RECOMBINATION OF HYDROGEN IN THE HOT MODEL OF THE UNIVERSE

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The considerable emission of energetic quanta during recombination of hydrogen in an expanding universe leads to a slowing down of the recombination and to a distortion of the relict radiation spectrum in the Wien region. The energy exchange between electrons and radiation in the Compton effect maintains the temperature of matter equal to that of the radiation up to a time corresponding to a red shift of $z \sim 150$, and this leads, in particular, to a change in the time dependence of the Jeans wavelength of the gravitational instability of a homogeneous medium in an expanding universe.

In the hot model of the universe[1,2] it is assumed that at an early stage of the expansion the fully ionized plasma is in equilibrium with radiation. Cooling as a result of expansion leads to recombination. Since in the course of the expansion there is conservation of specific entropy $T^3/n = 3 \times 10^6$ deg^3 cm^3 [2], it can be easily verified[1)] that according to the Saha equilibrium formula a 50% degree of ionization is attained at a temperature $T \sim 4100°$K and a density $n = p + H \approx 2.6 \times 10^4$ cm^{-3} (the letters p, H, and e denote the densities of the corresponding particles).

The density of quanta in the universe is much greater than the density of ions, electrons, and atoms ($n_\gamma/n \sim 10^8$). At first sight it seems to be impossible to have any sort of an inverse reaction of matter on the black-body radiation. As will be shown later, the distortion of the spectrum indeed turns out to be small, but owing to considerably more complicated causes.

The point is that the temperature for the effective recombination is considerably lower than the ionization potential: $kT \approx I/40 \approx h\nu_\alpha/30$ (ν_α is the frequency of the L_α-quantum), and therefore the density a of the energetic quanta with $\nu > \nu_\alpha = E_0/h$ constitutes a small part of the total radiation density and is approximately a factor of 200 lower than the density of protons and of atoms:

$$a = \frac{8\pi}{(hc)^3}\int_{E_0}^{\infty} E^2 e^{-E/kT}\,dE = \frac{8\pi E_0^2 kT}{(hc)^3}e^{-E_0/kT}.$$

For $T \sim 4100°$K the value of a is about 10^2 cm^{-3}. According to the Saha formula for e = p = H (50-percent ionization) we have

$$e = \frac{n}{2} = \frac{(2\pi m_e kT)^{3/2}}{h^3}e^{-I/kT}$$

and the ratio of the number of electrons to the number of quanta of energy above the ionization threshold I is equal to $[\pi(mc^2)^3 kT/2^3 I^4]^{1/2}$. This ratio does not depend on the absolute value of the hydrogen density: the smaller the value of n, the lower is the temperature for 50-percent ionization, and the smaller is the factor $\exp(-I/kT)$ which enters into the expression for the number of quanta of energy above threshold. The number of quanta with $\nu > \nu_\alpha$ is larger than the number of quanta with $h\nu > I$; since $h\nu_\alpha = 3I/4$, we get $a \sim n^{3/4}$ and $a/n \sim n^{-1/4}$, i.e., depends only weakly on n.

If each recombination act were accompanied by the emission of an energetic quantum, then the density of quanta would grow rapidly in comparison with the Planck spectrum. At the present time the temperature of the equilibrium radiation is equal to 3°K[2], so that all the wavelengths have increased since the time of the recombination by approximately a factor of 1400, and one might expect anomalies in the relict spectrum for wavelengths $\lambda \sim 10^{-2}$ cm. However, the assumption concerning the emission of an energetic quantum in the case of each recombination is incorrect in the present situation; the inverse absorption of L_α-quanta and of harder quanta takes place. As will be shown later a noticeable retardation of the recombination of electrons and protons occurs, since the super-equilibrium density of quanta with $\nu > \nu_\alpha$ leads to an above-equilibrium density of excited hydrogen atoms which are easily ionized by soft quanta ($I_2 = I/4 = 3.4$ eV). A decisive role is played by the two-quantum transition $H^* \to H + \gamma_1 + \gamma_2$, the astrophysical role of which was considered in[3,4]. Below we give an approximate theory of the recombination and of the spectrum arising as a result.

