THE ASTROPHYSICAL JOURNAL, 162:687–697, November 1970

STRUCTURE OF JUPITER: CHEMICAL COMPOSITION, CONTRACTION, AND ROTATION

W. B. HUBBARD
Department of Astronomy, University of Texas at Austin
Received 1970 February 16; revised 1970 June 8

ABSTRACT

A new model of Jupiter is constructed by means of an improved equation of state, derived in this paper, which includes the effects of electron screening on the vibrational modes of the metallic-hydrogen fluid. The proposed model has a hydrogen mass fraction of approximately 0.6 and a central temperature of the order of 7500° K. The observationally determined luminosity is then used to investigate the time-dependent properties of the model, on the assumption that the planet remains in complete convective equilibrium and rotates as a solid body. We find that, as a function of time, the radius and rotation period decrease, whereas the gravitational moments J and K and the rotational kinetic energy increase. Only a small fraction of the total energy release due to cooling goes into rotational kinetic energy, but the total energy reservoir is only just adequate to explain the observed excess flux of energy. The time rate of change of the rotation period and the gravitational moments appears to be unobservably small.

I. INTRODUCTION

The decametric rotation period of Jupiter is now known to within one part in a million (Donivan and Carr 1969), and a fairly definite value has now been assigned to the net Jovian luminosity (Aumann, Gillespie, and Low 1969). The hydrogen-to-helium ratio for Jupiter is still highly uncertain, although planned space experiments may yield improved results in the near future. With the present observational constraints, the purpose of this paper is (*a*) to construct thermally expanded model planets by means of an improved equation of state and (*b*) to study the time dependence of the planetary structure.

In Paper I (Hubbard 1968) it was shown that completely convective structure for Jupiter is implied by the observed net luminosity, and in Paper II (Hubbard 1969) adiabatic model planets were constructed. However, in Paper II it was assumed that the effect of electron screening on the thermal properties of metallic hydrogen at Jovian densities is negligible. In other words, the normal modes of vibration of the proton fluid were assumed to be represented by the normal modes of a one-component plasma on a uniform negative background. It now appears that this assumption is somewhat crude for Jovian densities, since it is necessary to include the effects of "yield" in the electron fluid to calculate accurate model planets. Inclusion of electron-screening effects can be expected to have an influence on the H/He ratio and the internal temperature distribution. In addition, the contraction of the planet is governed by the thermal terms in the equations of state, which are in turn influenced by electron screening.

In this paper, we calculate the effect of electron screening on the finite-temperature thermodynamics of metallic hydrogen (§ II), use the improved equations of state to construct a model of Jupiter (§ III), and study the time-dependent structure of the planet (§ IV). In regard to the latter, it should be noted that the model planets of Papers I and II actually imply a secular contraction, although the contraction rate was not explicitly calculated. Evidently, contraction implies a change in the moment of inertia, and therefore secular changes in the rotation period and gravitational moments J and K are to be expected. The rate of change of these quantities is calculated in § IV.

II. EFFECT OF ELECTRON SCREENING ON THE THERMODYNAMICS OF A HIGH-PRESSURE COULOMB FLUID

a) Introduction

The effect of electron screening on the vibrational modes of a Coulomb solid at high pressure has been considered by Kopyshev (1965), Trubitsyn (1966), and Ashcroft (1969). Kopyshev's calculation uses the Thomas-Fermi model, while Ashcroft employs the electron dielectric function. Trubitsyn's calculation considers the effect of local changes of the electron correlation energy and appears to omit the most dominant effects of the polarization of the electron fluid. Since the probable state of the Jovian interior is that of a dense fluid rather than a solid, we will consider this case in the following calculations, using a dielectric-function approach similar to that of Ashcroft.

