Corrected Derivation of Astronomical Constants from the Observations of Eros 1926–1945

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A new investigation of the motion of Eros and the mass of the earth+moon system (Rabe and Francis 1967) led to the discovery that a conceptual error was committed in the present author's earlier calculation of the mass coefficients (Rabe 1950). It is shown here that the correction of the errors in that earlier paper produces revised values for the reciprocal earth+moon mass of 328863±29 (mean error) or 328875±31, depending on the number of unknowns retained in the solution. These results are in reasonable agreement with the recent radar determinations of the astronomical unit, which point to a mass reciprocal very close to 328900. The corrected probable and mean errors of the masses of the other three inner planets come out much larger than those based on the erroneous coefficients. These masses affect the solution so unfavorably that it appears advisable to eliminate them as unknowns and to consider them as better known from other sources.

THE result $1/m_{\oplus+☾}=328452\pm64$ (mean error) for the reciprocal of the earth+moon mass had been found by the present author in an earlier investigation of the motion of Eros from 1926–45 (Rabe 1950). In the course of a new computer-based study of this motion and its dependence on the mass of the earth−moon system (Rabe and Francis 1967), it became clear that the Eros observations even of the extended arc 1926–65 agree very well with the quite different value $1/m_{\oplus+☾}=328912$, which is consistent with the recently adopted radar-based value of the astronomical unit. After the orbit of Eros had been integrated with an earth+moon mass of 1/328912, the mass correction resulting from a subsequent least-squares solution remained close to zero, regardless which one of two different procedures was used for computing the coefficients of the correction factor $\vartheta_{\oplus+☾}$ in the observation equations. However, at this point a complete reiteration of the earlier work showed that the application of the indicated differential correction did not produce either the improvement of the trajectory that was to be expected nor the improved residuals that had been indicated by the earlier least-squares solution.

At this point, credit should also be given to Dr. J. Schubart and Mr. G. Zech of the Astronomisches Rechen-Institut in Heidelberg, who independently discovered the erroneous nature of the old earth+moon coefficients, from integrations in which they varied this mass. A letter to this effect was received from Dr. Schubart just after our own experiments had eliminated any doubts in this matter.

The error which was then discovered had entered in the following way: on p. 117 of the earlier paper (Rabe 1950)

$$\frac{\partial\alpha}{\partial m}=\frac{\partial\alpha}{\partial M}\times(\text{pert. in } M)+\frac{\partial\alpha}{\partial n}\times(\text{pert. in } n)+\cdots. \quad (1)$$

The perturbations had been computed for the mean longitude L and the longitude of perihelion π. The osculating value of $M=L-\pi$, and the total perturbation in M was taken as the difference of the perturbations of L and π. However, if a perturbing mass is to be incremented, the mean motion n requires special attention. The total effect of the incremented mass upon the angular position in the orbit of Eros is fully provided by the total perturbation in M, which includes $\int\int(\partial n/\partial t)dt^2$. The only other effect to be provided is the radial "scaling" due to the osculating value of the semimajor axis a. But the total perturbation in n was used in the second term of Eq. (1), and thus it effectively entered twice and made the coefficients too large. This defect has now been remedied by removing from $\partial\alpha/\partial n$ the term which is factored by $(t-t_0)$, thus leaving only the "scaling" term in a. The erroneous inclusion of the total perturbation in n was due to a misconcept concerning the osculating nature of the orbital changes produced by the mass variations.

Once the error was realized, it was easy to correct all the mass coefficients listed in Table IV of the earlier investigation. The revised values are given in the present Table I. When the old work was reviewed, it was also realized that the coefficients for the corrections dl'', $d\epsilon$, de'', $e''d\pi''$ of the earth's orbit in the right ascension equations had inadvertently not yet been multiplied by $\cos\delta$. This was of little consequence, though, because of the smallness of these unknowns and their relatively large uncertainties. The factor $\cos\delta$ has been applied now before the revised solutions were obtained from the corrected observation equations.

