1973ApJ...183..637B

THE ASTROPHYSICAL JOURNAL, **183**:637–648, 1973 July 15
© 1973. The American Astronomical Society. All rights reserved. Printed in U.S.A.

STABILITY OF GENERAL-RELATIVISTIC POLYTROPES

S. A. BLUDMAN*
Department of Physics, University of Pennsylvania, Philadelphia
Received 1972 June 26

ABSTRACT

Tooper's calculations of the parameters of the Lane-Emden functions for polytropes in general-relativistic hydrostatic equilibrium are extended, particularly in the direction of greater incompressibility of the equation of state. The contraction of the nonrelativistic homology group when general relativity enters in is discussed. The stability limits for general-relativistic polytropes are obtained. Letting $\sigma = (P/\rho c^2)_c$ measure the effects of general relativity, for $\gamma - \frac{4}{3} \lesssim 1.73\sigma$ the polytrope $P = K\rho^\gamma$ will be unstable against radial collapse. Particularly if neutronic matter is stiff, these results directly determine the maximum mass and radius of massive neutron stars to 5–10 percent and 10–20 percent accuracy, respectively.

Subject headings: equation of state — interiors, stellar — neutron stars — relativity

I. INTRODUCTION: POLYTROPIC EQUATIONS OF STATE

In this paper we present some calculations of the mass, radius, and stability of fluid spheres obeying the polytropic equation of state

$$P = K\rho^\gamma , \tag{1.1}$$

and subject to the Tolman-Oppenheimer-Volkoff (TOV) equation for hydrostatic equilibrium in general relativity,

$$-\frac{dP}{dr} = G\frac{(m + 4\pi r^3 P/c^2)}{r^2(1 - 2Gm/c^2 r)}\left(\rho + \frac{P}{c^2}\right), \tag{1.2}$$

$$m(r) = \int_0^r \rho 4\pi r^2 dr . \tag{1.3}$$

Here ρ is the total mass density and $m(r)$ the gravitational mass interior to radius r.

The calculations extend to higher incompressibilities and central pressures than earlier calculations (Tooper 1964). Since, at the densities obtaining in the most massive neutron stars, neutronic matter is probably fairly incompressible, one of our purposes is to show how incompressible matter is obtained as the $\gamma \to \infty$ limit of a polytrope. Another purpose is to discuss the stability against radial collapse of stars obeying such an equation of state; our conclusions in this regard, differ from those of Tooper. Finally, we present a curve (fig. 3) of the limiting central density and mass beyond which superdense stars (white dwarfs or neutron stars) will be unstable against gravitation collapse. As shown in the following paper (Bludman 1973), this curve practically determines the limits of stability of stiff neutron stars obeying more realistic equations of state than equation (1.1).

Any equation of state (including that of incompressible matter) can be written in the form (1.1) by defining

$$\gamma(\rho) \equiv \frac{\rho dP}{P d\rho} . \tag{1.4}$$

* Supported in part by the U.S. AEC.

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

Over a limited domain of densities, $\gamma \approx$ constant so that the polytropic approximation (1.1) will apply, with some exponent $\gamma \equiv 1 + 1/n$ or polytropic index n in that density domain. Polytropes are characterized by exactly constant γ or n and are important as approximations to realistic equations of state, historically, and because nonrelativistic polytropes obey a homology or scaling law. We will also discuss the place of homology when dealing with relativistic polytropes.

In terms of the energy density $\epsilon = \rho c^2$,

$$P = -\frac{d(\epsilon/\tilde{n})}{d(1/\tilde{n})} = \tilde{n}\frac{d\epsilon}{d\tilde{n}} - \epsilon\,, \tag{1.5}$$

where $\tilde{n}$ is the number density of particles of mass $m_{\rm N}$ so that $m_{\rm N}\tilde{n}$ is the proper mass density. In terms of the chemical potential

$$\mu \equiv \frac{d\epsilon}{d\tilde{n}} = \frac{P + \epsilon}{\tilde{n}}\,, \tag{1.6}$$

the TOV equation (1.3) reads

$$-d\mu/\mu = d(\nu/2)\,, \tag{1.7}$$

where

$$\nu(r) \equiv \int_\infty^r \frac{G(m + 4\pi r^3 P/c^2)}{r^2(1 - 2Gm/c^2r)}\,dr\,. \tag{1.8}$$

The TOV equation therefore has the integral form

$$\mu e^{\nu(r)/2} = \text{constant} = m_{\rm N}c^2\left(1 - 2\frac{GM}{c^2R}\right)^{1/2}, \tag{1.9}$$

where $m_{\rm N}c^2$ is the value of $\mu(r)$ at the stellar radius R where the total mass is $M = m(R)$. This equation determines the metric $g_{00} = e^{\nu(r)}$ in terms of the local values of the chemical potential, which is itself determined by local state variables in equation (1.6).

