice of sign implies the same choice throughout a given equation. Then the free-particle, geodesic deviation equation takes the form

$$\frac{d^2x^i}{d\tau^2} = \frac{\partial}{\partial x^i}\{\tfrac{1}{2}\Omega_c^2[(x^1)^2 + (x^2)^2] - \Phi_t\} \pm 2\epsilon_{ij3}\Omega_c\frac{dx^j}{d\tau}\,. \tag{IV-6}$$

This equation is identical in form to the equation in Newtonian physics which describes, from the viewpoint of a rotating observer, particle motion in a potential Φ_t. In that case, the transition to the appropriate Euler equation of fluid flow is straightforward. We will proceed similarly and simply *impose* on the result the limitations of Newtonian physics. In a more sophisticated approach, one could perhaps exploit the post-Newtonian equations of hydrodynamics (Chandrasekhar 1965*a*, *b*); this would alleviate problems due to high densities in the orbiting body, but not those due to the limitation (IV-1) inherent in the use of the equation of geodesic deviation.

By analogy then, the Euler equation governing fluid flow in the orbiting frame, where equation (IV-6) governs free-particle motion, is (see Feynman, Leighton, and Sands 1964; and Chandrasekhar 1969)

$$\rho\frac{du^i}{d\tau} = -\frac{\partial p}{\partial x^i} + \rho\frac{\partial}{\partial x^i}\{\tfrac{1}{2}\Omega_c^2[(x^1)^2 + (x^2)^2] - \Phi_t - \Phi_G\} \pm 2\rho\epsilon_{ij3}\Omega_c u^j\,. \tag{IV-7}$$

Here, Φ_G is the self-gravitational potential of the fluid, ρ is its (constant) density, and p its pressure. The time derivative is the total time derivative of a fluid element

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau} + u^l\frac{\partial}{\partial x^l}\,,$$

and u^l is the velocity field of the fluid.

The Newtonian limit is the one of small masses and low velocities. The mass requirement must be imposed directly on the final results of any calculations. The most obvious velocity effects are those due to the rotation. Thus, with L characterizing the linear size of the fluid body, we demand that $L\Omega_c \ll 1$. But if condition (IV-1) is satisfied, so also is this. High particle velocities could also result from acceleration by the potentials. Since $|\Phi_G| \gtrless |\Phi_t| \geq |\Phi_c|$, the condition is $|\Phi_G|^{1/2} \ll 1$. But from equations (III-19),

$$|\Phi_t|^{1/2} \simeq \left[\frac{M}{r^3}\left(\frac{3\Delta - P}{P}\right)(x^1)^2\right]^{1/2} \leq \left[\frac{1}{M^2}\left(\frac{3\Delta - P}{P}\right)(x^1)^2\right]^{1/2}. \tag{IV-8}$$

If we choose $x^1 = L$ to correspond to the maximum possible acceleration, then the condition $|\Phi_t| \ll 1$ is satisfied if (IV-1) is. The only other velocity would be the sound velocity of the fluid resulting from a large $|\Phi_G|$. For the constant-density situations of interest to us, low sound velocities correspond to low pressure-density ratios; these depend in detail upon the mass m and size L of the final equilibrium configurations.

In summary, equation (IV-1) is the chief constraint on final solutions of the Euler equation.

V. THE ROCHE PROBLEM

The equilibrium problem of Roche is to determine how an incompressible fluid can maintain itself given equation (IV-7). At this point we can proceed via the elaborate and powerful virial method developed by Chandrasekhar and Lebovitz (Chandrasekhar 1969 and the references therein) or via a simpler method (see Jeans 1929). We choose the latter because it requires less formalism for an attack on the equilibrium problem.

Consider the situation with no fluid motions in the orbiting frame. Equation (IV-4) is then integrable and yields

$$\frac{p}{\rho} = \Phi_{t+c} + \Phi_G + \text{constant} . \tag{V-1}$$

The fluid body will be an equilibrium configuration if $\Phi_{t+c} + \Phi_G$ is constant on the surface. Because this potential is quadratic in the coordinates, ellipsoidal figures are possible.

The potential Φ_G is given by (see Chandrasekhar 1969, p. 43)

$$\Phi_G = -\pi G\rho\left[I - \sum_{l=1}^{3} A_l(x^l)^2\right] , \tag{V-2}$$

where

$$I = a_1a_2a_3 \int_0^\infty \frac{du}{Д} \tag{V-3}$$

and

$$A_l = a_1a_2a_3 \int_0^\infty \frac{du}{Д(a_l^2 + u)} , \tag{V-4}$$

with $Д = [(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)]^{1/2}$ and with a_1, a_2, and a_3 the three semiaxes of the ellipsoid whose surface satisfies

$$\frac{(x^1)^2}{a_1^2} + \frac{(x^2)^2}{a_2^2} + \frac{(x^3)^2}{a_3^2} = 1 . \tag{V-5}$$

