SOVIET ASTRONOMY—AJ VOL. 9, NO. 6, MAY-JUNE, 1966

SIZES OF THE LARGEST BODIES FALLING ONTO THE PLANETS DURING THEIR FORMATION

V. S. Safronov

Geophysics Institute, Academy of Sciences of the USSR
Translated from Astronomicheskii Zhurnal, Vol. 42, No. 6, pp. 1270-1276, November-December, 1965
Original article submitted March 25, 1965

Application of coagulation theory to the process of accumulation of the planets from solid matter leads to the conclusion that this matter was in the form of particles and bodies of different sizes. Falling onto the planets, the bodies imparted to them a rotational moment consisting of two components of different nature: a regular component ("direct" rotation), related to rotation of the system as a whole, and a random component, related to the random direction of velocity of the falling bodies relative to the planet and manifested in the inclinations of the axes of rotation of the planets. The largest bodies made the principal contribution to the random component of rotation. This article gives the derivation of expressions relating the values of the random component of rotation to the masses m_1 of the largest bodies falling onto a planet of mass m on the assumption of an exponential distribution function of the sizes of the bodies. Table 1 gives the values m_1/m determined from a comparison of the theoretically computed angles of inclination of the axes of rotation of the planets and the observed values. The largest bodies falling onto the earth had masses of about 10^{-3} of the earth's mass, that is, they were of the size of the largest asteroids. This same mechanism makes it possible to explain the anomalous rotation of Uranus if it is assumed that the random component of the rotation of Uranus was greater than the systematic component. The mass of the largest body falling onto the surface of Uranus in this case would have to be 0.05 of the mass of that planet.

An estimate of the size of the largest bodies falling onto the planets during their formation is important for determining the principal laws of the process of accumulation of the planets. It also is of considerable interest for geophysics because it is necessary for determining the earth's initial temperature and makes it possible to judge the scale of the initial inhomogeneities of the earth's mantle [1] which could exert an influence on the entire subsequent development of the earth. We already have noted [2] that the maximum size of the bodies from which the planets were formed can be determined from the inclinations of the axes of rotation of the planets. It was determined roughly that in the final stage of growth of a planet the masses of bodies falling onto it were less than 10^{-2} of the mass of the planet itself. This problem is considered in greater detail in this article, taking into account the probable distribution function of the protoplanetary bodies.

The observed rotation of the planets can be broken down into two components — systematic (regular) with the moment $\mathbf{K}_0$ (directed perpendicular to the central plane of the planetary system), characterizing direct rotation, and the random component with $\mathbf{K}_1$, manifested in the inclinations of the axes of rotation of the planets. The latter is related to the discreteness of the process of formation of the planets. It shows that a considerable part of the mass fell onto the planet in the form of individual bodies having randomly directed relative motion at the time of the impact. An identical order of magnitude of the angles of inclination of the axes of most of the planets is a characteristic property of a planetary system which has not yet been given due attention and which indicates a definite regularity in the growth process and a regularity in the size distribution of the bodies.

Assume m and r are the mass and the radius of the growing planet ("nucleus") and m'_i is the mass of the bodies falling onto it. First as a clarification we will consider the case when all the falling bodies have identical masses $m'_i = m'$ and move in the plane Oxy relative to the planet m, whose center is situated at O. Assume v is the velocity of the body relative to the planet prior to its approach to the planet. Then the momentum imparted to the planet by the mass m'_i

$$\Delta K_{1i} = m'vl_i \tag{1}$$

is directed along the z axis and represents a random value since the impact parameter l_i of the falling body is a random value with a constant density probability distribution in the interval $(-l_0, +l_0)$. The mathematical expectation l (mean value l) is equal to zero, but the mathematical expectation l^2 (l dispersion) is not equal to zero:

$$\mathrm{M}l = \bar{l} = 0, \quad \mathrm{D}l = \overline{l^2} = \frac{1}{2l}\int_{-l_0}^{+l_0} l^2\,dl = \frac{1}{3}l_0^2. \tag{2}$$

The value l_0 is the maximum impact parameter leading to a collision of m' with m and is related to the radii r and r' by the known relation

$$l_0^2 v^2 = (r + r')^2\left[v^2 + \frac{2G(m+m')}{r+r'}\right], \tag{3}$$

being an elementary corollary of the laws of conservation of energy and moment of momentum in a two-body system.