The corrected normal equations are given as Table II. Aside from the changes produced by the corrected coefficients of the observation equations, some minor differences may be noted which are due partly to rounding and partly to printing or manuscript errors in the original paper. When the revised normal equations are solved for all 16 unknowns, the mass reciprocals and their mean errors come out as

$$1/m_{\oplus+☾}=328631\pm133, \quad 1/m_{♀}=405120\pm4300$$
$$1/m_{♂}=3155200\pm40400, \quad 1/m_{☿}=5876000\pm53300.$$

TABLE I. Revised mass coefficients.

Eq. No.	$10^{-3}\vartheta_{\oplus+☾}$	$10^{-3}\vartheta_{♀}$	$10^{-4}\vartheta_{♂}$	$10^{-4}\vartheta_{☿}$
1	+ 2.437	−0.803	− 1.293	−0.433
2	+ 1.033	−0.321	− 0.184	−0.208
3	− 0.062	−0.004	− 0.088	+0.004
4	− 0.055	+0.003	− 0.093	+0.005
5	− 0.045	+0.011	− 0.097	+0.011
6	− 0.035	+0.020	− 0.100	+0.008
7	− 0.024	+0.028	− 0.097	+0.009
8	− 0.014	+0.033	− 0.095	+0.008
9	− 0.007	+0.035	− 0.089	+0.008
10	− 0.002	+0.033	− 0.082	+0.006
11	+ 0.006	+0.023	− 0.068	+0.004
12	+ 0.001	+0.016	− 0.046	+0.002
13	0.000	0.000	0.000	0.000
14	0.000	−0.019	+ 0.063	−0.001
15	− 0.001	−0.036	+ 0.134	−0.004
16	− 0.006	−0.048	+ 0.185	−0.004
17	− 0.015	−0.053	+ 0.211	−0.006
18	− 0.024	−0.052	+ 0.222	−0.007
19	− 0.030	−0.048	+ 0.209	−0.008
20	− 0.033	−0.043	+ 0.190	−0.009
21	− 0.030	−0.039	+ 0.168	−0.007
22	− 0.023	−0.034	+ 0.147	−0.005
23	− 0.012	−0.030	+ 0.126	−0.005
24	+ 0.721	+0.135	+ 0.021	+0.084
25	+ 1.549	+0.306	+ 0.068	+0.179
26	+ 1.263	+0.233	+ 0.087	+0.142
27	+ 2.199	+0.426	+ 0.708	+0.293
28	+ 3.177	+0.609	+ 1.134	+0.409
29	+ 1.410	+0.280	+ 0.460	+0.177
30	+11.680	+2.659	+ 7.488	−0.397
31	+18.500	+4.290	+12.423	+2.722
32	+12.291	+2.915	+ 8.428	+1.833
33	+ 2.888	+1.100	+ 2.456	+0.667
34	+ 1.861	+1.597	+ 2.632	+0.990
35	+ 0.062	+2.750	+ 4.065	+1.665
36	+ 0.085	+2.962	+ 4.603	+1.794
37	− 0.328	+2.636	+ 4.430	+1.622
38	+ 0.848	−0.276	− 0.623	+0.359
39	+ 0.605	−0.231	− 0.067	−0.129
40	+ 0.004	+0.032	+ 0.075	+0.003
41	+ 0.012	+0.036	+ 0.087	+0.004
42	+ 0.019	+0.036	+ 0.098	+0.003
43	+ 0.024	+0.032	+ 0.108	+0.002
44	+ 0.024	+0.025	+ 0.111	0.000
45	+ 0.022	+0.016	+ 0.108	−0.001
46	+ 0.017	+0.007	+ 0.096	−0.001
47	+ 0.010	0.000	+ 0.079	−0.002
48	− 0.003	−0.004	+ 0.053	−0.002
49	− 0.001	−0.004	+ 0.020	−0.001
50	0.000	0.000	0.000	0.000
51	− 0.001	+0.007	− 0.004	0.000
52	− 0.008	+0.015	0.000	−0.001
53	− 0.023	+0.025	− 0.003	−0.002
54	− 0.041	+0.033	− 0.020	−0.002
55	− 0.057	+0.040	− 0.042	−0.002
56	− 0.068	+0.043	− 0.059	−0.001
57	− 0.074	+0.043	− 0.068	0.000
58	− 0.076	+0.041	− 0.070	−0.001
59	− 0.075	+0.037	− 0.068	−0.002
60	− 0.071	+0.033	− 0.064	−0.002
61	+ 0.308	+0.045	+ 0.131	+0.035
62	+ 0.692	+0.119	+ 0.233	+0.075
63	+ 0.229	+0.009	+ 0.061	+0.020
64	+ 1.401	+0.279	+ 0.528	+0.181
65	+ 1.923	+0.401	+ 0.690	+0.244
66	+ 1.165	+0.246	+ 0.454	+0.154
67	− 0.352	−0.010	− 0.310	−0.020
68	+ 3.746	+0.911	+ 2.390	+0.588
69	− 6.657	−1.577	− 4.613	−0.968
70	+ 0.120	+0.014	+ 0.192	+0.007
71	+ 1.095	+0.912	+ 1.729	+0.534
72	− 0.079	+0.833	+ 1.460	+0.525
73	− 0.432	+3.467	+ 5.434	+2.124
74	− 0.184	+0.299	+ 0.379	+0.195