The potential satisfies, as it must, the inhomogeneous Poisson equation (Chandrasekhar 1969), $\nabla^2\Phi_G = 4\pi G\rho$, where

$$\nabla^2 = \sum_{i=1}^{3} [\partial^2/\partial(x^i)^2] .$$

We can simplify the problem of determining the shapes of the equilibrium ellipsoids by proceeding as follows. If $\Phi_{t+c} + \Phi_G$ is a constant on the surface, then the function Ψ satisfying

$$\Psi = \Phi_{t+c} + \Phi_G - C\pi G\rho\left[\frac{(x^1)^2}{a_1^2} + \frac{(x^2)^2}{a_2^2} + \frac{(x^3)^2}{a_3^2} - 1\right] \tag{V-6}$$

is the same constant on the surface if C is constant. Now demand that C be chosen so that $\nabla^2\Psi = 0$; this is always possible since ρ is constant. Then, the boundary condition that Ψ be constant on the surface of the ellipsoid requires, by uniqueness, that Ψ be constant everywhere inside the ellipsoid too (see Morse and Feshbach 1953, p. 706). But if this is true, the coefficient (in the expression for Ψ) of the square of each independently variable interior coordinate must vanish. We thus get, from equations (III-18), (V-2), and (V-6),

$$\frac{-\mu a_1^2}{G\pi\rho}\left[\frac{3\Delta}{P}\right] + 2A_1a_1^2 = 2C , \tag{V-7}$$

$$2A_2a_2^2 = 2C , \tag{V-8}$$

and

$$\frac{\mu a_3^2}{G\pi\rho}\left[\frac{3\Delta - 2P}{P}\right] + 2A_3a_3^2 = 2C\,, \tag{V-9}$$

where $\mu \equiv GM/r^3$. After we eliminate C, these yield

$$\frac{\mu}{G\pi\rho}\left[a_1^2\left(\frac{3\Delta}{P}\right) + a_3^2\left(\frac{3\Delta - 2P}{P}\right)\right] = 2[a_1^2A_1 - a_3^2A_3] \tag{V-10a}$$

and

$$\frac{\mu}{G\pi\rho}\left[a_3^2\left(\frac{3\Delta - 2P}{P}\right)\right] = 2[a_2^2A_2 - a_3^2A_3]\,. \tag{V-10b}$$

Finally, we take the ratio of equations (V-10) to obtain the "polarization equation" (so called because it is independent of the tidal intensity, proportional to μ, and depends only on the ratio of the intensities in different directions)

$$\frac{[3\Delta/(3\Delta - 2P)] + \alpha_3^2}{\alpha_3^2} = \frac{A_1 - \alpha_3^2A_3}{\alpha_2^2A_2 - \alpha_3^2A_3}\,, \tag{V-11}$$

where $\alpha_2 = a_2/a_1$ and $\alpha_3 = a_3/a_1$. As will be shown in Appendix B, A_1, A_2, and A_3 depend only on α_2 and α_3.

Note that

$$\frac{3\Delta}{3\Delta - 2P(r, a)} = \frac{3(r^2 - 2Mr + a^2)}{r^2 + 3a^2 \mp 4a(Mr)^{1/2}} = \frac{6\Delta}{3\Delta - S(r, a)}\,, \tag{V-12}$$

where, from equation (II-5), $S(r, a)$ vanishes at the last stable orbit. Thus, the term $3\Delta/(3\Delta - 2P)$ in the polarization equation is unity for the photon orbits, 2 for the last stable orbits, and 3 for orbits characteristic of the Newtonian limit ($r \to \infty$). A single series of ellipsoids will suffice for every orbit in a given class above.

The method of solution of the Roche problem is to specify values for r and a; the polarization equation then determines a one-parameter family of ellipsoids (a set of pairs $[\alpha_2, \alpha_3]$) which are in equilibrium at the given r and a. Substitution of this set of pairs (α_2, α_3), along with the same r and a, into the "density equation" (V-10b), results in a set of values for the "density function" $D(\alpha_2, \alpha_3) \equiv 2(A_2\alpha_2^2/\alpha_3^2 - A_3)$, and thence of the density. Appendix B contains a discussion of techniques used in solving the system. Table 1 lists the results for the photon and last stable orbits.

Recall from equations (III-15a, b) that the function $(3\Delta - 2P)/P$ in the density equation can be rewritten as

$$\frac{3\Delta - 2P}{P} = \frac{r^2 + 3a^2 \mp 4a(Mr)^{1/2}}{r^2 - 3Mr \pm 2a(Mr)^{1/2}} \tag{V-13a}$$

$$= \gamma^2\left\{\frac{r^2[r^2 + 3a^2 \mp 4a(Mr)^{1/2}]}{[r^2 - 2Mr \pm a(Mr)^{1/2}]^2}\right\}. \tag{V-13b}$$

We now introduce the dimensionless radius r_o, defined by $r = GMr_o/c^2$, into the expression for μ and obtain for the density equation (in conventional units, with $M_\odot$ the mass of the Sun)

$$\frac{\gamma^2}{\rho}\left[\frac{M_\odot}{M}\right]^2\left[\frac{c^6}{\pi G^3 M_\odot^2}\right] = \left\{\frac{r_0^3[r^2 - 2Mr \pm a(Mr)^{1/2}]^2}{r^2[r^2 + 3a^2 \mp 4a(Mr)^{1/2}]}\right\}D(\alpha_2, \alpha_3)\,, \tag{V-14}$$