The polytropic form (1.1) or

$$K_1\rho = (P/\rho)^n\,, \qquad K_1 \equiv K^n\,, \tag{1.10}$$

corresponds to the equation of state

$$\rho = \rho(\tilde{n}) = m_{\rm N}\tilde{n}[1 - (K_1 m_{\rm N}\tilde{n})^{1/n}/c^2]^{-n}\,, \tag{1.11}$$

for equation (1.5) gives

$$P = m_{\rm N}c^2\tilde{n}^2\frac{d}{d\tilde{n}}\left[1 - \frac{(K_1 m\tilde{n})^{1/n}}{c^2}\right]^{-n}$$

or

$$\frac{P}{\rho} = \frac{(K_1 m_{\rm N}\tilde{n})^{1/n}}{1 - (K_1 m_{\rm N}\tilde{n})^{1/n}/c^2} = (K_1\rho)^{1/n}\,.$$

The chemical potential

$$\frac{\mu}{m_{\rm N}c^2} = \frac{d\rho}{d(m_{\rm N}\tilde{n})} = \left(1 + \frac{P}{\rho c^2}\right)^{1+n}. \tag{1.12}$$

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

In the nonrelativistic limit

$$\rho \approx m_N\tilde{n} + n\frac{P}{c^2} = m_N\tilde{n} + \frac{\epsilon_{\rm int}}{c^2} \tag{1.13}$$

$$\mu - m_N c^2 \approx (1+n)\frac{P}{\tilde{n}} = \gamma\epsilon_{\rm int}\,, \tag{1.14}$$

where $\epsilon_{\rm int}/\tilde{n} = -\int Pd(1/\tilde{n})$ is the internal energy per particle and $\epsilon_{\rm int} = nP = (\gamma - 1)^{-1}P$ is the internal energy per unit volume.

The exact equations (1.10)–(1.12) show that in the limit $n \to 0$

$$\rho = \text{constant} = 1/K_1\,, \tag{1.15}$$

$$\frac{\mu}{m_N c^2} = 1 + \frac{P}{\rho c^2}\,. \tag{1.16}$$

This describes constant density matter which is incompressible. Note that in this limit, $\epsilon = \rho c^2 = m_N\tilde{n}$ = constant, but the chemical potential $\mu = d\epsilon/d\tilde{n}$ still varies with the pressure.

One might also define polytropes by the equation of state

$$P = B\tilde{n}^\Gamma\,. \tag{1.17}$$

Since $\gamma = \rho dP/Pd\rho$ and $\Gamma = \tilde{n}dP/Pd\tilde{n}$, the speed squared of low-frequency sound

$$c_s^2 = dP/d\rho = c^2\gamma\sigma = c^2\Gamma\sigma/(1+\sigma)\,, \tag{1.18}$$

where $\sigma = P/\rho c^2$. The two exponents are related by

$$\gamma = \Gamma/(1+\sigma)\,,$$

so that while they agree nonrelativistically, they disagree where σ is appreciable. In particular, $\gamma(\rho)$ and $\Gamma(n)$ cannot both be constant.

If we had used equation (1.17), then

$$\rho = m_N\tilde{n} + \frac{B\tilde{n}^\Gamma}{c^2(\Gamma - 1)}\,. \tag{1.19}$$

This agrees with equation (1.13) nonrelativistically and gives

$$P = (\Gamma - 1)(\rho - m_N\tilde{n})c^2\,. \tag{1.20}$$

One cannot, however, explicitly invert (1.19) to eliminate $\tilde{n}(\rho)$ and obtain $P = P(\rho)$ for use in the TOV equation. Equation (1.20), if it holds in the relativistic region $\rho \gg mn$, leads to an equation of state $P \approx (\Gamma - 1)\rho c^2$ which is always too soft for stability. For these reasons, at least in degenerate matter, equation (1.17) is not a suitable relativistic generalization of a polytrope, and we use equation (1.1) through the remainder of this paper.