1. DYNAMICS OF RECOMBINATION

Under dynamical equilibrium, the principal processes are the photoionization process $H + \gamma \to p + e$ and the inverse process of recombination with emission of a quantum. Regardless of whether the recombination proceeds directly to the ground state or by means of a cascade via excited states $p + e \to H^* + \gamma_1$, $H^* \to H + \gamma_2$, a single quantum is emitted, of energy equal to or greater than $h\nu_\alpha$. The same quantum is absorbed in the case of ionization[2)]. The processes indicated above leave constant the sum $\Sigma = e + a$, where e is the electron density and a is the density of quanta with $\nu > \nu_\alpha$.

We take the cosmological expansion into account; in this case the quantity

$$\Sigma' = V(e + a'),$$

is conserved, where V is the instantaneous physical volume corresponding to a constant co-moving volume,

[1)]Calculations are carried out on the basis of the following assumptions regarding the present-day situation: $T = 3^3$K, $n = 10^{-5}$ cm^{-3}.

[2)]Along with the direct ionization $h\nu + H = p + e$, we also consider the stepwise ionization: $h\nu + H \to H^*$, H^* + soft quantum = p + e.

Vn = const, and a′ is the number of quanta of frequency exceeding the frequency

$$\nu' = \frac{\nu_\alpha(1+z_0)}{1+z'} = \nu_\alpha\left(\frac{n'}{n_0}\right)^{1/3},$$

which varies in accordance with the red shift in the course of expansion. It is well known that the expansion transforms the equilibrium spectrum with temperature T_0 into an equilibrium spectrum with $T' = T_0(n'/n_0)^{1/3}$. We choose the instant of time T_0 which corresponds to a 50-percent ionization with $e/a_0 \sim 100$, and the instant T' such that $e/n \ll 1$ when recombination has practically been finished. In this case it follows from $\Sigma' = \text{const}$ that

$$a' = a_0\left(1+\frac{e}{a_0}\right)\left(\frac{n'}{n_0}\right)^{1/3}.$$

The equilibrium spectrum corresponds to a density $a' = a_0(n'/n_0)^{1/3}$, and consequently in the approximation assumed by us one would obtain at the later stages a spectrum with a′ 100 times larger than the value for the equilibrium distribution. At the present time $\nu'_\alpha \approx 10^{-3}\nu_\alpha$, i.e., this maximum possible violation of equilibrium pertains to radiation of wavelength of the order of 10^{-2} cm. The expansion, leaving Σ' unchanged, alters Σ in accordance with the fact that a) all the densities are decreased and b) a fraction of the quanta "falls below the threshold ν_α" as a result of the red shift (their energy becomes less than the energy of a L_α-quantum). It is easy to construct the equation for Σ under this assumption

$$d\Sigma/dt = -3\mathcal{H}\Sigma - \nu_\alpha\mathcal{H}f(\nu_\alpha),$$

since $d\nu/dt = -\mathcal{H}\nu$, where f is the function describing the distribution of the quanta with respect to frequencies, $\mathcal{H}$ is the instantaneous value of the Hubble constant, $\mathcal{H} = -d\ln n^{1/3}/dt$. Stipulating the Wien form of the spectrum

$$f \sim e^{-h\nu/kT}, \quad a = f(\nu_\alpha)kT/h,$$

we obtain

$$\frac{d\Sigma}{dt} = -3\mathcal{H}\Sigma - \mathcal{H}\frac{h\nu_\alpha}{kT}a.$$

Knowing the value of Σ, it is easy to reconstruct the whole situation. Indeed, the fast processes guarantee dynamic equilibrium between the electron density and the density of different excited states of the hydrogen atoms; transitions between them occur as a result of the absorption and emission of low-energy quanta, the number of such quanta is great and they, doubtlessly, are in an equilibrium corresponding to the overall temperature of the radiation. This means that

$$H^* = Kep = Ke^2, \tag{1}$$

where $K = K(I^*, T)$ is evaluated by means of the Saha formula for the binding energy of the excited state. However, such a relation does not hold between e and H; the density e is greater than would correspond to the Saha formula. Consequently, $f(\nu_\alpha)$ likewise does not correspond to equilibrium with a common temperature T.