TABLE 1

CHARACTERISTICS OF THE RELATIVISTIC ROCHE ELLIPSOIDS IN KERR-METRIC ORBITS

α_2	α_3	$D(\alpha_2, \alpha_3)$	$(\alpha_2\alpha_3)^{-1/3}$	$\alpha_2(\alpha_2\alpha_3)^{-1/3}$	$\alpha_3(\alpha_2\alpha_3)^{-1/3}$
Ellipsoids Near the Photon Orbits					
0.250..........	0.22850	0.17340	2.5965	0.6491	0.5933
0.300..........	0.26976	0.20159	2.3119	0.6936	0.6237
0.350..........	0.31038	0.22370	2.0958	0.7335	0.6505
0.400..........	0.35073	0.23965	1.9245	0.7698	0.6750
0.460..........	0.39927	0.25084	1.7592	0.8093	0.7024
0.470..........	0.40741	0.25188	1.7350	0.8154	0.7068
0.480..........	0.41558	0.25268	1.7115	0.8215	0.7112
0.490..........	0.42377	0.25326	1.6887	0.8275	0.7156
0.500..........	0.43199	0.25361	1.6667	0.8333	0.7200
0.510..........	0.44024	0.25373	1.6453	0.8391	0.7243
0.520..........	0.44853	0.25363	1.6246	0.8448	0.7287
0.530..........	0.45685	0.25330	1.6044	0.8503	0.7330
0.540..........	0.46521	0.25275	1.5848	0.8558	0.7373
0.600..........	0.51645	0.24489	1.4778	0.8867	0.7632
0.650..........	0.56094	0.23249	1.3998	0.9099	0.7852
0.700..........	0.60764	0.21492	1.3297	0.9308	0.8080
0.750..........	0.65725	0.19227	1.2659	0.9494	0.8320
0.800..........	0.71069	0.16457	1.2071	0.9657	0.8579
0.850..........	0.76923	0.13174	1.1521	0.9793	0.8863
0.900..........	0.83469	0.09364	1.1000	0.9900	0.9182
0.950..........	0.90997	0.04992	1.0497	0.9973	0.9552
Ellipsoids in the Last Stable Orbits					
0.250..........	0.23815	0.08958	2.5610	0.6402	0.6099
0.300..........	0.28307	0.10461	2.2751	0.6825	0.6440
0.350..........	0.32753	0.11648	2.0586	0.7205	0.6742
0.400..........	0.37176	0.12511	1.8875	0.7550	0.7017
0.460..........	0.42487	0.13119	1.7232	0.7927	0.7321
0.470..........	0.43375	0.13176	1.6991	0.7986	0.7370
0.480..........	0.44265	0.13221	1.6758	0.8044	0.7418
0.490..........	0.45156	0.13253	1.6533	0.8101	0.7466
0.500..........	0.46050	0.13272	1.6316	0.8158	0.7513
0.510..........	0.46945	0.13280	1.6105	0.8213	0.7560
0.520..........	0.47842	0.13275	1.5900	0.8268	0.7607
0.530..........	0.48742	0.13258	1.5701	0.8322	0.7653
0.540..........	0.49645	0.13229	1.5509	0.8375	0.7699
0.600..........	0.55131	0.12808	1.4459	0.8676	0.7972
0.650..........	0.59821	0.12141	1.3701	0.8905	0.8196
0.700..........	0.64657	0.11198	1.3025	0.9117	0.8421
0.750..........	0.69680	0.09986	1.2415	0.9311	0.8651
0.800..........	0.74942	0.08511	1.1859	0.9487	0.8888
0.850..........	0.80505	0.06776	1.1348	0.9646	0.9136
0.900..........	0.86453	0.04783	1.0872	0.9785	0.9400
0.950..........	0.92899	0.02527	1.0425	0.9904	0.9685

a form especially useful for unstable, high-energy orbits, and

$$\frac{1}{\rho}\left[\frac{M_\odot}{M}\right]^2\left[\frac{c^6}{\pi G^3 M_\odot^2}\right] = \left\{\frac{r_0^3[r^2 - 3Mr \pm 2a(Mr)^{1/2}]}{r^2 + 3a^2 \mp 4a(Mr)^{1/2}}\right\} D(\alpha_2, \alpha_3)\,, \qquad \text{(V-15)}$$

a form useful for all other orbits.

The constant $c^6/(\pi G^3 M_\odot^2)$ has the value 1.969×10^{17} g cm^{-3}; its precision is limited by that of the gravitational constant.