II. RELATIVISTIC LANE-EMDEN EQUATION

We begin by defining dimensionless radial, mass, and density coordinates

$$\xi = Ar\,, \tag{2.1}$$

$$v(\xi) = \frac{A^3}{4\pi\rho_c}m(r)\,, \tag{2.2}$$

$$\theta^n(\xi) = \rho/\rho_c\,, \tag{2.3}$$

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

where

$$A \equiv \left(\frac{4\pi G}{c^2}\frac{1}{n+1}\frac{\rho_c}{\sigma c^2}\right)^{1/2}, \tag{2.4}$$

$$\sigma = (P/\rho c^2)_c\,, \qquad \frac{\rho_c}{\sigma c^2} = \left(\frac{c^2}{K}\right)^n \sigma^{n-1}\,, \tag{2.5}$$

and subscript c refers to central ($r = 0$) values. Then equations (1.1), (1.2), and (1.3) take the form

$$\frac{P}{\rho c^2} = \frac{K}{c^2}\rho^{1/n} = \sigma\theta \tag{2.6}$$

$$\frac{dv}{d\xi} = \xi^2\theta^n\,, \tag{2.7}$$

$$-\xi^2\frac{d\theta}{d\xi} = \frac{(v + \sigma\theta\xi dv/d\xi)(1+\sigma\theta)}{1 - 2\sigma(n+1)v/\xi}\,. \tag{2.8}$$

The Lane-Emden functions are the solutions of equations (2.7) and (2.8) satisfying the condition $\theta = 1$ at $\xi = 0$ and $\theta = 0$ at $\xi = \xi_1$.

The polytropic and general-relativity features appear only in equation (2.8). The index σ is an index of the role of general relativity since for $\sigma \to 0$, equation (1.3) reduces to

$$-dP/dr = Gm\rho/r^2\,, \tag{2.9}$$

and equation (2.8) to

$$-\xi^2 d\theta/d\xi = v\,, \tag{2.10}$$

which together with equation (2.7) reproduces the ordinary (nonrelativistic) Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0\,. \tag{2.11}$$

a) Homology Transformations

The nonrelativistic Lane-Emden equation admits the homology or scale transformation

$$\xi \to \lambda_1\xi\,, \qquad \theta \to \lambda_2\theta\,, \qquad v \to \lambda_3 v\,, \tag{2.12}$$

provided

$$\lambda_3 = \lambda_1{}^3\lambda_2{}^n\,, \qquad \lambda_1\lambda_2 = \lambda_3\,,$$

i.e., provided

$$\lambda_2 = \lambda_1{}^{\varpi}\,, \qquad \lambda_3 = \lambda_1{}^{\varpi+1} \tag{2.13}$$

where $\varpi = 2/(n-1)$ (Chandrasekhar 1939). This homology theorem asserts that any solution $\theta(\xi)$ of the Lane-Emden equation of fixed index n generates a whole

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

equivalence class of solutions $\lambda_1^{\varpi}\theta(\lambda_1\xi)$ which correspond to different values of K and distributions of mass $m(r)$. It determines the evolution of stars of fixed γ or polytropic index.

Now the relativistic Lane-Emden equations (2.8), (2.7) depend on the general-relativity index σ as well as the polytropic index n. Under the scale transformations (2.12) a solution of the relativistic Lane-Emden equation of indices n, σ transforms generally into a solution with different indices n, $\lambda_2\sigma$. Only for $\varpi = 0$ ($n = \infty$ or isothermal equation of state $P = K\rho$) is the restricted scale transformation $\lambda \to \lambda_1\xi$, $v \to \lambda_1 v$ admitted.[1]

b) *Incompressible Matter*

For example, when $n = 0$ (incompressible matter of constant density ρ), the TOV equations (2.7), (2.8) have the solutions

$$v = \tfrac{1}{3}\xi^2 \tag{2.14}$$

$$\frac{P}{\rho c^2} = \sigma\theta = \frac{(1 + 3\sigma)(1 - x^2)^{1/2} - (1 + \sigma)}{3(1 + \sigma) - (1 + 3\sigma)(1 - x^2)} \tag{2.15}$$

where

$$x^2 = \tfrac{2}{3}\sigma\xi^2 = \frac{8\pi}{3}\frac{G\rho r^2}{c^2}\,. \tag{2.16}$$

The central pressure is given by

$$(P/\rho c^2)_c = \sigma\,,$$

and the value of r for which $P = 0$ determines the stellar radius

$$X^2 \equiv \frac{8\pi}{3}\frac{G\rho R^2}{c^2} = 4\sigma\frac{1 + 2\sigma}{(1 + 3\sigma)^2}\,. \tag{2.17}$$

From equation (2.14) the total energy or gravitational mass is

$$M = \tfrac{4}{3}\pi\rho R^3 = \frac{4\pi\rho}{3}\left(\frac{3}{8\pi}\frac{c^2}{G\rho}\right)^{3/2} X^2\,. \tag{2.18}$$

The baryon number A or proper mass is

$$m_{\rm N}A = \int\frac{m_{\rm N}\tilde{n}4\pi r^2 dr}{[1 - 2Gm(r)/c^2r]^{1/2}}$$

$$= 2\pi m_{\rm N}\tilde{n}\left(\frac{3}{8\pi}\frac{\rho^2}{G\rho}\right)^{3/2}[\sin^{-1} X - X(1 - X^2)^{1/2}]\,. \tag{2.19}$$

There is no homology group. Instead, if $R \to \lambda R$, $M \to \lambda M$ so that $2GM/c^2R = X^2$ stays invariant, then according to equations (2.17), $\rho \to \lambda^{-2}\rho$, $\tilde{n} \to \lambda^{-2}\tilde{n}$, and according to equation (2.19) $A \to \lambda A$, i.e., the star is transformed into one of different baryon number.

