A NOTE ON THE EVALUATION OF THE LATITUDE OF THE MOON

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SUMMARY

Investigations of a discrepancy of about $0''\!\cdot\!034 \sin(F{-}2D)$ between two methods for the evaluation of the fundamental lunar ephemeris have revealed a numerical error in a coefficient used in the reverse transformation by Eckert, Walker and Eckert of Brown's series for the latitude of the Moon. As a consequence, the series given by them for the differential correction of the Improved Lunar Ephemeris requires amendment. The investigations have also shown that the interpretation of Brown's formula for the latitude that was used in the Improved Lunar Ephemeris does not give the best representation of Brown's original series.

INTRODUCTION

The fundamental lunar ephemeris that is currently published in *The Astronomical Ephemeris*, in *The American Ephemeris* and elsewhere is based on the use of the series and precepts that were given by Eckert, Jones & Clarke (1954) to define the ' Improved Lunar Ephemeris ' (ILE 1954; designated $j = 0$), which replaced the ephemeris obtained by the use of Brown's ' Tables of the Motion of the Moon ' (1919). Since Brown did not explicitly make allowance for the effects of aberration two additional longitude terms were included to give an incomplete differential correction for aberration (Clemence, Porter & Sadler 1952). Further differential corrections to take into account the change to the IAU system of astronomical constants (IAU 1966) and to allow for a numerical error in Brown's series (Eckert 1966) were introduced to give the ephemeris designated $j = 1$ (Supplement to *Astronomical Ephemeris* 1968 & to *American Ephemeris*). The ILE, however, does not represent the full precision of Brown's original theory (1908) of the solar perturbations, particularly in regard to the series for the sine parallax. A new transformation of Brown's series in rectangular coordinates to series for longitude, latitude and sine parallax was therefore made by Eckert, Walker & Eckert (1966; afterwards referred to as EWE), who also gave series for the differential correction of the ILE. It has since been agreed (IAU 1968) that the ephemerides for 1972 onwards should be based on the new series and designated $j = 2$; accordingly, these differential corrections have been applied to the ephemerides that are in press for the year 1972.

Independent computations that by-passed these differential corrections have been made at the U.S. Naval Observatory and at the Royal Greenwich Observatory, and they have shown the presence of an error, which behaves like $0''\!\cdot\!034 \sin(F-2D)$, in the differential corrections in latitude. At first it was supposed that this was due to the use in EWE of an alternative interpretation from that used in ILE of a formula given by Brown for the series in latitude. This is essentially

the same explanation as that given by Woolard to account for part of the discrepancy between the values obtained by the use of Brown's Tables and those given by the direct evaluation of the ILE series. Further investigation by Eckert has, however, shown that the error was caused by the use in EWE of an insufficient number of significant figures in the relevant coefficient in a formula taken from ILE. The effect is, coincidentally, almost identical to that given by the alternative interpretation and these studies have shown that in this respect the interpretation used in ILE was not that which Brown intended.

Because of the error just described a correction $\delta\beta$ given in Table I of this paper must be applied to the series β in Table III of EWE to make it consistent with the values of the latitude in the ILE. Since the series $\Delta\beta$ in Table III was obtained by subtracting β from the new series for the latitude from Brown's rectangular coordinates, the new value of the latitude may be obtained directly without correction from $\beta + \Delta\beta$ or by applying $\Delta\beta - \delta\beta$ to the ILE. The alternative interpretation of Brown's formulae requires a correction to the ILE of $-\delta\beta - 0''\!\cdot\!0002 \sin (F - 2D)$; i.e. the values of β in Table III are by coincidence what Brown intended.

ORIGIN OF THE DISCREPANCY

To understand how such a discrepancy could arise it is necessary to realize that Brown did not tabulate his original series for the latitude directly, but instead transformed it in such a way that some of the tables for the solar perturbations in longitude could also be used for the latitude. He expressed the series in the form (Tables, I, p. 5)

$$(1 + C)(\gamma_1 \sin S + \gamma_2 \sin 3S + \gamma_3 \sin 5S + N),\tag{1}$$

where S is the sum of the fundamental argument F and a series of terms whose arguments also occur in the longitude series, C is a series with small coefficients, and N is a short series with arguments dependent on F and with fairly large coefficients. The form of the series and the values of the constants were chosen to give the most convenient method of tabulation (Brown 1911, p. 651), and Brown's final choice of coefficients γ_1, γ_2, γ_3 was given in the 'table of principal terms' (Tables, I, p. 16) in the form

$$\begin{cases} + 18\ 518\!\cdot\!511 \sin S \\ + \qquad 1\!\cdot\!189 \sin S \\ - \qquad 6\!\cdot\!241 \sin 3S \\ + \qquad 0\!\cdot\!004 \sin 5S. \end{cases}\tag{2}$$