With the falling of several bodies m' onto m, in accordance with the theorem of addition of dispersion as the sum of the independent random values [3], when m'v = const we have

$$\mathrm{D}\sum_{i=1}^{n}\Delta K_{1i} = (m'v)^2\,\mathrm{D}\sum_{i=1}^{n} l_i$$

$$= (m'v)^2\sum_{i=1}^{n}\mathrm{D}l_i = (m'v)^2\frac{nl_0^2}{3}. \tag{4}$$

Therefore, the mean value of the square of momentum, imparted by n bodies m' with the total mass $\Delta m = nm'$, is equal to

$$\overline{\Delta K_1^2} = (m'v)^2\frac{nl_0^2}{3} = (vl_0)^2\frac{m'}{3}\Delta m. \tag{5}$$

Since $\overline{\Delta K_1} \propto \bar{l} = 0$, the random component of the moment of momentum ΔK_1 imparted to the planet by the falling bodies obviously is determined by its mean-square deviation, related to m' by expression (5). Expression (5) shows that the imparted momentum is the greater the larger the body m'. Small particles make virtually no contribution to ΔK_1.

In the more general case of motion of bodies in all possible directions an estimate of the imparted momentum can be made in the following way. Assume one third of all the bodies (n/3) move parallel to the x axis, a third parallel to the y axis and a third parallel to the z axis. This method is applied in the kinetic theory of gases.

We will consider bodies moving toward the surface of a planet in the direction of the z axis. Upon falling onto the planet they will impart to it the momentum components K_{1ix} and K_{1iy} along the x and y axes respectively. Obviously

$$K_{1ix} = m'vl\sin\varphi, \quad K_{1iy} = m'vl\cos\varphi, \tag{6}$$

where φ is the angle between the plane Oxz and the orbital plane of the body relative to the planet. The dispersion of the random value K_{1ix} is equal to

$$\mathrm{D}K_{1ix} = \overline{K_{1ix}^2} = (m'v)^2\frac{\int_0^{l_0}\int_0^{2\pi}(l\sin\varphi)^2\,l\,dl\,d\varphi}{\int_0^{l_0}\int_0^{2\pi}l\,dl\,d\varphi} = \frac{(m'vl_0)^2}{4}. \tag{7}$$

Similarly

$$\mathrm{D}K_{1iy} = \frac{(m'vl_0)^2}{4}.$$

The component of momentum along the x axis is also introduced by bodies moving parallel to the y axis toward the surface of a planet; in this case the dispersion $\mathrm{D}K_{1ix}$ is determined by expression (7). The dispersion of the sum of the random values K_{1ix} is equal to the sum of the dispersions of the terms

$$\mathrm{D}\sum_{i=1}^{n}K_{1ix} = 2\,\frac{n}{3}\,\mathrm{D}K_{1ix} = \frac{n}{6}\,(m'vl_0)^2. \tag{8}$$

The dispersion of the components of momentum along the y and z axes will be the same. According to (8), the mathematical expectation of the square of the component of momentum along the x axis is

$$\Delta K_{1x}^2 = \frac{v^2 l_0^2 m'\Delta m}{6}. \tag{9}$$

Therefore,

$$\Delta K_1^2 = \Delta K_{1x}^2 + \Delta K_{1y}^2 + \Delta K_{1z}^2 = \frac{1}{2} v^2 l_0^2 m' \Delta m. \quad (10)$$

We substitute here $v^2 l_0^2$ from (3), on the right-hand side of the latter, assuming $v^2 = Gm/\theta r$, where θ is of the order of unity, and omitting the terms m' and r', which, as will be demonstrated below, are small in comparison with m and r. Then

$$\Delta K_1^2 = \left(1 + \frac{1}{2\theta}\right) Gmrm'\Delta m = \left(1 + \frac{1}{2\theta}\right) Gmrnm'^2. \quad (11)$$