[1] Cf. Harrison *et al.* (1965), who discuss the scale invariance of the TOV equation but apparently fail to emphasize that the general-relativity homology group is restricted (to $n = \infty$) as compared with the nonrelativistic homology group (all n). Their reduction of the TOV equation to a single first-order equation between the "Bondi invariants" $m(r)/r$ and $4\pi r^2 P$ is applicable only to the Γ-law equation of state $P = (\Gamma - 1)\rho c^2$. Since for this equation $\gamma = 1$, such stars will always be unstable unless surrounded by an envelope of much stiffer material such that the average $\langle\gamma\rangle > \gamma_{\rm CR} > \frac{4}{3}$.

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

III. NUMERICAL CALCULATION OF RELATIVISTIC LANE-EMDEN PARAMETERS

The relativistic Lane-Emden functions $\theta(\xi)$, $v(\xi)$ depend on two indicial parameters n and σ. For $\sigma \to 0$ these reduce to the nonrelativistic Lane-Emden functions (Chandrasekhar 1939) and for $n \to 0$ to the case of incompressible matter, for which analytic solutions are possible in both the nonrelativistic and relativistic cases. While the nonrelativistic Lane-Emden equation can also be solved analytically for $n = 1, 5$, this is not true relativistically, and recourse must be had to numerical integration. This has been done by Tooper (1964) for $n = 3.0, 2.5, 2.0, 1.5, 1.0$ and for $\sigma < 1/\gamma = n/(n+1)$.

The speed of low-frequency sound waves is $c_s^2 = dP/d\rho = \gamma P/\rho = c^2\gamma(P/\rho c^2)$, so that at the star's center, where $P/\rho c^2$ has its maximum value σ, $c_s^2 = c^2\gamma\sigma$. If $\sigma > 1/\gamma$, the sound speed so calculated would exceed that of light. Especially because such a superluminal core would in any case be surrounded by ordinary subluminal matter, it is not clear that any such occurrence would have serious consequences except for the maximum M/R and surface redshift which it would permit (Bludman and Ruderman 1968).

TABLE 1

PARAMETERS OF THE RELATIVISTIC LANE-EMDEN FUNCTIONS

σ	ξ_1	$v(\xi_1)$	$\tilde{M}$
	$n = 3.0$, $\sigma_{CR} = 0$		
0..........	6.8999	2.0182	2.0182
0.1........	6.8289	1.0787	1.0787
0.2........	7.9525	0.7133	0.7133
0.3........	10.830	0.5388	0.5388
0.4........	17.799	0.4517	0.4517
0.5........	37.104	0.4214	0.4214
0.6........	90.749	0.4490	0.4490
0.7........	162.34	0.5260	0.5260
0.8........	187.27	0.5964	0.5964
0.9........	187.17	0.6372	0.6372
	$n = 2.5$, $\sigma_{CR} = 0.039$		
0..........	5.3570	2.1869	0
0.1........	4.7838	1.1694	0.6576
0.2........	4.7223	0.7608	0.5088
0.3........	4.9868	0.5558	0.4113
0.4........	5.5453	0.4387	0.3489
0.5........	6.4325	0.3665	0.3082
0.6........	7.7234	0.3203	0.2819
0.7........	9.5149	0.2905	0.2657
0.8........	11.884	0.2721	0.2573
0.9........	14.825	0.2618	0.2550
	$n = 2.0$, $\sigma_{CR} = 0.097$		
0..........	4.3538	2.4105	0
0.1........	3.7002	1.2987	0.4107
0.2........	3.3993	0.8404	0.3758
0.3........	3.3721	0.6056	0.3317
0.4........	3.2489	0.4681	0.2961
0.5........	3.2970	0.3801	0.2688
0.6........	3.3996	0.3202	0.2480
0.7........	3.5468	0.2774	0.2321
0.8........	3.7329	0.2457	0.2198
0.9........	3.9524	0.2217	0.2103