TABLE 2

Densities of Relativistic Roche Ellipsoids near the Photon Orbits of Kerr Gravitational Fields*

α_2	a/M Retrograde 1.0	Retrograde 0.4	0.0	Prograde 0.1	Prograde 0.5	Prograde 0.9	Prograde 0.99	Prograde 0.999	Prograde 0.99999
0.250	.2129 + 18	.2892 + 18	.3785 + 18	.4101 + 18	.6183 + 18	.1404 + 19	.2499 + 19	.3078 + 19	.3363 + 19
0.300	.1831 + 18	.2488 + 18	.3256 + 18	.3527 + 18	.5318 + 18	.1207 + 19	.2149 + 19	.2647 + 19	.2893 + 19
0.350	.1650 + 18	.2242 + 18	.2934 + 18	.3179 + 18	.4793 + 18	.1088 + 19	.1937 + 19	.2386 + 19	.2607 + 19
0.400	.1541 + 18	.2093 + 18	.2739 + 18	.2967 + 18	.4474 + 18	.1016 + 19	.1808 + 19	.2227 + 19	.2434 + 19
0.460	.1472 + 18	.1999 + 18	.2617 + 18	.2835 + 18	.4274 + 18	.9703 + 18	.1727 + 19	.2128 + 19	.2325 + 19
0.470	.1466 + 18	.1991 + 18	.2606 + 18	.2823 + 18	.4256 + 18	.9663 + 18	.1720 + 19	.2119 + 19	.2316 + 19
0.480	.1461 + 18	.1985 + 18	.2597 + 18	.2814 + 18	.4243 + 18	.9632 + 18	.1715 + 19	.2112 + 19	.2308 + 19
0.490	.1458 + 18	.1980 + 18	.2592 + 18	.2808 + 18	.4233 + 18	.9611 + 18	.1711 + 19	.2107 + 19	.2303 + 19
0.500	.1456 + 18	.1978 + 18	.2588 + 18	.2804 + 18	.4227 + 18	.9597 + 18	.1708 + 19	.2104 + 19	.2300 + 19
0.510	.1455 + 18	.1977 + 18	.2587 + 18	.2803 + 18	.4225 + 18	.9593 + 18	.1708 + 19	.2103 + 19	.2299 + 19
0.520	.1456 + 18	.1977 + 18	.2588 + 18	.2804 + 18	.4227 + 18	.9597 + 18	.1708 + 19	.2104 + 19	.2300 + 19
0.530	.1457 + 18	.1980 + 18	.2591 + 18	.2807 + 18	.4232 + 18	.9609 + 18	.1710 + 19	.2107 + 19	.2302 + 19
0.540	.1461 + 18	.1984 + 18	.2597 + 18	.2813 + 18	.4242 + 18	.9630 + 18	.1714 + 19	.2111 + 19	.2307 + 19
0.600	.1508 + 18	.2048 + 18	.2680 + 18	.2904 + 18	.4378 + 18	.9939 + 18	.1769 + 19	.2179 + 19	.2382 + 19
0.650	.1588 + 18	.2157 + 18	.2823 + 18	.3059 + 18	.4611 + 18	.1047 + 19	.1864 + 19	.2295 + 19	.2509 + 19
0.700	.1718 + 18	.2334 + 18	.3054 + 18	.3309 + 18	.4988 + 18	.1132 + 19	.2016 + 19	.2483 + 19	.2714 + 19
0.750	.1920 + 18	.2608 + 18	.3414 + 18	.3698 + 18	.5576 + 18	.1266 + 19	.2253 + 19	.2776 + 19	.3033 + 19
0.800	.2243 + 18	.3048 + 18	.3988 + 18	.4321 + 18	.6515 + 18	.1479 + 19	.2633 + 19	.3243 + 19	.3544 + 19
0.850	.2802 + 18	.3807 + 18	.4982 + 18	.5398 + 18	.8138 + 18	.1848 + 19	.3289 + 19	.4051 + 19	.4427 + 19
0.900	.3943 + 18	.5356 + 18	.7009 + 18	.7594 + 18	.1145 + 19	.2599 + 19	.4627 + 19	.5699 + 19	.6229 + 19
0.950	.7395 + 18	.1005 + 19	.1315 + 19	.1424 + 19	.2148 + 19	.4876 + 19	.8679 + 19	.1069 + 20	.1168 + 20

* The upper left entry in this table gives the density, $0.2129 \times 10^{18}\gamma^2(M_\odot/M)^2$ g cm^{-3}, of an equilibrium ellipsoid whose second largest axis is 0.25 of the length of the largest. The ellipsoid's orbit, parametrized by its energy-at-infinity per unit mass γ, is near the retrograde, circular, photon orbit of a Kerr black hole with mass M and angular-momentum parameter $a/M = 1.0$.