b) Model of the Coulomb Fluid

We will assume that the Jovian matter consists of a densely packed fluid in which there exists short-range order but no long-range order. We focus attention on one ion of charge $+Ze$ located at the center of a Wigner-Seitz cell of radius

$$a = (3/4\pi n_i)^{1/3} , \tag{1}$$

where n_i is the number density of ions. The remainder of the fluid is assumed to be described by a distribution of positive charge of density Zn_ie spread uniformly in space except within the Wigner-Seitz cell, where the density is zero. To this we add a distribution of electrons which consists of a uniform background density together with a self-consistent polarization charge induced by the presence of the Wigner-Seitz cell and the point charge. We then displace the ion by an infinitesimal amount ξ from the center of the Wigner-Seitz cell and again calculate the self-consistent electron-charge distribution. The difference in energy between the perturbed and unperturbed states gives the restoring force and thus the vibrational frequency of the ion. The calculation is a routine exercise in linear-response theory, and leads to the result for the difference in energy:

$$\delta E = \tfrac{2}{3}\pi n_i Z^2 e^2 \xi^2 (1 - I_d) . \tag{2}$$

Here

$$I_d = (8\pi^3 Z n_i)^{-1} \int_0^\infty 4\pi k^2 dk Q_k f_k / \epsilon_k^2 , \tag{3}$$

where

$$f_k = 4\pi Z n_i (\sin ka - ka \cos ka)/k^3 \tag{4}$$

and

$$\epsilon_k = 1 + Q_k = 1 + \frac{4}{\pi a_0 k} \int_0^{k_F/k} x dx \ln \frac{1 + 1/2x}{|1 - 1/2x|} , \tag{5}$$

the usual static dielectric function of a zero-temperature electron gas. Here a_0 is the Bohr radius and k_F is the electron Fermi wavenumber.

The above calculation does not use antisymmetrized electron wave functions, and therefore it is necessary to add an exchange correction to the energy difference. This can be done in two ways: one can use a dielectric function with an approximate exchange correction included (J. Hubbard 1957), or one can calculate the exchange interaction between the perturbed electron clouds resulting from the displacement of the ion, following the method of Wigner and Huntington (1935). In this discussion we have chosen to use the latter method, with the result

$$\delta E = (2\pi/3) n_i Z^2 e^2 \xi^2 (1 - I_d - I_x) , \tag{6}$$

where

$$I_x = \frac{2}{\pi}\int_0^\infty \frac{dx}{x}\frac{Q_x^2}{(1+Q_x)^2}\, g(x)(\sin ax - ax\cos ax)\ . \tag{7}$$

Here

$$x = k/k_F\ , \qquad a = (\tfrac{9}{4}\pi Z)^{1/3}\ ,$$

and

$$g(x) = 9\int_0^\infty dy \sin y[\sin(y/x) - (y/x)\cos(y/x)]^2(y/x)^6\ . \tag{8}$$

The exchange interaction between the perturbed and unperturbed electron clouds can be shown to vanish identically.

For completeness, one should perhaps include the change in energy due to the change of electron correlation energy. Trubitsyn has estimated this (the only screening effect he has taken into account) and has shown it to be negligibly small, and therefore we will not include it.

The vibrational frequency of the ion in its unit cell thus becomes

$$\omega = 3^{-1/2}\omega_P(1 - I_d - I_x)^{1/2}\ , \tag{9}$$

where ω_P is the ion plasma frequency. Since I_d and I_x are both positive, the effect of electron screening is of course to reduce the vibrational frequency. If the sum of I_d and I_x were to become greater than unity, an imaginary frequency would result, implying instability of the unit cell and the fluid metallic phase. The present calculations do not indicate that this occurs at densities where metallic hydrogen is thought from other considerations to be stable (De Marcus 1958).

c) Expansion in Powers of r_s

It is possible to expand expression (9) for the vibrational frequency in powers of $r_s = a/a_0$, in a manner analogous to the expansion obtained by Salpeter (1961) for the pressure of a zero-temperature plasma. We find

$$1 - I_d - I_x = 1 - r_s/r_s^0 + O(r_s^{3/2})\ , \tag{10}$$

where

$$1/r_s^0 = \frac{2}{\pi^2}\int_0^\infty \frac{dx}{x^3}\left[1 - \frac{\sin 2ax}{2ax}\right]\ln\frac{1+x}{|1-x|}\ . \tag{11}$$

In Figure 1 we have plotted r_s^0 as a function of Z. At densities of the order of and less than the density corresponding to r_s^0, the effect of electron screening on ion vibration can be expected to be dominant. The effect does not appear to be important at white-dwarf densities, except possibly for the case of very high Z.