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

TABLE 1—*Continued*

σ	ξ_1	$v(\xi_1)$	$\tilde{M}$
$n = 1.5,\ \sigma_{CR} = 0.20$			
0..........	3.6541	2.7131	0
0.1........	3.0390	1.4821	0.2636
0.2........	2.7000	0.9604	0.2872
0.3........	2.4937	0.6884	0.2790
0.4........	2.3617	0.5271	0.2651
0.5........	2.2755	0.4227	0.2513
0.6........	2.2198	0.3506	0.2390
0.7........	2.1853	0.2985	0.2285
0.8........	2.1663	0.2594	0.2195
0.9........	2.1589	0.2292	0.2118
$n = 1.0,\ \sigma_{CR} = 0.42$			
0..........	3.1416	3.1402	0
0.1........	2.5993	1.7510	0.1751
0.2........	2.2773	1.1425	0.2285
0.3........	2.0645	0.8191	0.2457
0.4........	1.9135	0.6249	0.2500
0.5........	1.8012	0.4981	0.2491
0.6........	1.7146	0.4101	0.2461
0.7........	1.6461	0.3461	0.2423
0.8........	1.5906	0.2979	0.2383
0.9........	1.5450	0.2605	0.2345
$n = 0.9,\ \sigma_{CR} = 0.50$			
0..........	3.0553	3.2473	0
0.1........	2.5289	1.8199	0.1622
0.2........	2.2129	1.1899	0.2196
0.3........	2.0018	0.8537	0.2411
0.4........	1.8507	0.6512	0.2488
0.5........	1.7371	0.5187	0.2505
0.6........	1.6486	0.4267	0.2495
0.7........	1.5777	0.3597	0.2473
0.8........	1.5198	0.3092	0.2446
0.9........	1.4715	0.2700	0.2417
$n = 0.8,\ \sigma_{CR} = 0.61$			
0..........	2.9736	3.3640	0
0.1........	2.4631	1.8955	0.1506
0.2........	2.1534	1.2423	0.2115
0.3........	1.9447	0.8920	0.2372
0.4........	1.7940	0.6804	0.2483
0.5........	1.6798	0.5418	0.2527
0.6........	1.5901	0.4453	0.2539
0.7........	1.5177	0.3750	0.2533
0.8........	1.4580	0.3219	0.2519
0.9........	1.4078	0.2807	0.2500
$n = 0.7,\ \sigma_{CR} = 0.75$			
0..........	2.8960	3.4916	0
0.1........	2.4016	1.9784	0.1401
0.2........	2.0986	1.3003	0.2043
0.3........	1.8927	0.9347	0.2341
0.4........	1.7429	0.7131	0.2486
0.5........	1.6285	0.5676	0.2558
0.6........	1.5382	0.4662	0.2591
0.7........	1.4647	0.3922	0.2603
0.8........	1.4037	0.3364	0.2603
0.9........	1.3522	0.2930	0.2596

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

TABLE 1—*Continued*

σ	ξ_1	$v(\xi_1)$	$\tilde{M}$
$n = 0.6$, $\sigma_{CR} = 0.97$			
0..........	2.8224	3.6316	0
0.1........	2.3439	2.0709	0.1307
0.2........	2.0478	1.3648	0.1978
0.3........	1.8450	0.9823	0.2316
0.4........	1.6965	0.7497	0.2497
0.5........	1.5824	0.5966	0.2597
0.6........	1.4918	0.4898	0.2653
0.7........	1.4177	0.4118	0.2684
0.8........	1.3559	0.3529	0.2700
0.9........	1.3034	0.3071	0.2706
$n = 0.5$, $\sigma_{CR} = 1.24$			
0..........	2.7523	3.7860	0
0.1........	2.2898	2.1735	0.1222
0.2........	2.0008	1.4358	0.1920
0.3........	1.8013	1.0350	0.2298
0.4........	1.6544	0.7910	0.2516
0.5........	1.5409	0.6295	0.2647
0.6........	1.4503	0.5164	0.2727
0.7........	1.3759	0.4340	0.2779
0.8........	1.3136	0.3717	0.2812
0.9........	1.2604	0.3231	0.2833
$n = 0.4$			
0..........	2.6857	3.9582	0
0.1........	2.2389	2.2882	0.1147
0.2........	1.9572	1.5182	0.1874
0.3........	1.7613	1.0963	0.2292
0.4........	1.6160	0.8378	0.2546
0.5........	1.5034	0.6664	0.2707
0.6........	1.4130	0.5471	0.2816
0.7........	1.3385	0.4596	0.2891
0.8........	1.2759	0.3932	0.2942
0.9........	1.2224	0.3416	0.2979
$n = 0.3$			
0..........	2.6222	4.1498	0
0.1........	2.1911	2.4178	0.1080
0.2........	1.9166	1.6102	0.1833
0.3........	1.7244	1.1653	0.2294
0.4........	1.5811	0.8916	0.2588
0.5........	1.4695	0.7099	0.2785
0.6........	1.3795	0.5824	0.2922
0.7........	1.3050	0.4885	0.3018
0.8........	1.2424	0.4182	0.3094
0.9........	1.1886	0.3631	0.3150
$n = 0.2$			
0..........	2.5617	4.3664	0
0.1........	2.1461	2.5648	0.1021
0.2........	1.8789	1.7160	0.1803
0.3........	1.6905	1.2447	0.2307
0.4........	1.5493	0.9535	0.2644
0.5........	1.4387	0.7596	0.2878
0.6........	1.3493	0.6233	0.3049
0.7........	1.2751	0.5228	0.3173
0.8........	1.2124	0.4470	0.3271
0.9........	1.1585	0.3881	0.3349