Among the other 12 quantities, the rather unlikely result $dl'' = +0''.91 \pm 0''.56$ was obtained for the correction of the earth's longitudes. The earth+moon mass reciprocal differs by about twice its mean error from the value 328912, which is consistent with the recently adopted radar-based astronomical unit.

When the mass of Mercury is omitted as an unknown, the 15-unknowns solution yields

$$1/m_{\oplus+☾} = 328677 \pm 136,$$
$$1/m_{♂} = 3151100 \pm 41900,$$
$$1/m_{♀} = 405990 \pm 4440$$
$$dl'' = +0''.54 \pm 0''.56, \text{ etc.,}$$

not very different from the preceding results. The further omission of the mass correction for Mars, however, produces the quite different 14-unknowns solution which has been listed as Solution I in Table III. In this solution,

$$1/m_{\oplus+☾} = 328853 \pm 46, \quad 1/m_{♀} = 406740 \pm 4440,$$

and

$$dl'' = +0''.23 \pm 0''.52.$$

The preceding results suggest that the masses of Mercury and Mars may not only be poorly determined from Eros, but that their inclusion exerts a rather unfavorable effect on the results for the other unknowns. It seems advisable, therefore, to assume at least the mass of Mars as already better known from other sources, and to forego its determination from Eros. Solution I is based on this assumption. The 16-unknowns result for $1/m_{☿}$ shows a surprisingly small formal error, in strong contrast to the subsequent result from the extended 1926–65 arc (Rabe and Francis 1967). This appears to be a consequence of the fact that at the zero epoch 1931 January 18.0 the perturbation produced by Mercury in the mean motion n of Eros is close to an extremum of its more or less periodic oscillations, causing relatively large perturbations in longitude to build up in the double integration. Since it seems curious that the choice of the zero epoch should play such an important role in the accuracy of a mass determination, the present result should be accepted with reservations. It may also have been affected by correlations with other unknowns.

The normalized matrix exhibiting all correlation coefficients is given as Table IV. It reveals strong correlations between the mass corrections for Venus, Mars, and Mercury, as well as between dn and these three masses. Fortunately, the mass of earth+moon is less strongly correlated to the four unknowns just mentioned, except for the coefficient +0.85 due to Mars. This may explain the drastic change in the result for $1/m_{\oplus+☾}$ from Solution I, as compared to the values obtained from the 16- and 15-unknowns

TABLE II. Corrected normal equations (unit = 0.1).