Unfortunately, Brown did not make it clear how the partitioning of the term in sin S is to be interpreted. In the evaluation of ILE it has been assumed that the contributions to the coefficients of sin S are simply to be added, i.e. that $\gamma_1 = 18\ 519\!\cdot\!700$. Further, since Brown lists the coefficients of $\gamma_1 C$, rather than of C itself, the expression (1) has been evaluated in the form

$$(\gamma_1 + \gamma_1 C)[\sin S + (\gamma_2/\gamma_1) \sin 3S + (\gamma_3/\gamma_1) \sin 5S + (1/\gamma_1)N].\tag{3}$$

This complicated expression for the latitude involves unnecessary approximations and so it is better, as well as more convenient with present-day methods of computation, to work in terms of the series obtained directly from the conversion to spherical coordinates of the fundamental series for the rectangular coordinates of the Moon. Eckert, Walker & Eckert obtained such a series (here denoted by β_2)

TABLE I

Coefficients in the series for the latitude of the Moon

Arguments +50				Series $j = 2$	EWE Table III		Corrections to EWE			Brown (adjusted)	
l	l'	F	D	β_2	β	$\Delta\beta$	$\delta\beta$	β'	$\Delta\beta'$	β_0	$\Delta\beta_0$
50	50	51	48	−623·6552	−623·6553	+1	+336	−623·6217	−335	−623·6536	−16
50	50	51	46	−3·6745	−3·6791	+46	+2	−3·6789	+44	−3·6750	+5
51	50	51	48	−166·5767	−166·5729	−38	−28	−166·5757	−10	−166·5752	−15
51	50	51	46	−6·5797	−6·5813	+16	+4	−6·5809	+12	−6·5799	+2
49	50	51	50	−999·6945	−999·6848	−97	−13	−999·6861	−84	−999·6845	−100
49	50	51	48	−33·3579	−33·3628	+49	+20	−33·3608	+29	−33·3586	+7
50	51	51	48	−29·6525	−29·6546	+21	+14	−29·6532	+7	−29·6518	−7
50	49	51	48	+12·1247	+12·1245	+2	−7	+12·1238	+9	+12·1248	−1
48	50	51	50	−31·7596	−31·7627	+31	+16	−31·7611	+15	−31·7626	+30
48	50	51	48	−2·1464	−2·1500	+36	+1	−2·1499	+35	−2·1460	−4

This table contains only those terms, out of a total of 685, for which $\delta\beta$ differs from zero. All coefficients are for the system of parameters used in the lists of terms in Brown's Tables and in the ILE.

β_2: EWE transformation of Brown's series in rectangular coordinates; the adopted basis for the lunar ephemeris $j = 2$.

β: EWE series intended to represent the latitude of the ILE but actually corresponding to the precepts for use of Brown's Tables.

$\Delta\beta = \beta_2 - \beta$: the correction to be applied to β to give β_2.

β': a series corresponding to the precepts given in ILE for the evaluation of the latitude.

$\delta\beta = \beta' - \beta$: the correction applied to β to give β'.

$\Delta\beta' = \beta_2 - \beta' = \Delta\beta - \delta\beta$: the correction to be applied to the latitude of ILE to give β_2.

β_0: Brown's original series for the latitude after adjustment to the adopted system of parameters.

$\Delta\beta_0 = \beta_2 - \beta_0$.

but, in order to be able to give a series $\Delta\beta$ for the differential correction of the ILE, they had first to form a new series β that would contain all the errors of the special series used in that ephemeris. In making the reverse transformation from the expression (3) the value of the coefficient of N (i.e., $1/\gamma_1$) was inadvertently taken as 0·000 054, instead of the more precise value 0·000 053 996 555. The coefficients in the series β (EWE, Table III) should therefore receive the correction ($\delta\beta$) of $-0·3445 \times 10^{-8}(\gamma_1+\gamma_1 C)N$. Since the sum of the series $\gamma_1 C$ is small compared with γ_1, this correction is given by

$$\delta\beta = -0·000\ 063\ 80\ N = +0·03356 \sin (F-2D)-0·0028 \sin (l+F-2D)+ \dots$$

The same correction $\delta\beta$ must be subtracted from the series $\Delta\beta$ for the differential corrections. The corrected series may be denoted by β' and $\Delta\beta'$, and the relevant coefficients of the various series are given in Table I.