The expansion (10) is not particularly useful except as a criterion, due to its extremely slow convergence (in contrast to Salpeter's pressure expansion).

d) Thermodynamics

We will assume that the thermodynamics of the fluid metallic hydrogen can be described by the high-temperature ($T > \Theta$) Debye theory, with the characteristic Debye temperature

$$\Theta = (\hbar/k_B)C_\theta\omega\ . \tag{12}$$

Here k_B is Boltzmann's constant and C_θ is a dimensionless constant of order unity. The density dependence of Θ is of prime importance for the convective Jovian models and the precise value of C_θ is necessary only for determining the absolute value of the entropy corresponding to a particular adiabat. We set $C_\theta = 1$ hereafter.

For a mixture of hydrogen and helium, the entropy per heavy particle is assumed, following Debye theory, to be given by

$$S/Nk_B = 3 \ln (k_B T/\hbar\bar{\omega}) , \tag{13}$$

where N is the total number of heavy particles, T is the absolute temperature,

$$\ln \bar{\omega} = \phi_H \ln \omega_H + \phi_{He} \ln \omega_{He} ,$$

and ϕ_H, ϕ_{He} are the number fractions of hydrogen and helium, respectively. Thus, we assume that the protons and α-particles vibrate independently, each with the frequency of its own Wigner-Seitz cell. We have $\omega_H = \omega_H(r_{s,H})$, $\omega_{He} = \omega_{He}(r_{s,He})$ through equation (9).

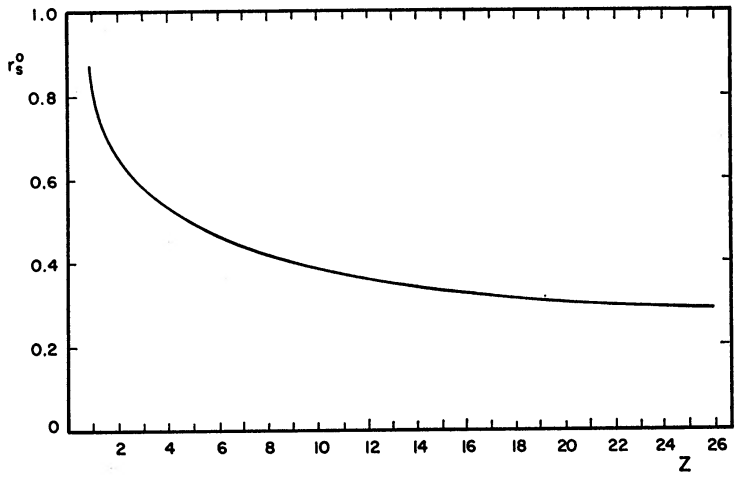

FIG. 1.—Density parameter r_s^0 as a function of the atomic number Z. Electron screening strongly affects the ion vibrations for $r_s \gtrsim r_s^0$.

Under these assumptions, the course of an adiabat in the temperature-density plane is given by

$$\ln T = \ln C + \tfrac{1}{2}\phi_H \ln \rho_H + \tfrac{1}{2}\phi_{He} \ln \rho_{He} + \phi_H \ln q_H + \phi_{He} \ln q_{He} , \tag{14}$$

where

$$q_{H,He} = (1 - I_d - I_x)_{H,He}^{1/2} , \tag{15}$$

$\rho_{H,He}$ is the partial density in grams per cm^3, and C is a constant which labels the adiabat.

There is at present no theoretical guide to the effect of a chemical mixture on adiabats under the temperature and pressure conditions encountered in the interior of Jupiter; therefore, we have considered an alternative adiabatic law to test the sensitivity to the assumed multicomponent adiabats. The alternative form is

$$\ln T = \ln C + \tfrac{1}{2} \ln \rho + \phi_H \ln q_H + \phi_{He} \ln q_{He} . \tag{16}$$

Here ρ is the total mass density. In the absence of screening, the alternative form reduces to the adiabatic law used in Paper II. For a single-component system, e.g., pure hydrogen, forms (14) and (16) are of course identical.