dM_0	100 dn	ds	dp	dq	$d\varphi$	dl''	$d\epsilon$
+8767	+ 33208	+ 5894	+ 161	+ 197	+ 3089	− 4169	+ 718
	+1051744	+25158	−2434	− 107	−31552	−15197	+ 17
		+ 4104	+ 77	+ 216	+ 2006	− 2822	+ 505
			+1320	− 272	− 849	− 410	+ 327
				+3269	− 42	+ 653	+2143
					+ 7289	− 1307	− 294
						+ 2299	+ 73
							+1714

de''	$e''d\pi''$	$\Delta\alpha_0$	$\Delta\delta_0$	$10^{-3}\vartheta_{\oplus+☾}$	$10^{-3}\vartheta_{♀}$	$10^{-4}\vartheta_{♂}$	$10^{-4}\vartheta_{☿}$
− 4702	+ 6012	+1095	− 601	+ 2606	+ 822	+ 2116	+ 454 = −4337
+12326	+24546	+6776	+1825	+62373	+28957	+63943	+15903 = +3454
− 2959	+ 3637	+ 796	− 389	+ 1942	+ 628	+ 1542	+ 347 = −2889
+ 446	+ 493	+ 111	− 31	− 63	− 56	− 90	− 34 = − 240
+ 452	− 878	− 330	− 472	+ 81	− 4	+ 20	+ 8 = − 588
− 5774	+ 961	+ 229	− 390	− 2004	− 978	− 1995	− 471 = −2070
+ 2193	− 3040	− 572	+ 235	− 1245	− 374	− 971	− 200 = +2019
+ 274	+ 25	− 104	− 388	+ 159	+ 2	+ 49	+ 4 = − 794
+ 5534	− 3047	− 461	+ 322	+ 666	+ 437	+ 714	+ 189 = +2627
	+ 5900	+ 683	− 165	+ 1929	+ 639	+ 1727	+ 340 = −2875
		+ 277	0	+ 493	+ 177	+ 395	+ 101 = − 446
			+ 370	+ 41	+ 59	+ 85	+ 39 = + 410
				+ 7388	+ 1683	+ 4762	+ 826 = −1307
					+ 800	+ 1753	+ 439 = + 153
						+ 4193	+ 934 = − 142
							+ 282 = + 123

solutions involving $m_{♂}$. Additional strong correlations are seen to connect dM_0, ds, dl'', and $e''d\pi''$.

The additional Solutions II and III shown in Table III are those for 13 and 7 unknowns, respectively. Solution II omits $m_{♀}$ as an unknown, but retains all others as in Solution I, while Solution III determines only the six Eros elements in addition to the earth +moon mass. The square-sum $\Sigma[\Delta\Delta]$ of the final residuals and the mean representation error μ of the average observation equation of unit weight are listed with each solution. The omission of $m_{♀}$ is seen to result in a marked decrease of the formal errors of dn and $m_{\oplus+☾}$. This may be due to the strong correlations involving $m_{♀}$.

TABLE III. Solutions for 14, 13, and 7 unknowns.

	Solution I	Solution II	Solution III
$1/m_{\oplus+☾}$	328853±46	328863±29	328875±31
$1/m_{♀}$	406740±4440		
100 dn	+0″0216±0″0624	+0″0394±0″0018	+0″0407±0″0019
dM_0	+0″70±0″60	+0″64±0″56	−0″49±0″07
ds	−1.17±0.77	−1.12±0.74	−0.12±0.10
dp	+0.09±0.15	+0.09±0.15	−0.06±0.04
dq	−0.15±0.19	−0.14±0.19	−0.14±0.02
$d\varphi$	−0.15±0.13	−0.15±0.12	+0.04±0.02
dl''	+0.23±0.52	+0.23±0.52	
$d\epsilon$	−0.24±0.15	−0.24±0.15	
de''	−0.12±0.19	−0.12±0.19	
$e''d\pi''$	−0.47±0.21	−0.44±0.19	
$\Delta\alpha_0$	−0.18±0.52	−0.15±0.51	
$\Delta\delta_0$	0.00±0.18	−0.01±0.18	
$\Sigma[\Delta\Delta]$	6.66	6.67	9.09
μ	±0″333	±0″331	±0″368