TABLE 3

DENSITIES OF RELATIVISTIC ROCHE ELLIPSOIDS IN THE LAST STABLE ORBITS OF KERR GRAVITATIONAL FIELDS*

	a/M								
	Retrograde			Prograde					
α_2	1.0	0.4	0.0	0.1	0.5	0.9	0.99	0.999	0.99999
0.250.......	.6031 + 16	.1152 + 17	.2035 + 17	.2413 + 17	.5796 + 17	.3517 + 18	.1429 + 19	.2664 + 19	.3963 + 19
0.300.......	.5164 + 16	.9861 + 16	.1743 + 17	.2066 + 17	.4963 + 17	.3011 + 18	.1223 + 19	.2281 + 19	.3393 + 19
0.350.......	.4637 + 16	.8856 + 16	.1565 + 17	.1855 + 17	.4457 + 17	.2704 + 18	.1099 + 19	.2048 + 19	.3047 + 19
0.400.......	.4318 + 16	.8245 + 16	.1457 + 17	.1727 + 17	.4150 + 17	.2518 + 18	.1023 + 19	.1907 + 19	.2837 + 19
0.460.......	.4118 + 16	.7863 + 16	.1390 + 17	.1647 + 17	.3957 + 17	.2401 + 18	.9755 + 18	.1819 + 19	.2706 + 19
0.470.......	.4100 + 16	.7829 + 16	.1384 + 17	.1640 + 17	.3940 + 17	.2391 + 18	.9713 + 18	.1811 + 19	.2694 + 19
0.480.......	.4086 + 16	.7803 + 16	.1379 + 17	.1635 + 17	.3927 + 17	.2383 + 18	.9680 + 18	.1805 + 19	.2685 + 19
0.490.......	.4076 + 16	.7784 + 16	.1376 + 17	.1631 + 17	.3918 + 17	.2377 + 18	.9657 + 18	.1800 + 19	.2678 + 19
0.500.......	.4070 + 16	.7772 + 16	.1374 + 17	.1628 + 17	.3912 + 17	.2373 + 18	.9642 + 18	.1798 + 19	.2674 + 19
0.510.......	.4068 + 16	.7768 + 16	.1373 + 17	.1627 + 17	.3910 + 17	.2372 + 18	.9637 + 18	.1797 + 19	.2673 + 19
0.520.......	.4069 + 16	.7771 + 16	.1373 + 17	.1628 + 17	.3911 + 17	.2373 + 18	.9641 + 18	.1797 + 19	.2674 + 19
0.530.......	.4074 + 16	.7781 + 16	.1375 + 17	.1630 + 17	.3916 + 17	.2376 + 18	.9653 + 18	.1800 + 19	.2677 + 19
0.540.......	.4083 + 16	.7798 + 16	.1378 + 17	.1634 + 17	.3925 + 17	.2381 + 18	.9674 + 18	.1804 + 19	.2683 + 19
0.600.......	.4218 + 16	.8054 + 16	.1423 + 17	.1687 + 17	.4054 + 17	.2459 + 18	.9992 + 18	.1863 + 19	.2771 + 19
0.650.......	.4449 + 16	.8496 + 16	.1502 + 17	.1780 + 17	.4276 + 17	.2594 + 18	.1054 + 19	.1965 + 19	.2924 + 19
0.700.......	.4824 + 16	.9212 + 16	.1628 + 17	.1930 + 17	.4636 + 17	.2813 + 18	.1143 + 19	.2131 + 19	.3170 + 19
0.750.......	.5409 + 16	.1033 + 17	.1826 + 17	.2164 + 17	.5199 + 17	.3154 + 18	.1282 + 19	.2389 + 19	.3555 + 19
0.800.......	.6347 + 16	.1212 + 17	.2142 + 17	.2539 + 17	.6100 + 17	.3701 + 18	.1504 + 19	.2803 + 19	.4171 + 19
0.850.......	.7972 + 16	.1522 + 17	.2690 + 17	.3189 + 17	.7662 + 17	.4648 + 18	.1889 + 19	.3521 + 19	.5238 + 19
0.900.......	.1129 + 17	.2157 + 17	.3812 + 17	.4519 + 17	.1086 + 18	.6586 + 18	.2676 + 19	.4989 + 19	.7422 + 19
0.950.......	.2138 + 17	.4083 + 17	.7215 + 17	.8553 + 17	.2055 + 18	.1247 + 19	.5065 + 19	.9443 + 19	.1405 + 20

* The upper left entry in this table gives the density, $0.6031 \times 10^{16}(M_\odot/M)^2$ g cm^{-3}, of an equilibrium ellipsoid whose second largest axis is 0.25 of the length of the largest. The ellipsoid is in the last stable, retrograde, circular orbit of a Kerr black hole with mass M and angular-momentum parameter $a/M = 1.0$.

Consider the photon orbits. Since $P(r, a) = 0$ then holds, the right side of (V-14) simplifies via $\pm a(Mr_\gamma)^{1/2} = -\frac{1}{2}(r_\gamma^2 - 3Mr_\gamma)$, and the equation becomes (with $r_{0\gamma} \equiv GMr_\gamma/c^2$)

$$\frac{\gamma^2}{\rho}\left[\frac{M_\odot}{M}\right]^2\left[\frac{c^6}{\pi G^3 M_\odot^2}\right] = \left[\frac{r_{0\gamma}^3(r_\gamma - M)^2}{12(r_\gamma^2 - 2Mr_\gamma + a^2)}\right]D(\alpha_2, \alpha_3)\,. \tag{V-16}$$