The pressure in this model, for either adiabatic law, is given by

$$P = P_0 + P_T , \tag{17}$$

where P_0 is the pressure at zero temperature, given in Paper II, and

$$P_T = 3\gamma n_i k_B T , \tag{18}$$

where γ is the well-known Grüneisen parameter:

$$\gamma = d \ln \bar{\omega} / d \ln \rho . \tag{19}$$

In the absence of screening, $\gamma = 0.5$. The effect of screening is to increase γ.

The internal energy is unaffected by screening and is given by

$$E = E_0 + 3 n_i k_B T , \tag{20}$$

where E_0 is the internal energy at zero temperature.

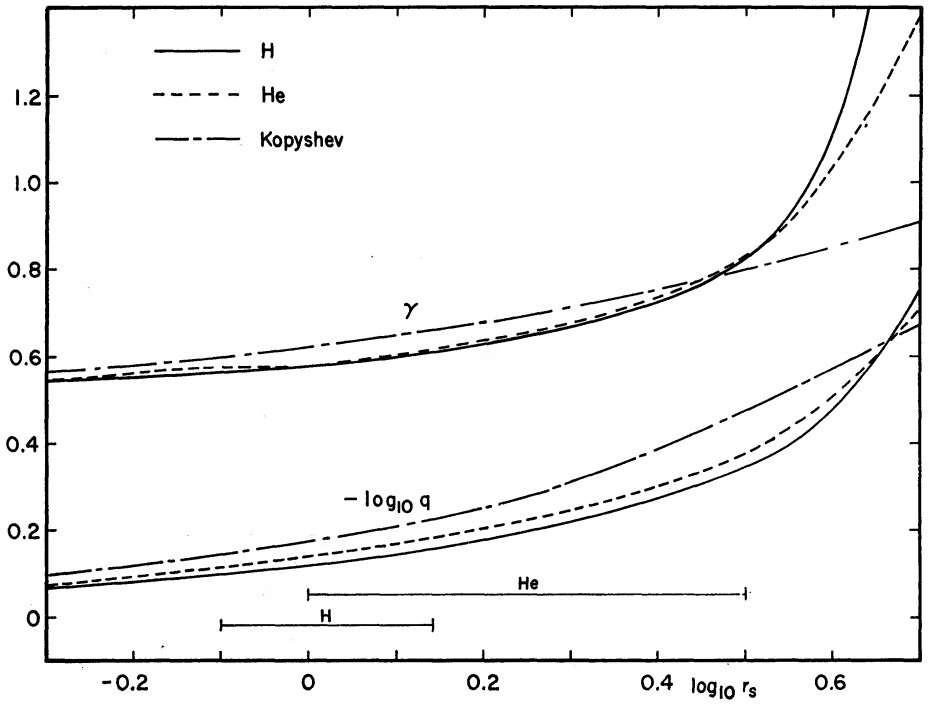

FIG. 2.—The Grüneisen parameter, γ, and correction factor to the ion frequency, q, as a function of the density parameter r_s. Range of r_s for hydrogen and helium in Jovian models is indicated by the bars.

In Figure 2 are presented the results of the present calculations for hydrogen and helium, together with Kopyshev's result for a hydrogen lattice. Since Kopyshev's calculation uses Thomas-Fermi theory without exchange, there exists a simple scaling law, such that Kopyshev's curves for any other Z should be displaced horizontally to the left by an amount $\frac{1}{3} \log_{10} Z$. Our results have been normalized to Kopyshev's by assuming that C_θ is the same in the lattice and dense-liquid phase; this assumption does not affect the position of the curves for γ.

e) Transport Coefficients

The calculation of thermal conductivity for the high-pressure solid state given by Hubbard and Lampe (1969) can be easily modified to include electron screening. We will assume that this result is also approximately correct for the dense liquid. Screening

modifies the mean square amplitude of an ion about its equilibrium position and also modifies the cross-section for electron-ion collisions. The result of the calculation is that the thermal conductivities given by Hubbard and Lampe must be multiplied by a correction factor

$$G/G_0 = 2q^2 \Big/ \int_0^2 x dx \epsilon^{-2}(ax) \, . \tag{21}$$

The correction factors for hydrogen and helium are plotted in Figure 3 and are seen to be of order unity for the densities of interest. This agrees with the estimate given in Paper I for the thermal conductivity of metallic hydrogen.