The final residuals of Solutions I, II, and III have been listed in Table V. The residuals of Solution I have been given only in order to illustrate the minuteness of the changes caused by the inclusion of $m_{♀}$ as an unknown. None of the residuals is affected by more than ±0″03. Since the two values of $1/m_{♀}$ involved are 406740 and 408000, respectively, this finding underlines the rather poor determination even of the mass of Venus. Much larger are the changes produced in the residuals of Solution III in consequence of the omission of the elements of the earth's orbit and of $\Delta\alpha_0$ and $\Delta\delta_0$. The considerable increases in $\Sigma[\Delta\Delta]$ and μ, as one goes from Solution II to Solution III, indicate the true relevancy of some of these corrections. The pronounced changes in the values of some of the remaining 7 unknowns, as shown in Table III, have not been able to prevent the large increase of $\Sigma[\Delta\Delta]$. In the light of all these facts, it appears that Solution II is the most preferable one, especially since its results agree very well with those of the comparable solution from the extended 1926–65 arc (Rabe and Francis 1967). The present Solution II includes $1/m_{\oplus+☾} = 328863\pm29$, while the 13-unknowns solution 1926–65 leads to 328890±16. It should be noted that the first result is based on the hand-computed but obviously excellent perturbations by G. Stracke, while the second one rests on IBM-7094 integrations of the Eros orbit.

In conclusion, it can be stated that the corrected Eros observation equations 1926–45 lead to an earth +moon mass fairly close to that value of about 1/328900 which would be consistent with the most recent radar

TABLE IV. Normalized correlation coefficients.

	dM_0	dn	ds	dp	dq	$d\varphi$	dl''	$d\epsilon$
dM_0	+1.00							
dn	+0.35	+1.00						
ds	+0.98	+0.38	+1.00					
dp	+0.05	−0.07	+0.03	+1.00				
dq	+0.04	0.00	+0.06	−0.13	+1.00			
$d\varphi$	+0.39	−0.36	+0.37	−0.27	−0.01	+1.00		
dl''	−0.93	−0.31	−0.92	−0.23	+0.24	−0.32	+1.00	
$d\epsilon$	+0.19	0.00	+0.19	+0.22	+0.90	−0.08	+0.04	+1.00
de''	−0.68	+0.16	−0.62	+0.17	+0.11	−0.91	+0.61	+0.09
$e''d\pi''$	+0.84	+0.31	+0.74	+0.18	−0.20	+0.15	−0.82	+0.01
$\Delta\alpha_0$	+0.70	+0.39	+0.75	+0.18	−0.34	+0.16	−0.70	−0.14
$\Delta\delta_0$	−0.33	+0.09	−0.32	−0.04	−0.43	−0.24	+0.26	−0.49
ϑ⊕+☾	+0.32	+0.71	+0.35	−0.02	+0.02	−0.27	−0.30	+0.04
ϑ♀	+0.31	+1.00	+0.35	−0.06	0.00	−0.41	−0.27	0.00
ϑ♂	+0.35	+0.96	+0.37	−0.04	+0.01	−0.36	−0.31	+0.02
ϑ☿	+0.29	+0.92	+0.33	−0.05	+0.01	−0.33	−0.25	0.00

	de''	$e''d\pi''$	$\Delta\alpha_0$	$\Delta\delta_0$	ϑ+☾	ϑ♀	ϑ♂	ϑ☿
de''	+1.00							
$e''d\pi''$	−0.53	+1.00						
$\Delta\alpha_0$	−0.37	+0.53	+1.00					
$\Delta\delta_0$	+0.22	−0.11	0.00	+1.00				
ϑ⊕+☾	+0.10	+0.29	+0.34	+0.02	+1.00			
ϑ♀	+0.21	+0.30	+0.38	+0.11	+0.69	+1.00		
ϑ♂	+0.15	+0.35	+0.37	+0.07	+0.85	+0.96	+1.00	
ϑ☿	+0.15	+0.26	+0.36	+0.12	+0.58	+0.94	+0.85	+1.00

TABLE V. Residuals.