In the limit when $a = M(1 - \epsilon)$, equation (II-6b) gives the radius, so the limiting density relation is

$$\frac{\gamma^2}{\rho}\left[\frac{M_\odot}{M}\right]^2\left[\frac{c^6}{\pi G^3 M_\odot^2}\right] \simeq \tfrac{1}{3}[1 + O(\epsilon^{1/2})]D(\alpha_2, \alpha_3)\,. \tag{V-17}$$

Similarly, since $S(r, a) = 0$ holds for the last stable orbits, the right side of (V-15) reduces to yield the equation ($r_{0s} \equiv GMr_s/c^2$)

$$\frac{1}{\rho}\left[\frac{M_\odot}{M}\right]^2\left[\frac{c^6}{\pi G^3 M_\odot^2}\right] = \tfrac{1}{2}r_{0s}^3 D(\alpha_2, \alpha_3)\,, \tag{V-18}$$

with a limiting form via (II-6c) of

$$\frac{1}{\rho}\left[\frac{M_\odot}{M}\right]\left[\frac{c^6}{\pi G^3 M_\odot^2}\right] \simeq \tfrac{1}{2}[1 + 3(4\epsilon)^{1/3}]D(\alpha_2, \alpha_3)\,. \tag{V-19}$$

Notice from equation (V-18) that the requisite equilibrium density is finite for bodies in the last stable orbit of the Kerr metric for any value of the parameter a of interest here. This important conclusion has been demonstrated independently by Stewart and Walker (1973).

Tables 2 and 3 respectively list the density of an equilibrium ellipsoid with the specified shape for the photon and last stable orbits.

Jeans (see 1929) used the parameters $\bar{a}_i = a_i(a_1a_2a_3)^{-1/3}$ to characterize the shapes of triaxial ellipsoids. Figure 4 shows the families of ellipsoids characteristic of the photon, last stable, and Newtonian orbits (see Chandrasekhar 1969, for the last family). In the original Newtonian calculation, the equation analogous to equation (V-11) is

$$\frac{(3 + \mathrm{p}) + \alpha_3^2}{\mathrm{p}\alpha_2^2 + \alpha_3^2} = \frac{A_1 - \alpha_3^2 A_3}{\alpha_2^2 A_2 - \alpha_3^2 A_3}\,,$$

while the density equation is

$$\frac{\mu}{\pi\rho G} = \frac{2(\alpha_2^2 A_2 - \alpha_3^2 A_3)}{\mathrm{p}\alpha_2^2 + \alpha_3^2}\,.$$

Here, p is the freely variable ratio of the mass of the fluid body to that of a rigid sphere with which it is in orbit.

Use of the equation of geodesic deviation in the relativistic situation, where $\mathrm{p} = 0$ is assumed, forbids this degree of freedom there. A more elaborate treatment of the two-body problem, currently nonexistent, is needed to incorporate this variability into the relativistic domain.

Note from tables 2 and 3 that the columns exhibit minima along the sequences of ellipsoids. These minima are the Roche limits for the cases in question. They represent the minimum density required for the existence of an ellipsoidal fluid body in equilibrium in the particular orbit. The Roche limits for bodies in the last stable orbits and near the photon orbits (Fishbone 1972*a*) appear in figures 2 and 3, respectively.