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

TABLE 1—*Continued*

σ	ξ_1	$v(\xi_1)$	$\tilde{M}$
	$n = 0.1$		
0..........	2.5040	4.6089	0
0.1........	2.1037	2.7308	0.09689
0.2........	1.8438	1.8377	0.1781
0.3........	1.6593	1.3367	0.2333
0.4........	1.5202	1.0254	0.2716
0.5........	1.4108	0.8172	0.2991
0.6........	1.3220	0.6710	0.3199
0.7........	1.2482	0.5634	0.3359
0.8........	1.1856	0.4816	0.3485
0.9........	1.1317	0.4176	0.3585
	$n = 0$, $\sigma_{CR} = \infty$		
0..........	2.449	4.898	0
0.1........	2.064	2.931	0.0926
0.2........	1.811	1.980	0.1770
0.3........	1.63	1.45	0.2382
0.4........	1.493	1.110	0.2808
0.5........	1.39	0.890	0.3146
0.6........	1.30	0.728	0.3384
0.7........	1.22	0.610	0.3573
0.8........	1.161	0.522	0.3735
0.9........	1.108	0.453	0.3868
1.0........	1.061	0.398	0.3977

We have extended Tooper's calculations to smaller n (stiffer matter) and larger σ. The results, calculated by Mr. Richard Adams using the classical Runge-Kutta method, are given in table 1. Here ξ_1 is the first zero of the Lane-Emden function $\theta(\xi)$ so that the stellar radius is

$$R = A^{-1}\xi_1 = \left[\frac{c^2}{4\pi G}(n+1)\sigma^{1-n}\left(\frac{K}{c^2}\right)^n\right]^{1/2}\xi_1\,, \tag{3.1}$$

and the stellar mass

$$M = \frac{4\pi\rho_c}{A^3}\,v(\xi_1) = \left[\frac{1}{4\pi}\left(\frac{(n+1)c^2}{G}\right)^3\left(\frac{K}{c^2}\right)^n\right]^{1/2}\tilde{M}\,, \tag{3.2}$$

$$\tilde{M} \equiv \sigma^{(3-n)/2}v(\xi_1)\,. \tag{3.3}$$

The tabulated values for ξ_1 and $v(\xi_1)$ agree with the nonrelativistic ($n = 0$) and Tooper values [for $n = 1.0$ and $\sigma < n/(n + 1)$] to better than 1 percent.

Other useful functions that we do not tabulate are $GM/c^2R = \sigma(n + 1)v(\xi_1)/\xi_1$, which determines the surface redshift, and $\rho_c/\bar{\rho} = \frac{1}{3}\xi_1{}^3/v(\xi_1)$, which expresses how much central compression of density takes place.

In figures 1 and 2 we plot $v(\xi_1)$ and $\tilde{M}$ as functions of n and σ. $\tilde{M}$ and M increase with $\sigma = K\rho_c{}^{1/n}$ up to some maximum value σ_{CR}. Since a necessary condition of stability is $dM/d\rho_c > 0$, σ_{CR} marks the onset of the first mode of radial instability.[2]

[2] Tooper (1964) considers the case $n = 3.0$ and observes the *minimum* in $v(\xi_1) = \tilde{M}$ at $\sigma = 0.5$, implying that the $n = 3$ relativistic polytrope is stable for $\sigma < 0.5$. This is misleading since the $n = 3$ polytrope is already marginally unstable nonrelativistically and is rendered unstable by the smallest effects of general relativity. $\tilde{M}$ has a maximum at $\sigma = 0$; the minimum at $\sigma = 0.5$ marks the onset of the next mode of nonradial instability.