Owing to the large cross section for the absorption of L_α-quanta, the equilibrium $e + p \rightarrow H^* + \gamma_1$, $H^* \rightleftarrows H + \gamma_\alpha$ is established, at which we have

$$f(\nu_\alpha) = \text{const}\cdot\frac{H_{2P}^*}{H} = \text{const}\cdot\frac{ep}{H}\exp\left(\frac{I_{2P}}{kT}\right),$$
$$a = \frac{kT}{h}f(\nu_\alpha) = \text{const}\cdot T\frac{ep}{H}\exp\left(\frac{I_{2P}}{kT}\right)$$

(for details cf. the Appendix). Here H^*_{2P} is the density of hydrogen atoms in the 2P-state.

The relation connecting a with e and p together with the trivial conditions e = p and e + p + H = n, enables us to express in an elementary fashion in terms of Σ, n, and T all the quantities e, p, f, and a of interest to us, as well as the hydrogen atom density in all the excited states[3]. In particular, this pertains also to H^*_{2S}—the density of hydrogen atoms in the 2S-state. Taking the two-quantum radiation into account, the equation for Σ assumes the form

$$\frac{d\Sigma}{dt} = -3\mathcal{H}\Sigma - \mathcal{H}\frac{h\nu_\alpha}{kT}a - w\left[H_{2S}^* - H\exp\left(-\frac{h\nu_\alpha}{kT}\right)\right], \tag{2}$$

where w is the known transition probability, $w = 8\ \text{sec}^{-1}$, while the last term takes into account the possibility of the two-quantum excitation of an atom from the ground state to H^*_{2S}. Strictly speaking, one should add to w the contributions of all other two-quantum processes 3S → 1S, 3D → 1S, ..., and also of the transitions from the continuum, leaving aside, however, only those processes in which the energy of each of the two quanta is less than the energy of the L_α-quantum.

In the problem of recombination during a cosmological expansion, n and T are known functions of the time. In the approximate theory the problem reduces to the integration of one differential equation with several algebraic relations between the quantities entering in it.[4]

Thus, we obtain all the quantities of interest to us as functions of the time. In order to obtain the radiation spectrum at the present time we note that at each instant the spectral density has no discontinuity at a frequency ν_α; on the other hand, at later times the quanta of frequency $\nu < \nu_\alpha$ undergo only a red shift in the approximation under consideration. Therefore to each value of $f(\nu_\alpha)$ at the present moment t′ there corresponds at the present time $f((\nu = \nu_\alpha)/(1+z')) = (1+z')^{-2}f_\alpha(t')$. Here z′ is the red shift corresponding to t′, and the power of (1 + z′) corresponds to the fact that f is given as the density of quanta per unit volume (physical, not comoving!) per unit frequency.

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[3]These densities are small and may be left out of account in the sums used to evaluate Σ and n.

[4]At each instant of time, for appropriate values of T and n, there is a definite equilibrium value of Σ_e which would be realized in the course of time if the expansion were stopped, i.e., at $\mathcal{H} = 0$. It is evident that at $\Sigma = \Sigma'_e$ the solution of all equations for e, p, H_{2S}^*, etc. will identically give equilibrium values, and for the present time one would obtain exactly the Planck expression, or more exactly its limiting expression – the Wien formula $f(\nu) = \text{const}\cdot\exp(-h\nu/kT)$, since stimulated processes have not been taken into account. Since $h\nu/kT \gtrsim 30$, this is negligibly small.