It may be noticed that the principal coefficient in $\Delta\beta'$ is much larger than any coefficient in $\Delta\beta$ (including the terms not affected by this correction). This implies, quite unexpectedly, that the series β is closer to the improved series β_2 than is the corrected series β', which is the equivalent of the ILE series. Further examination of Brown's work suggests, in fact, that Brown almost certainly intended that the evaluation of his Tables should correspond more closely to β than to β'. As Woolard (1954) realized, it is possible to interpret Brown's precepts for the use of his Tables to imply that the latitude is to be calculated from

$$(1+C+\Delta C)(\gamma_1^0 \sin S+\gamma_2 \sin 3S+\gamma_3 \sin 5S+N) \qquad (4)$$

where $\gamma_1^0 = +18\ 518''·511$ and $\Delta C = +0·000\ 062\ 2$, so that $\Delta C.\gamma_1^0 = 1''·189$. Since the difference between $\gamma_1^0 C$ and $\gamma_1 C$ is negligible, this expression can be written as

$$(\gamma_1+\gamma_1 C)[\sin S+(\gamma_2/\gamma_1^0) \sin 3S+(\gamma_3/\gamma_1^0) \sin 5S+(1/\gamma_1^0)N]. \qquad (5)$$

If Brown had intended that the use of his Tables would be equivalent to the evaluation of expression (3) rather than of (5), it would have been necessary for him to have made adjustments to remove at least the effect of the additional terms in $\Delta C.N$. This he did not do, but he gave no explicit statement that this was a deliberate decision and not an oversight as was tentatively assumed by Woolard. It is of interest to note that the value of the coefficient $(1/\gamma_1^0)$ is $+0·000\ 054\ 000\ 022$ and so we can see that, by chance, the series β given by EWE corresponds much more closely to expression (5) than to expression (3).

FURTHER INVESTIGATIONS OF BROWN'S TRANSFORMATION

It is not easy to follow through Brown's derivation of his final tabulations for the latitude from the original series given in the theory, since he used an intermediate form of series, he made several changes in notation and in the arbitrary parameters, and he rearranged and partitioned the expressions to facilitate the construction of the Tables. We have, however, started with the expressions in Section 9 of Brown's intermediate paper (1911), applied the effects of changing the parameters from those used in the theory to the preliminary set used for the Tables and ILE, and compared the results with the corresponding series in Chapter I of the Tables. The new values for γ_1 and γ_3 agree exactly, but there is an unexplained difference of $-0''·001$ for γ_2. The principal characteristic of each coefficient may be inferred from the argument or taken from the Tables; the powers and products of the changes in the parameters

above the first may be neglected in the principal characteristics. The effects of the changes in the parameters on the higher-order characteristics were estimated from the original expressions for the latitude in Brown's theory (1908) and were also found to be negligible. When adjusted the coefficients of the series C agree exactly with those of the ' Terms in $\gamma_1 C$ ', but five of the coefficients of S differ from those of the ' Terms in S ' by more than one unit. These discrepancies (in the sense ' Tables—adjusted paper ') are:

$$+0''\cdot03 \sin (l-2D), \quad +0''\cdot03 \sin (l+l'), \quad +0\cdot05 \sin (l-l'-2D),$$
$$+0\cdot03 \sin (l-l'), \quad -0''\cdot10 \sin (l-l'+2D).$$

The corresponding contributions to latitude are approximately one-tenth as large. Agreement between the adjusted values of the terms in $k_0 N$ given in the paper and the ' Terms in N ' is only obtained if the former are multiplied by γ_1^0/k_0, rather than γ_1/k_0; even so the following discrepancies are found:

$$-0''\cdot002 \sin (F-4D) \quad \text{and} \quad -0''\cdot001 \sin (F-l'-2D).$$

Thus this rederivation shows that Brown intended his series of ' Terms in N ' to be used in expression (3) and not in expression (5); we can see no explanation for the discrepancies in the terms in S, but though intriguing they are of no practical importance.

Further confirmation of this follows from a comparison of the series β_2, β and β' with the series β_0, say, obtained by adjusting Brown's original latitude series for the changes in the arbitrary parameters. It is found that β_0 is much closer to β_2 than is β', and is in general closer even than β since it avoids the approximations made in the special transformation to the series used in the Tables. Some of the coefficients of β_0 are given in our Table I.

The presence in the ILE latitude of an error that is similar to $\delta\beta$ has also been indicated by a study by Morrison & Sadler (1969) of the observations of occultations during 1960–66.

CONCLUSIONS

These investigations have shown that the differential corrections $\Delta\beta$ given by EWE require amendment and that the series used in the ILE itself is based on an incorrect, but reasonable, interpretation of Brown's precepts. The published form of the fundamental lunar ephemeris for $j = 2$ will be based on the direct evaluation of the series β_2; it will not be economical to correct the figures now printed for the ephemeris for 1972, but corrections will be given. The new ephemeris will also be free from the approximations involved in the adjustment to the IAU system of astronomical constants and in the correction for aberration. The change from Brown's preliminary constants to the IAU constants is made in one step and Brown's elements are adjusted so as to give a geometric ephemeris, from which the aberration correction is calculated rigorously.

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