To the extent that inelastic electron scattering is unimportant, the electrical conductivity (given in Paper I) and the shear viscosity (calculated by Hubbard 1966) must be multiplied by approximately the same factor G/G_0.

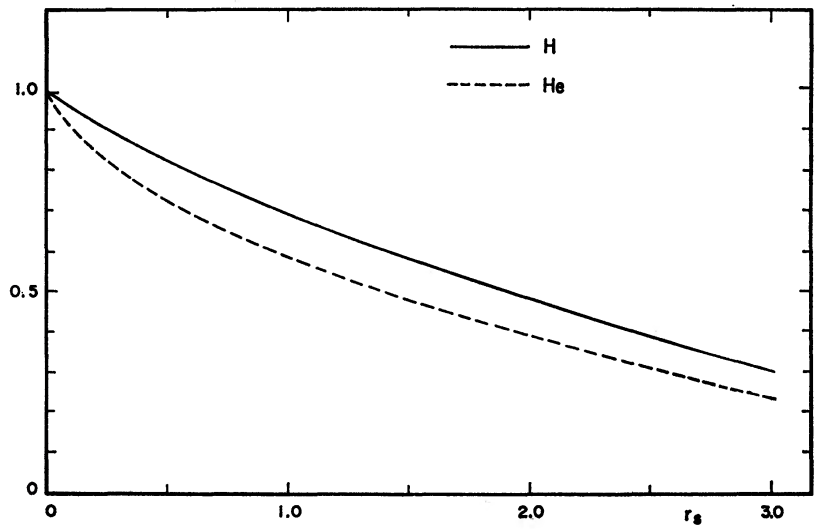

FIG. 3.—Correction factor for the thermal conductivity of metallic hydrogen and helium

f) Net Effect of Electron Screening

Electron screening should not have an extreme effect on static Jovian models. The adiabats become steeper in the (T, ρ)-plane, but the adiabatic (P, ρ)-relations are not so greatly affected since the increase of the Grüneisen parameter with decreasing density tends to offset the more rapid temperature drop. One conclusion reached in Paper I seems to be vitiated: it was concluded there that the adiabatic gradient $\nabla_s = d \ln T / d \ln P$ becomes smaller than 0.3 with decreasing density. This conclusion was based upon the assumption of $\gamma = 0.5$ at all densities. In the present calculation, we find that $\nabla_s \simeq 0.3$ at all densities. This result does not affect the general conclusion of Paper I, that Jupiter must be completely convective with the observed energy flux.

III. JUPITER MODELS

In order to facilitate comparison, we have calculated models of Jupiter under the same assumptions as in Paper II, except that the effect of electron screening is included in the thermal-perturbation terms for the metallic core. A zero-temperature equation of state can be readily calculated from the physics presented in § II and can be shown to agree with the equation of state of De Marcus (1958) and Salpeter and Zapolsky (1967)

to order r_s^2. However, we will continue to use the Salpeter-Zapolsky equation of state, both for convenience and because of their very careful evaluation of the correlation pressures. The input to the present models is thus identical to the input of model J7 in Paper II, except for the inclusion of electron screening. In addition, the present models are calculated on the basis of the radio rotation period of Donivan and Carr, which is 0.7 percent smaller than the mean optical rotation period used in Paper II.

If the adiabatic constant C and the hydrogen abundance by mass X are chosen as the only free parameters, the best fit to the observed mass, radius, and gravitational moments J and K is obtained for $X = 0.66$, $C = 4600$. This model (model J8) is computed on the basis of equation (16). To check its sensitivity to the adiabatic law, we have constructed model J9 using equation (15). Model J9 has a hydrogen abundance of 0.59, and $C = 5130$. Other properties of models J8 and J9 are given in Table 1 and Figure 4.