2. NUMERICAL CALCULATIONS

As shown in the Appendix, the rate of passage of L_α-quanta below the threshold is given by

$$w_\alpha H_{2P}^* = \frac{H_{2P}^*}{H}\frac{8\pi\mathcal{H}}{3\lambda_\alpha^2}\left[\frac{1}{\text{cm}^3\text{sec}}\right].$$

In the case of weak ionization (when $H \sim n$)

$$w_\alpha = \frac{8\pi\mathcal{H}}{3\lambda_\alpha^3 H} \approx \frac{8\pi\mathcal{H}}{3\lambda_\alpha^3 n} = \frac{8\pi\mathcal{H}_0}{3\lambda_\alpha^3 n_0(1+z)^{3/2}} = \frac{1{,}55\cdot 10^3}{(1+z)^{3/2}}\left[\frac{1}{\text{sec}}\right]$$

(Here we have taken into account that $\mathcal{H} = \mathcal{H}_0(1+z)^{3/2}$ and $n = n_0(1+z)^3$, where $\mathcal{H}_0$ and n_0 are the present values of Hubble's constant and of the density). It is evident that in the region of interest to us $z \sim 10^3$, $w_\alpha \ll w$, and the rate of recombination is determined by the two-quantum decays of the 2S level.

Neglecting the passage of L_α-quanta below the threshold and taking into account the fact that $a < e$, we simplify Eq. (2):

$$\frac{dp}{dt} = -3\mathcal{H}p - w\left[H_{2S} - H\exp\left(-\frac{h\nu_\alpha}{kT}\right)\right]. \tag{3}$$

We introduce dimensionless variables: the red shift z and the degree of ionization $x = e/(e+H) = e/n$; substituting $H_{2S} = K(I_2^*, T)e^2$ from (1), will obtain

$$\frac{dx}{dz} = \Omega^{1/2} w n_c z^{1/2} \exp\left(\frac{I}{4kT_0 z}\right)\left[x^2 - \frac{(2\pi m_e k T_0)^{3/2}}{\Omega h^3 n_c z^{3/2}}(1-x)\exp\left(-\frac{I}{kT_0 z}\right)\right], \tag{4}$$

where $n_c = \rho_c/m_p = 3\mathcal{H}_0^2/8\pi G m_p = 10^{-5}\ \text{cm}^{-3}$ is the present critical density, $\Omega = n_0/n_c$; and it has been taken into account that

$$\frac{dt}{dz} = \frac{\mathcal{H}_0^{-1}}{(1+z)^2(1+\Omega z)^{1/2}} \approx \frac{\mathcal{H}_0^{-1}}{z^{5/2}\Omega^{1/2}}.$$

The expression in brackets in (4) is evidently equal to zero in the case of thermodynamic equilibrium, since it represents the Saha formula.

The results of numerical integration are shown in Fig. 1, which shows the equilibrium degree of ionization and the degree of ionization obtained upon integrating (4).

3. THE ASYMPTOTIC BEHAVIOR OF RECOMBINATION

When the temperature has dropped significantly, the energy of the equilibrium quanta is insufficient to maintain equilibrium between excited levels of the hydrogen atom and the free electrons, and each recombination event leads to the emission of an energetic quantum. The rate of photoionization and upwards diffusion along

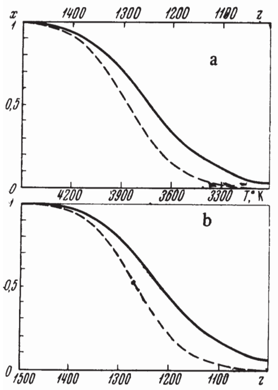

FIG. 1. Deviation of the degree of ionization from the equilibrium value $x = p/(p+H)$. For comparison recombination curves are given for different values of the present density of matter in the universe: a – $\Omega = 1$, $\rho = 2 \times 10^{-29}$ g/cm³ and b – $\Omega = 0.05$, $\rho = 10^{-30}$ g/cm³.