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

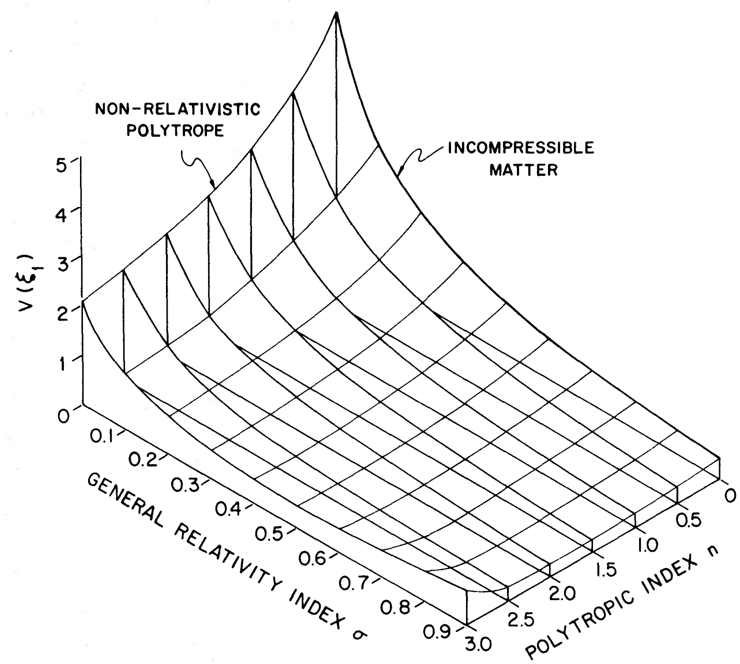

FIG. 1.—Relativistic Lane-Emden function $v(\xi_1)$ determining the stellar mass plotted as a function of polytropic index n and general-relativity index σ. The curve for $\sigma = 0$ reduces to the nonrelativistic Lane-Emden function and for $n = 0$ to incompressible matter for which an analytic solution is possible. The function $v(\xi_1)$ decreases with increasing n as the equation of state softens and with increasing σ as the effects of general relativity become more important.

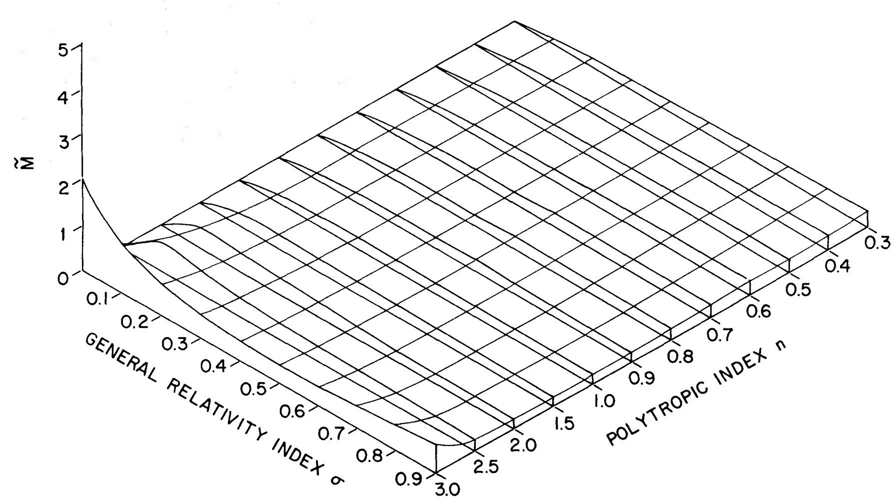

FIG. 2.—The function $\tilde{M} = \sigma^{(3-n)/2} v(\xi_1)$ which determines the stellar mass by equation (3.2). The maxima in M as a function of σ determine the values of σ and n or γ plotted in fig. 3. Except for large n and small σ, $\tilde{M}$ shows a broad maximum value about 0.25.

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

The maxima in $\tilde{M}$ were located by using a finer mesh than the 0.1 intervals in σ appearing in table 1 and are plotted in figure 3. The stability curve plotted is, for $\sigma < 1$,

$$\gamma - \tfrac{4}{3} \simeq 1.73\sigma - 0.31\sigma^2 \quad (3.4)$$

to within a few percent. Since $\sigma \sim GM/c^2R$, this implies the adequacy of a first post-Newtonian approximation. Equation (3.4) shows how, for a given value of σ, the minimum value of γ necessary for stability is raised above the value $\frac{4}{3}$ which would be sufficient for stability in Newtonian theory. Conversely, a polytrope of given exponent γ or polytropic index n becomes unstable when the dimensionless general-relativity index σ exceeds $\sigma_{\rm CR}$. The numerical value of the central density ρ_c at which this happens is then determined by the dimensional parameter K through $\sigma = K\rho_c^{1/n}$.