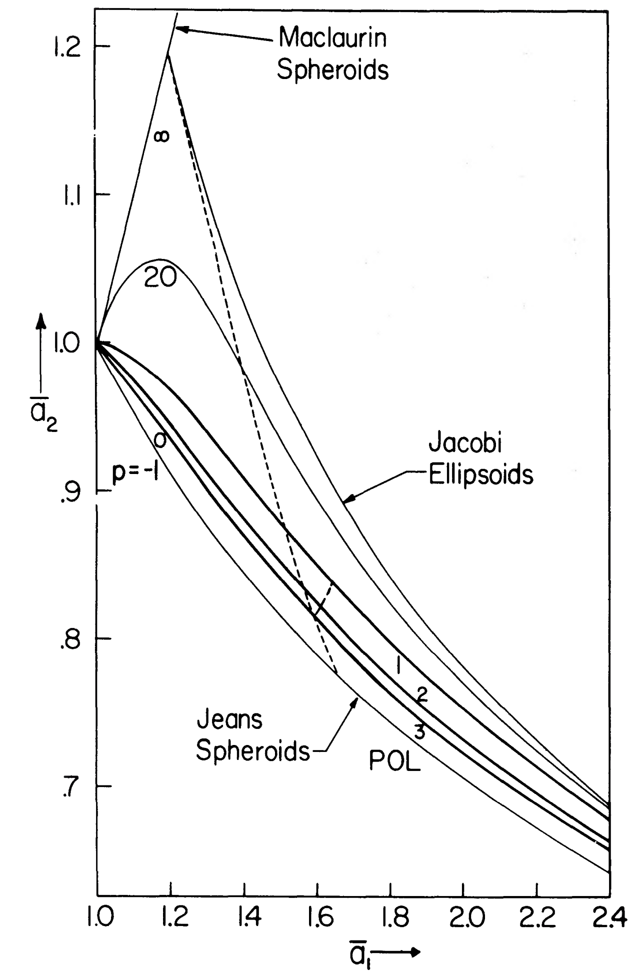

FIG. 4.—Families of triaxial ellipsoids labeled by their Jeans parameters $\bar{a}_1 = a_1(a_1a_2a_3)^{-1/3}$ and $\bar{a}_2 = a_2(a_1a_2a_3)^{-1/3}$, where a_i are the semiaxes of an ellipsoid. The parameter p $= -1$, 0, 20, or ∞ below a curve is the ratio of the mass of a deformable fluid body to that of a rigid sphere with which the former is in orbit; the families represented by these curves are realized in Newtonian theory. The p $= \infty$ sequence begins at $\bar{a}_1 = \bar{a}_2 = 1$, follows the Maclaurin sequence to its conjunction with the Jacobi sequence, and continues along the latter. The long, dashed line connects the Roche-limit configurations of these families. Three dark curves labeled by (POL) = 3, 2, 1 depict the families of relativistic Roche ellipsoids. These families are characteristic of infinitesimal bodies in, respectively, the Newtonian, last stable, and photon orbits in Kerr geometries with $0 \leq a \leq M$. The short, dashed line segment connects the Roche-limit configurations for the relativistic families.

We finally show the Roche limits for general Kerr equatorial, circular, geodesics in figure 5. The difference between the values in the graph and 0.0901, the Newtonian value, is a manifestation of general relativitity. The additional numerical results from which this last figure is derived appear in the author's thesis (Fishbone 1972*b*).

For a complete understanding of the applicability of the Roche-limit analysis, a study of the stability properties of the ellipsoids is necessary. In the Newtonian limit, the following holds for stability with respect to small perturbations from equilibrium: only those equilibrium configurations slightly more deformed than the Roche-limiting one are unstable (Chandrasekhar 1969). Whether this plausible fact also holds for the relativistic ellipsoids will be the subject of a future paper.

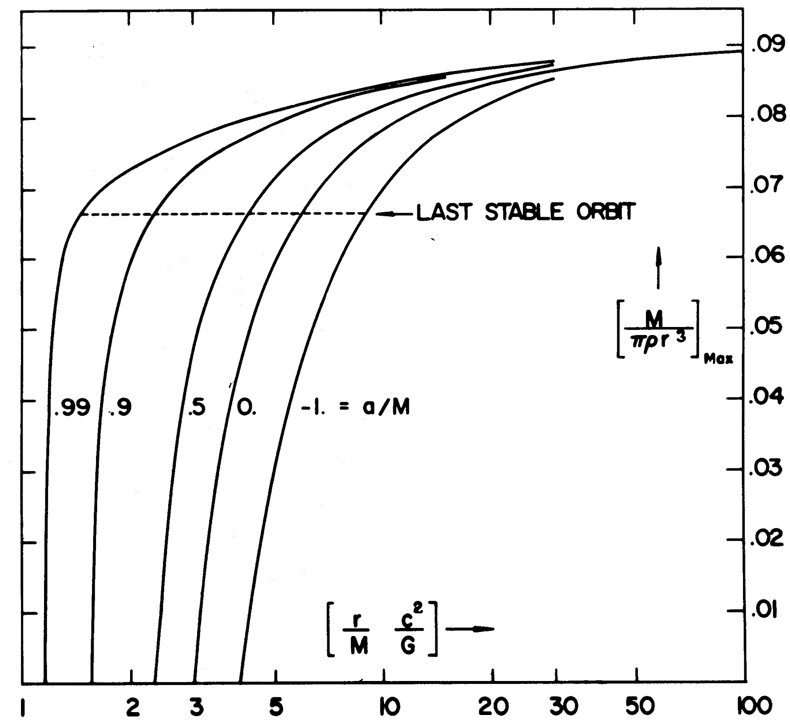

FIG. 5.—Roche limits for incompressible fluid ellipsoids in equatorial, circular, geodesic orbit in several Kerr geometries. The maximum value of the function $M/(\pi\rho r^3)$ for a given orbit defines a minimum density for the ellipsoidal body. In the Newtonian limit, $r \to \infty$, the maximum value is 0.0901. At the last stable orbit the maximum value is 0.0664.

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APPENDIX A

THE SCHWARZSCHILD CIRCULAR GEODESIC FRAME

Consider a Lorentz transformation from the standard, orthonormal Schwarzschild basis to one in circular, geodesic orbit, moving in the direction of increasing Φ on the equator, $\theta = \pi/2$. With the velocity relation (II-8) implicit, the new basis is

$$\omega^{\tau} = \left(1 - \frac{2M}{r}\right)\left(1 - \frac{3M}{r}\right)^{-1/2} dt - \left(\frac{M}{r}\right)^{1/2}\left(1 - \frac{3M}{r}\right)^{-1/2} r \sin\theta d\Phi\,, \tag{A1a}$$

$$\omega^{r} = \left(1 - \frac{2M}{r}\right)^{-1/2} dr\,, \tag{A1b}$$

$$\omega^{\theta} = r d\theta\,, \tag{A1c}$$

$$\omega^{\phi} = \left(1 - \frac{2M}{r}\right)^{1/2}\left(1 - \frac{3M}{r}\right)^{-1/2} r \sin\theta d\Phi - \left(\frac{M}{r}\right)^{1/2}\left(1 - \frac{2M}{r}\right)^{1/2}\left(1 - \frac{3M}{r}\right)^{-1/2} dt. \tag{A1d}$$