the energy axis (which decreases exponentially as the temperature falls) becomes smaller than the rate of two quantum decay of the 2S level starting with $z \sim 870$ ($T \sim 2500°$K). At $z \sim 700$, the rate of the passage of L_α-quanta below the threshold and the rate of photoionization of the 2P level become equal, i.e., at $z \sim 700$ each recombination event[5)] leads to a decrease in the degree of ionization. Utilizing the results of integration of (4) up to $z \sim 870$, we obtain the asymptotic behavior of the recombination from the equation

$$dp/dt = -\alpha(t)p^2 \tag{5}$$

or

$$dx/dz = \alpha(z)x^2\Omega^{1/2}z^{1/2}\mathcal{H}_0^{-1}n_c.$$

In the temperature range where the Compton effect for the quanta of the relict radiation incident on electrons maintains the electron temperature equal to the radiation temperature (cf. Sec. 5), $T \approx T_0(1+z)$ and the recombination coefficient[5]

$$\alpha = \alpha_{2P} + \alpha_{2S} = \alpha_0/\sqrt{T} \approx 2{,}5\cdot 10^{-11}/\sqrt{T}\ [\text{cm}^3/\text{sec}]$$

is proportional to $z^{-1/2}$, i.e.,

$$\frac{dx}{dz} = \frac{\alpha_0}{\sqrt{T_0}}\Omega^{1/2}\mathcal{H}_0^{-1}n_c x^2 = Ax^2,$$

whence

$$x(z) \doteq 1/(C - Az).$$

For $\Omega = 1$ numerical calculation gives for $z \sim 870$ the value $x \sim 2\cdot 10^{-3}$. Taking into account the fact that up to $z \sim 700$ the 2P level is in equilibrium with the radiation and that $\alpha_{2S} \sim \alpha/3$[5], we obtain for $z \sim 700$ the value $x \sim 3.3 \times 10^{-4}$. For smaller values of z we have $x(z) = [3 \times 10^3 + 43(700 - z)]^{-1}$. However, at $z \sim 150$ the temperature of matter is no longer the same as the temperature of the radiation (cf. Sec. 5), and beginning from that instant, $\alpha(z) \propto z^{-1}$ and $x(z = 0) = 2.5 \times 10^{-5}$.[6)]

The ratio of the probabilities of the processes $H + p \to H_2^+ + h\nu$ and $H + e^- \to H^- + h\nu$ to the probability of radiative recombination in the temperature range of interest to us is much smaller than the ratio $p/H = e/H = x$[5]. Therefore these processes can be neglected. The weakness of the processes $H + p \to H_2^+ + h\nu$ and $H + H \to H_2 + h\nu$ indicates the negligible ratio of the density of H_2 to that of H in the intergalactic medium.

4. DISTORTION OF THE RELICT RADIATION SPECTRUM

Emission of energetic quanta in two-quantum decays of the 2S level must distort the shape of the spectrum of the observed relict radiation $F(\nu_0)$ in the Wien region. If j_ν is the spectral brightness per unit volume and $\nu = \nu_0(1+z)$, then according to[7] the distortion of the spectrum is

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5) As a result of cascade transitions, the recombining electron must fall into a 2S level or a 2P level.

6) The residual degree of ionization in the hot model of the universe has been obtained earlier by Ozernoĭ under the assumption of complete thermodynamic equilibrium, subsequent quenching, and recombination at $T \propto (1+z)^2$; he obtained $x(z=0) = 10^{-4}$ [6], It is evident that our discussion should increase $x(z = 0)$ considerably.

$$\Delta F(\nu_0) = \frac{c}{\mathcal{H}_0} \int_0^{z_{max}} \frac{j_\nu(z)\,dz}{(1+z)^5(1+\Omega z)^{1/2}}.$$

In the case of two-quantum emission

$$j_\nu(z) = \frac{h\nu}{4\pi} A\left(\frac{\nu}{\nu_\alpha}\right) N_{2S}(z),$$

where $A(\nu/\nu_\alpha)$ is the probability of emission of a quantum of frequency ν per unit frequency interval and is tabulated in[4], with

$$w = \int_0^{\nu_\alpha} A\left(\frac{\nu}{\nu_\alpha}\right) d\nu.$$

Therefore we have

$$\Delta F(\nu_0) = \frac{ch\,\nu_0}{4\pi\mathcal{H}_0} \int_0^{\nu_\alpha/\nu - 1} \frac{N_{2S}(z) A(\nu_0(1+z)/\nu_\alpha)}{(1+z)^4(1+\Omega z)^{1/2}}\,dz.$$