Also drawn on figure 3 is the "causality limit" curve $\sigma < 1/\gamma$. This intersects the $\sigma_{\rm CR}$ curve at $\sigma_{\rm CR} = 0.48$, where $\gamma = 2.084$, $n = 0.926$. The region of high γ and low σ for which the equation of state is causal and the star is stable is marked. The maximum value of σ is set by general relativity for $\gamma < 2.084$ and by causality for $\gamma > 2.084$. As discussed in the following paper, in a massive neutron star, most of the mass is contained in the core region, for which, especially if the equation of state is stiff, a polytropic approximation applies. Hence, in dealing with stiff equations of state, the maximum neutron-star mass will obtain at that central density $\rho_c \sim 4 \times 10^{15}$ g cm^{-3} for which $\sigma = \sigma_{\rm CR}$ in figure 3.

IV. CONCLUSIONS

We have extended to smaller n and larger σ Tooper's calculations of the parameters of the Lane-Emden functions for polytropes in general-relativistic hydrostatic

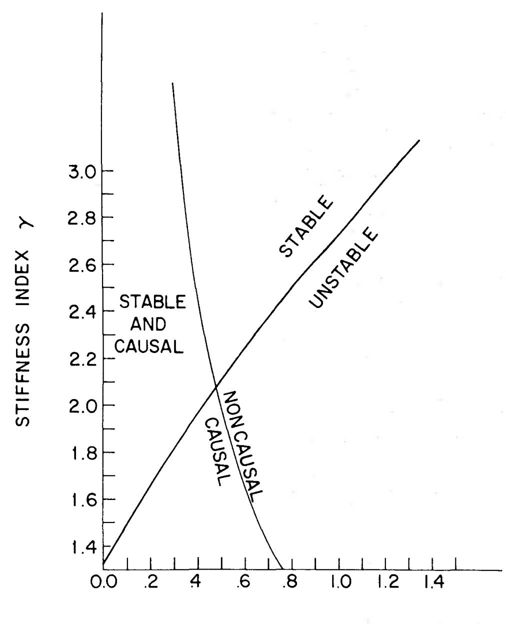

FIG. 3.—Critical value of the general-relativity index $\sigma = (P/\rho c^2)_c$ above which a polytrope is destabilized by general relativity effects. (Calculated by Mr. Richard Adams.) Also shown is the curve $\sigma = 1/\gamma$ above which the sound speed at the center of the star $c_s^2 = dP/d\rho = \gamma P/\rho$ will exceed that of light. The two curves intersect at $\sigma = 0.48$, $\gamma = 2.084$. Regions in which the equation of state at the star's center is causal and noncausal and for which the star is stable or unstable are indicated.

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1973ApJ...183..637B

equilibrium. Given in table 1 are ξ_1, the zero of $\theta(\xi)$ which determines the stellar radius R by equation (3.1), and $v(\xi_1)$ and $\tilde{M}$, which determine the stellar mass M by equations (3.2) and (3.3). Figures 1 and 2, showing $v(\xi_1)$ and $\tilde{M}$, respectively, as functions of n and σ, illustrate how the stellar mass increases with stiffness of equation of state and decreases with the effects of general relativity.

For given $\sigma = (P/\rho c^2)_c$ there is a minimum $\gamma > \frac{4}{3}$, or for given γ a maximum σ, beyond which instability against radial collapse takes place. These critical values of σ or γ are plotted in figure 3 and are approximately related by $\gamma - \frac{4}{3} = 1.73\sigma$.

If neutronic matter is fairly stiff at transnuclear densities then, especially for neutron stars of nearly maximum mass, most of the mass is contributed by a central volume over which the equation of state is approximately polytropic. The results of this paper are therefore applicable to the determination of the maximum mass of such stiff neutron stars (Bludman 1973).

I am indebted to Mr. Richard Adams for the careful numerical calculations, particularly of table 1 and figure 3.

REFERENCES

Bludman, S. A. 1973, *Ap. J.*, **183**, 649.
Bludman, S. A., and Ruderman, M. A. 1968, *Phys. Rev.*, **170**, 1176.
Chandrasekhar, S. 1939, *An Introduction to the Study of Stellar Structure* (Chicago: University of Chicago Press; New York: Dover Publications, 1957).
Harrison, B. K., Thorne, K., Wakano, M., and Wheeler, J. A. 1965, *Gravitational Theory and Gravitational Collapse* (Chicago: University of Chicago Press).
Tooper, R. F. 1964, *Ap. J.*, **140**, 434.

© American Astronomical Society • Provided by the NASA Astrophysics Data System