TABLE 1

JUPITER MODELS J8 AND J9

Parameter	J8	J9
Central density (g cm^{-3})	4.23	4.25
Central temperature (° K)	7300	7500
Mass (10^{30} g)	1.902	1.902
Equatorial radius (10^9 cm)	7.14	7.14
J	0.02203	0.02206
K	0.00250	0.00250
X	0.66	0.59
Y	0.34	0.41
Total angular momentum (10^{45} g cm^2 sec^{-1})	4.53	4.52
Rotation kinetic energy (10^{41} ergs)	3.98	3.97
Thermal energy (10^{41} ergs)	10.2	9.7

NOTE.—X and Y are the hydrogen and helium abundances by mass; negligible heavy-element content is assumed. These models are chemically homogeneous and do not possess a high-density core.

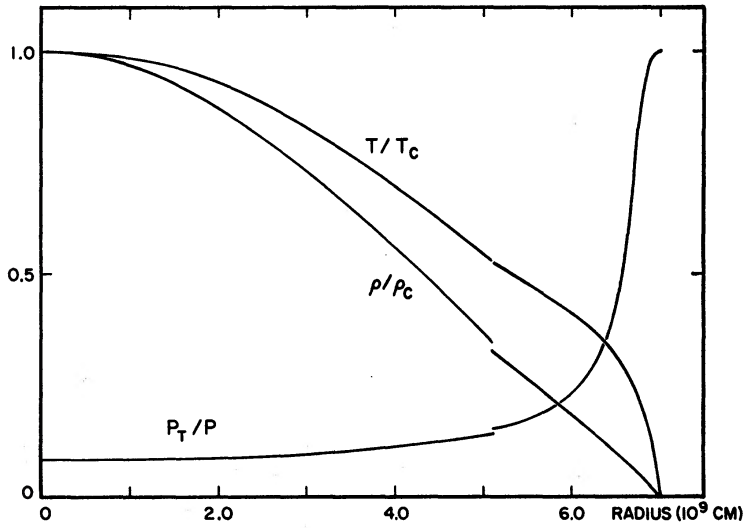

FIG. 4.—Run of temperature T, density ρ, and thermal pressure perturbation P_T/P as a function of radius for model J8. Here T_c and ρ_c are the central values.

Thus we find that, as expected, the inclusion of electron screening does not greatly change the hydrogen abundance of Jupiter. However, the models are still somewhat underabundant in hydrogen compared with the Sun ($X \simeq 0.75$).

The important effect of electron screening is that the central temperature is lower by nearly a factor of 2 than in the models without screening. This has an important effect on the thermal-energy reservoir, since it reduces it by essentially a factor of 2.

IV. CONTRACTION AND ROTATION

Ultimately, when the constituent relations become better known, it will be desirable to follow the evolution of a Jovian-mass object by means of a Henyey-type program of stellar evolution. As a preliminary step, we consider the current evolution of Jupiter under the assumption of completely adiabatic structure. To this end, we calculate adiabatic models J8+ with $C = 5000$ and J8− with $C = 4200$. The moment of inertia I of the model is calculated to second order in the oblateness by means of the formula

$$I = \tfrac{2}{3}\int_0^M dm s^2(1 - \epsilon_2 + \tfrac{6}{35}\epsilon_2^2 - \tfrac{1}{5}s\epsilon_2') , \tag{22}$$

where dm is the increment of mass $4\pi s^2\rho ds$, s is the mean radius of a surface of constant density, and ϵ_2 is the oblateness coefficient defined by Peebles (1964). We then assume that the model J8± rotates rigidly and calculate a new period of rotation which gives the model the same angular momentum as the model J8. Using the new period of rotation, we calculate a new model J8±, and the iterations proceed until angular momentum is conserved to within one part in 10^{-5}.

Next, we evaluate the difference in energy between J8 and J8±. The difference in total energy is given by

$$\delta E_{\text{total}} = \int_0^M dm(\delta\mathfrak{E} - P\delta\rho/\rho^2) + \delta E_{\text{rot}} , \tag{23}$$

where $\mathfrak{E}$ is the internal energy per gram and E_{rot} is the rotational kinetic energy. Now we have assumed that $\mathfrak{E} = \mathfrak{E}_0 + \mathfrak{E}_T$ and $P = P_0 + P_T$, where the first term is the zero-temperature part and the second term is the temperature-dependent part. For small density changes $\delta\rho$, we have

$$\delta\mathfrak{E}_0 = P_0\delta\rho/\rho^2 , \tag{24}$$

so

$$\delta E_{\text{total}} = \delta E_T + \delta E_G + \delta E_{\text{rot}} , \tag{25}$$

where

$$\delta E_T = \int_0^M dm\delta\mathfrak{E}_T , \tag{26}$$

the change in thermal energy, and

$$\delta E_G = -\int_0^M dm P_T\delta\rho/\rho^2 , \tag{27}$$

the change in the *available* gravitational energy.