Starting with $z \sim 700$, each recombination event leads to emission of an energetic quantum, and therefore the asymptotic behavior of $F(\nu_0)$ has the form[7] for $\nu_\alpha/\nu_0 = z_\alpha \ll 700$:

$$F(\nu_0) = \frac{ch}{4\pi\mathcal{H}_0} \Omega^{3/2} z_\alpha^{3/2} \alpha(z_\alpha) x^2(z_\alpha) \approx 6\cdot 10^{-29} \Omega^{3/2} \frac{\nu_\alpha}{\nu},$$

and for $\nu_\alpha/\nu \sim 700$

$$F(\nu_0) = 6\cdot 10^{-28}(\nu_\alpha/\nu).$$

The calculated shape of the spectrum is given in Fig. 2. Instead of the expected excess by a factor of 100—200 compared to the equilibrium spectral density in a narrow region, we have an entirely different picture.

For $\lambda < 1.6\ 10^{-2}$ cm, the radiation due to the recombinations always is considerably greater than the equilibrium radiation, but is small in absolute value. Apparently in this domain even such weak factors as radiation by dust and by galaxies in the infrared range turn out to be more essential. Moreover it is necessary to take into account the distortion of the spectrum as a result of the inverse Compton effect of the radiation involving hot electrons for $z \sim 100$[8], if by that time the secondary heating of the gas has already taken place, and also the bremsstrahlung pertaining to the hot intergalactic gas.

It should be emphasized once more that the obtained considerable intensification of the spectrum compared to the equilibrium value refers to the region of the spectrum in which the total number of quanta and their energy constitute a very small part of the total equilibrium radiation, therefore the change in the intensity referred to above corresponds to a negligible change in the effective temperature. However, as can be seen from Fig. 2, a measurement of the flux for $\lambda < 2 \times 10^{-2}$ cm enables us to establish an upper limit for the density of matter in the universe.

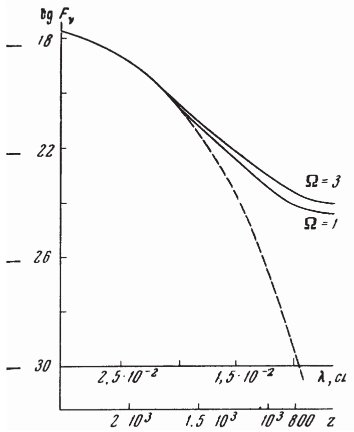

FIG. 2. Distortion of the relict radiation spectrum as a result of hydrogen recombination (F_ν in erg/cm^2 sec-Hz-sr). The dotted curve shows the Planck curve.

5. ENERGY EXCHANGE BETWEEN ELECTRONS AND RADIATION

A slowing down of the rate of recombination compared to the Saha formula leads to the fact that the density of the non-recombined electrons (which is practically independent of the total density of matter) remains sufficient, up to $z \sim 150$, to maintain the temperature of the matter equal to the temperature of the radiation. The energy is transferred from the quanta to the electrons by the Compton effect; collisions equalize the temperature of the electrons and the hydrogen atoms.