This basis has no particular significance off the equator. (This is the $a = 0$ form of equations [III-6].) In terms of this basis, the connection forms determined via equation (III-9) are

$$\omega^{\tau}{}_{r} = -(M/r^3)^{1/2}\omega^{\phi}\,, \tag{A2a}$$

$$\omega^{\tau}{}_{\theta} = -\left(\frac{M}{r^3}\right)^{1/2}\left(1 - \frac{2M}{r}\right)^{1/2}\left(1 - \frac{3M}{r}\right)^{-1} \cot\theta\left[\omega^{\phi} + \left(\frac{M}{r}\right)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1/2}\omega^{\tau}\right], \tag{A2b}$$

$$\omega^{\tau}{}_{\phi} = -\frac{1}{2}\left(\frac{M}{r^3}\right)^{1/2}\left(1 - \frac{3M}{r}\right)^{-1}\omega^{r}\,, \tag{A2c}$$

$$\omega^{r}{}_{\theta} = -\frac{1}{r}\left(1 - \frac{2M}{r}\right)^{1/2}\omega^{\theta}\,, \tag{A2d}$$

$$\omega^{r}{}_{\phi} = -\left(\frac{M}{r^3}\right)^{1/2}\left[\omega^{\tau} + \left(\frac{M}{r}\right)^{-1/2}\left(1 - \frac{M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-1/2}\omega^{\phi}\right], \tag{A2e}$$

$$\omega^{\theta}{}_{\phi} = -\frac{1}{r}\cot\theta\left(1 - \frac{2M}{r}\right)\left(1 - \frac{3M}{r}\right)^{-1}\left[\omega^{\phi} + \left(\frac{M}{r}\right)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1/2}\omega^{\tau}\right]. \tag{A2f}$$

The components of the Riemann tensor in this basis follow from a fourfold Lorentz transformation $R_{\alpha\beta\gamma\delta} = \Lambda_{\alpha}{}^{(\alpha)}\Lambda_{\beta}{}^{(\beta)}\Lambda_{\gamma}{}^{(\gamma)}\Lambda_{\delta}{}^{(\delta)}R_{(\alpha)(\beta)(\gamma)(\delta)}$ of the components in the standard basis. The nonzero ones are

$$R_{\phi\theta\phi\theta} = -R_{\tau r\tau r} = \frac{2M}{r^3}\left(1 - \frac{3M}{r}\right)^{-1}\left(1 - \frac{3M}{2r}\right), \tag{A3a}$$

$$R_{\phi r\phi r} = -R_{\tau\theta\tau\theta} = \frac{-M}{r^3}\left(1 - \frac{3M}{r}\right)^{-1}, \tag{A3b}$$

$$R_{r\theta r\theta} = -R_{\tau\phi\tau\phi} = -M/r^3\,, \tag{A3c}$$

$$R_{\tau\theta\phi\theta} = -R_{\tau r\phi r} = \frac{3M}{r^3}\left(\frac{M}{r}\right)^{1/2}\left(1 - \frac{2M}{r}\right)^{1/2}\left(1 - \frac{3M}{r}\right)^{-1}. \tag{A3d}$$

APPENDIX B

REMARKS ON THE NUMERICAL METHODS EMPLOYED

For automated computation, we use a form for the gravitational potentials different from that of equations (V-4). For example,

$$A_1 = a_1a_2a_3 \int_0^\infty \frac{du}{[(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)]^{1/2}(a_1^2 + u)}$$

$$= \alpha_2\alpha_3 \int_0^\infty \frac{dz}{[(1 + z)(\alpha_2^2 + z)(\alpha_3^2 + z)]^{1/2}(1 + z)},$$

where $z = u/a_1^2$. Now let $z = \tan^2 x$; then

$$A_1 = 2\alpha_2\alpha_3 \int_0^{\pi/2} \frac{\sin x \cos^2 x dx}{[(\alpha_2^2 \cos^2 x + \sin^2 x)(\alpha_3^2 \cos^2 x + \sin^2 x)]^{1/2}}.$$

Simpson's $\frac{1}{3}$-rule was the scheme employed to evaluate the integrals. A solution for the polarization equation was then found via the Newton-Raphson method of iteration. The University of Maryland Univac 1108 computer performed the calculations; the aforementioned numerical methods are part of its "Math-Pack" library (University of Maryland Computer Science Center 1970). In all cases, the integrals were evaluated with an absolute accuracy of 10^{-6}; the subsequent solution of the polarization equation was carried out with an absolute accuracy of 10^{-5}.

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