The time rate of change of Jovian structure is obtained by averaging the energy difference between J8 and J8±, or similarly for J9 and J9±, and dividing by the observed luminosity of Aumann *et al.* The luminosity contributions by the three terms in equation (25) are in the ratio (thermal energy output/gravitational energy output/rotational energy output) = 1.00/0.07/−0.01 for model J8 and 1.00/0.28/−0.05 for model J9. Thermal energy, as defined by equation (26), is therefore the dominant source

of Jupiter's luminosity. The distribution of energy production in model J8 is plotted in Figure 5; it is quite similar to that of a highly degenerate white dwarf. However, the contribution due to the release of gravitational energy, as defined by equation (27), appears to be nonnegligible. Its exact proportion depends rather sensitively on the adiabatic relation.

In Table 2 we list the predicted e-folding times for several quantities. The e-folding time is defined as the present value of the quantity divided by its first derivative with respect to time, evaluated at present. Evidently, the time rate of change of observable quantities is too small to be detected with present techniques.

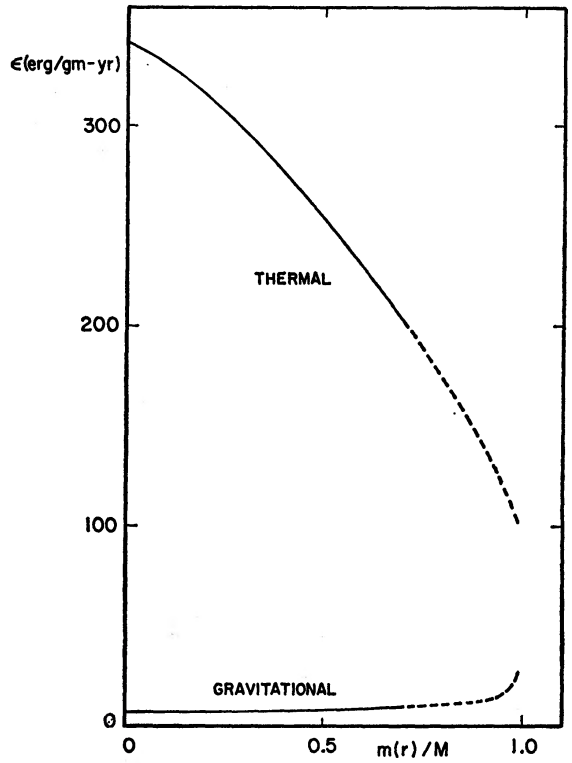

FIG. 5.—Production of energy per gram in the Jovian interior for cooling (*upper curve*) and gravitational-energy release (*lower curve*). Dashed line corresponds to the molecular envelope where energy release is relatively uncertain.

TABLE 2

PREDICTED e-FOLDING TIMES* FOR JUPITER MODELS J8 AND J9

Parameter	J8	J9
Rotation period.....	-100×10^9	-20×10^9
J.....	$+40\times10^9$	$+20\times10^9$
K.....	$+2\times10^9$	$+9\times10^9$
Thermal energy.....	-4×10^9	-4×10^9

* In years. Rigid-body rotation is assumed, and the observed net luminosity of Aumann *et al.* is used.

In Figure 6 is plotted the contraction velocity for model J8 as a function of mass shell $m(r)$ which is approximately a Lagrangian variable. The dashed curve shows the contraction velocity for homologous contraction, where the contraction velocity is assumed to be proportional to the radius of the mass shell. The homologous-contraction curve is normalized to the same contraction velocity as the true curve at the surface. Since the Jovian contraction is decidedly nonhomologous, angular momentum must be transferred from the outer mass shells to the inner mass shells for rigid-body rotation to be maintained. It seems doubtful that any significant differential rotation could be set up by the contraction, due to the combined inhibiting effects of the magnetic field and convection. However, the tendency to differential rotation may, along with the convective motions, play a role in the maintenence of the Jovian magnetic field.