The rate of energy exchange between the radiation and the electrons[9] is given by

$$L = 4\sigma_0 \frac{k(T_\gamma - T_e)}{m_e c} \mathcal{E}_\gamma,$$

where σ_0 is the Thomson cross section, $\mathcal{E}_\gamma \sim (1+z)^4$ is the radiation energy density, T_e and T_γ are the temperatures of the electrons and of the radiation. If there were no Compton exchange of energy between electrons in the radiation, then the electron temperature after recombination would fall adiabatically with $\gamma = 5/3$, while the radiation temperature would fall adiabatically with $\gamma = 4/3$, i.e., $T_e \sim \rho^{2/3} \sim (1+z)^2$, while $T_\gamma \sim \rho^{1/3} \sim (1+z)$. From this it follows that

$$\frac{d(T_\gamma - T_e)}{dz} = \frac{T_\gamma}{1+z} - \frac{2T_e}{1+z}.$$

The limiting value z_{min} up to which $T_\gamma \sim T_e$, could be obtained by equating $d(T - T_e)/dz$ and

$$L(z) = L(t)\frac{dt}{dz} = \frac{4\sigma_0 kT(z)}{m_e c\mathcal{H}_0} \frac{(1+z)^2}{(1+\Omega z)^{1/2}}.$$

However, there are many more neutral hydrogen atoms than electrons and collisions with the gas-kinetic cross section entirely suffice to transfer energy to them from the electrons, and therefore in order to maintain the temperature of the matter it is necessary to transfer energy from the radiation to the electrons at a rate which is bigger by a factor $1/x$, i.e.,

$$\frac{kT(z)}{1+z} \approx \frac{4\sigma_0 kT(z)}{m_e c\mathcal{H}_0} x \frac{(1+z)^2}{(1+\Omega z)^{1/2}}$$

and from this it follows that

$$\frac{4\sigma_0 \Omega^{-1/2}}{mc\mathcal{H}_0} x(\Omega, z) z^{3/2} = 1.$$

Consequently z_{min} depends only weakly on γ and is equal to 150—200.

Adiabatic variation of the temperature of matter with $\gamma = 4/3$ during such a long period must have an effect on the Jeans wavelength of gravitational instability in a homogeneous expanding universe. The Jeans mass remains constant after recombination, while in the absence of interaction between the relict radiation and plasma it falls off in accordance with a power law[10].

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APPENDIX

We obtain the rate of passage of quanta below the threshold accompanying an expansion of the universe. The equation for the spectral density of quanta in the line L_α has the following form:

$$\frac{df_\nu}{dt} = A_\nu \varphi(\nu) H_{2P}^* - B_\nu \varphi(\nu) f_\nu H,$$

where A_ν and B_ν are the Einstein coefficients, and $\varphi(\nu)$ is the line profile. Since for an expanding universe $\nu = \nu_0(t_0/t)^{2/3}$ and

$$\frac{\partial \nu}{\partial t} = -\frac{2}{3}\nu_0 \frac{t_0^{2/3}}{t_0^{5/3}} \approx -\frac{2}{3}\nu\frac{1}{t} = -\mathcal{H}\nu,$$

it follows that

$$\frac{df_\nu}{dt} = \frac{\partial f_\nu}{\partial t} - \frac{\partial f_\nu}{\partial \nu}\cdot\mathcal{H}\nu.$$

Solving the characteristic equation

$$\frac{df_\nu}{B_\nu \varphi(\nu) H (f^* - f_\nu)} = -\frac{d\nu}{\mathcal{H}\nu},$$

where

$$f^* = \frac{H_{2P}^*}{H}\cdot\frac{A_\nu}{B_\nu} = \frac{H_{2P}^*}{H}\frac{8\pi h\nu^3}{3c^3},$$

we obtain

$$f_\nu = f^*\left(1 - \exp\left\{-\frac{(n-p)B_{\nu_\alpha}}{\mathcal{H}\nu_\alpha}\right\}\int\limits_{+\infty}^{\nu}\varphi(\nu)\,d\nu\right),$$

whence it can be seen that outside the profile $f_\nu \approx f^*$.[7)] Now one can find the number of quanta passing below the threshold. It is given by

$$-\frac{f^* d\nu/dt}{h\nu} = \frac{\mathcal{H}}{h}f^* = \frac{H_{2P}^*}{H}\frac{8\pi\nu^3\mathcal{H}}{3c^3}\cdot [\text{cm}^{-3}\cdot\text{sec}^{-1}].$$

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7) One need not take the induced radiation into account, since $H_2P \ll H$ as a result of the low temperature.