	$\Delta\alpha\cos\delta$ of solution				$\Delta\delta$ of solution		
Eq.	I	II	III	Eq.	I	II	III
1	+0″.18	+0″.15	+0″.14	38	+0″.01	−0″.01	−0″.31
2	−0.02	+0.01	+0.12	39	−0.40	−0.41	−0.52
3	−0.17	−0.18	+0.12	40	−0.14	−0.14	−0.36
4	+0.10	+0.10	+0.37	41	+0.02	+0.03	−0.17
5	0.00	0.00	+0.23	42	+0.05	+0.05	−0.11
6	−0.07	−0.06	+0.12	43	+0.17	+0.18	+0.06
7	+0.03	+0.04	+0.17	44	−0.32	−0.31	−0.38
8	+0.07	+0.08	+0.17	45	+0.02	+0.02	+0.02
9	+0.25	+0.26	+0.29	46	−0.16	−0.16	−0.10
10	−0.33	−0.32	−0.34	47	+0.04	+0.04	+0.18
11	+0.25	+0.25	+0.16	48	+0.08	+0.08	+0.28
12	+0.37	+0.37	+0.20	49	−0.16	−0.17	−0.01
13	−0.03	−0.03	−0.21	50	+0.24	+0.24	+0.37
14	−0.08	−0.08	−0.36	51	+0.25	+0.25	+0.32
15	+0.31	+0.30	+0.07	52	−0.22	−0.21	−0.18
16	−0.46	−0.46	−0.57	53	−0.28	−0.27	−0.27
17	−0.10	−0.10	−0.08	54	−0.25	−0.24	−0.30
18	−0.05	−0.05	+0.09	55	−0.12	−0.11	−0.21
19	−0.25	−0.25	−0.05	56	+0.15	+0.15	−0.02
20	0.00	0.00	+0.21	57	+0.04	+0.04	−0.20
21	−0.12	−0.12	+0.06	58	−0.09	−0.09	−0.39
22	−0.04	−0.05	+0.08	59	+0.03	+0.02	−0.31
23	+0.26	+0.25	+0.32	60	+0.23	+0.23	−0.12
24	−0.97	−0.98	−1.20	61	+0.61	+0.61	+0.60
25	−0.15	−0.13	−0.28	62	−0.21	−0.20	−0.45
26	+0.34	+0.34	+0.08	63	−0.13	−0.14	−0.28
27	−0.38	−0.39	−0.29	64	+0.02	+0.03	−0.26
28	+0.67	+0.66	+0.74	65	+0.58	+0.59	+0.39
29	+0.53	+0.52	+0.64	66	+0.10	+0.11	+0.13
30	−0.60	−0.60	−0.51	67	−0.46	−0.46	−0.54
31	−0.17	−0.18	−0.25	68	−0.24	−0.25	−0.54
32	+0.88	+0.88	+1.09	69	+0.31	+0.28	+0.14
33	+0.11	+0.12	−0.18	70	−0.34	−0.35	−0.50
34	−0.46	−0.46	−0.47	71	−0.09	−0.06	−0.33
35	−0.02	0.00	+0.24	72	+0.25	+0.22	+0.06
36	+0.03	+0.04	+0.08	73	+0.40	+0.39	+0.06
37	−0.37	−0.40	−0.14	74	+0.02	+0.02	−0.04

determinations of the astronomical unit. The formal mean errors associated with the present corrected results for $1/m_{\oplus+☾}$ (from the 13- and 7-unknowns solutions) are smaller than one half of the ±64 obtained with the original erroneous result. The new mean errors of the other planetary masses are much larger, however, at least for Venus and Mars, than the fictitiously small uncertainties found in 1950 from the erroneous mass coefficients. While the new result for the mass of Mercury looks surprisingly good, it should not be accepted at face value. In this connection, the finding of a systematic trend in the 1930–31 residuals of solutions that are not so predominated by this one close approach (Rabe and Francis 1967) becomes of particular interest. It appears that any unaccounted-for systematic errors, such as those first noted by Witt (1933) in the solar coordinates based on Newcomb's Tables, may be more easily absorbed by the unknowns (leading to distorted results especially for the more poorly determined ones) when magnified in one approach represented by 21 normals out of a total of only 37.

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