V. SUMMARY

The introduction of electron screening in the finite-temperature equation of state of dense hydrogen revises the Jovian hydrogen abundance upward slightly from 54 percent

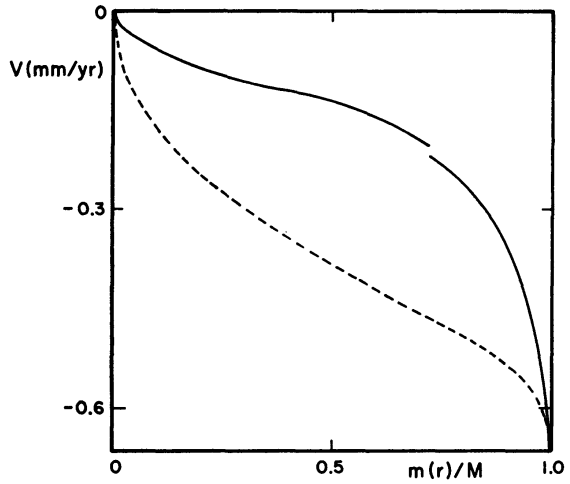

FIG. 6.—Contraction velocity in the Jovian interior as a function of mass shell $m(r)$. *Dashed curve*, contraction velocity for homologous contraction.

to about 60 percent by mass. More important, the average internal temperature is reduced by almost a factor of 2, which reduces the available energy by essentially the same factor. The Kelvin-Helmholtz time for the present Jovian model, if rotation and the available reservoir of gravitational energy are taken into account, is about 4×10^9 years, which is close to the accepted age of the solar system. If electron screening were not taken into account, the Kelvin-Helmholtz time would be roughly 8×10^9 years. Thus the present model of Jupiter and the most recent determination of the net Jovian energy flux are consistent, but only marginally so. A revision upward of the net flux might necessitate a search for an additional source of energy, or admit the possibility that Jupiter was not formed as early as other solar-system objects.

The calculations of this paper indicate that the rotation period of Jupiter should be essentially constant, with only an undetectable secular decrease. Any observable intrinsic change in the radio rotation period would therefore not be due to a gross change in planetary structure, but would more probably be due to a migration of the magnetic-field geometry with respect to the corotating frame. If the Jovian magnetic field is at all analogous to the terrestrial magnetic field, such migrations should occur and should be nearly detectable at present.

The programming assistance of Wayne L. Slattery contributed significantly to this investigation. This work was supported by National Science Foundation grant NSF USDP GU-1598 and National Aeronautics and Space Administration grant NGR 44-012-152.

REFERENCES

Ashcroft, N. W. 1969, paper given at Aspen Conference on Quantum Crystals, August.
Aumann, H. H., Gillespie, C. M., Jr., and Low, F. J. 1969, *Ap. J.* (*Letters*), **157**, L69.
De Marcus, W. C. 1958, *A.J.*, **63**, 2.
Donivan, F. F., and Carr, F. D. 1969, *Ap. J.* (*Letters*), **157**, L65.
Hubbard, J. 1957, *Proc. Roy. Soc. London*, *A*, **243**, 336.
Hubbard, W. B. 1966, *Ap. J.*, **146**, 858.
———. 1968, *ibid.*, **152**, 745 (Paper I).
———. 1969, *ibid.*, **155**, 333 (Paper II).
Hubbard, W. B., and Lampe, M. 1969, *Ap. J. Suppl.*, No. 163, **18**, 297.
Kopyshev, V. P. 1965, *Soviet Phys. Doklady* (transl.), **10**(**4**), 338.
Peebles, P. J. E. 1964, *Ap. J.*, **140**, 328.
Salpeter, E. E. 1961, *Ap. J.*, **134**, 669.
Salpeter, E. E., and Zapolsky, H. S. 1967, *Phys. Rev.*, **158**, 876.
Trubitsyn, V. P. 1966, *Soviet Phys.—Solid State* (transl.), **8**, 688.
Wigner, E., and Huntington, H. B. 1935, *J. Chem. Phys.*, **3**, 765.