The imparted specific momentum is inversely proportional to the root of n:

$$\Delta K_1 / \Delta m = \sqrt{(1 + 1/2\theta) Gmr/n}. \quad (11')$$

On the basis of the rule of addition of dispersion it is easy to obtain an expression for ΔK_1^2 in the more general case when the masses m'_j of falling bodies are different. This requires that expression (11) be summed for all m'_j. Assume n(m') is the mass distribution of bodies falling onto the planet; these bodies have the total mass

$$\Delta m = \int_0^{m_1} m' n(m') dm'. \quad (12)$$

Integrating (11) for all m' and substituting Δm from (12), we obtain

$$\Delta K_1^2 \approx \left(1 + \frac{1}{2\theta}\right) Gmr \frac{\int_0^{m_1} n(m') m'^2 dm'}{\int_0^{m_1} n(m') m' dm'} \Delta m, \quad (13)$$

where m_1 is the mass of the largest body, not counting the planet itself. Obviously, this relation makes sense when $m_1 \ll \Delta m \ll m$.

In general, the expression

$$J(m, m_1) = \frac{\int_0^{m_1} n(m') m'^2 dm'}{m \int_0^{m_1} n(m') m' dm'} \quad (14)$$

is a function of planetary mass m, since n(m') is dependent on time. If

$$n(m', t) = c(t) m'^{-q}, \quad (15)$$

then, when $q < 2$,

$$J(m, m_1) = \frac{\int_0^{m_1} m'^{2-q} dm'}{m \int_0^{m_1} m'^{1-q} dm'} = \frac{2-q}{3-q} \frac{m_1}{m}. \quad (16)$$

The masses m' of the falling bodies increase parallel with the growth of the planet; therefore, in the first approximation it can be assumed that m_1/m is constant. Then J = const. Assuming the density of the planet to be constant and integrating (13) for m, we find the value of the square of the random component of the rotational moment of the planet:

$$K_1^2 = \sum \Delta K_1^2 = \left(1 + \frac{1}{2\theta}\right) GJ \int_0^m m^2 r\, dm \approx \left(1 + \frac{1}{2\theta}\right) GJ \frac{3}{10} m^3 r$$

and

$$K_1 = m \sqrt{\frac{3}{10}\left(1 + \frac{1}{2\theta}\right) JGmr}. \quad (17)$$

Allowance for an increase of density of the planet with m exerts virtually no influence on the results: the right-hand side of (17) increases only by a value of about 1%. It is possible that m_1/m increased in the final stage of growth of m. Then the masses of the largest falling bodies could be several tens of percent greater than the values determined below on the basis of (20) on the assumption J = const.

It was assumed in (13) that Δm is the total increment of the earth's mass, since we assume that virtually all bodies falling onto the earth imparted both regular and random components of rotation. According to the A. V. Artem'ev-V. V. Radzievskii hypothesis [4], the regular component of rotation was imparted by bodies not falling directly onto the planet but trapped by it as a result of their inelastic collisions in its zone of attraction, that is, by essentially the same mechanism which according to E. L. Ruskol [5] led to the formation of a satellite swarm around the planet. If the largest bodies were not present in the matter trapped in this way, or if these bodies were greatly broken down during collisions, the random rotational component which they imparted to the planet was small. Then (if the authors' hypothesis is accepted) these bodies should not be included in Δm. However, according to the authors' own estimate, the part of the matter falling onto the planet by such a "two-stage" method should be only several percent. The corresponding correction to

m_1 therefore falls within the limits of accuracy of our estimate.

The vector $\mathbf{K}_1$ has a random direction in space. Assume the angle between the systematic component of momentum $\mathbf{K}_0$ directed perpendicular to the orbital plane and $\mathbf{K}_1$ is equal to δ, and the angle between $\mathbf{K}_0$ and the vector of the total moment of momentum of the planet $\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1$ (inclination of the axis of rotation) is equal to ε. Then the component of momentum perpendicular to $\mathbf{K}_0$ is equal to

$$K_1 \sin \vartheta = K \sin \varepsilon. \tag{18}$$

The right-hand side of (18) is known from observations. On the left-hand side K_1 represents the relation (17) and the angle ϑ can have any value between 0 and π. As the probable value $\sin\vartheta$ in (18) it is natural to use its mean value. In the case of a uniform distribution of vectors K_1 over the sphere,

$$\overline{\sin \vartheta} = \frac{1}{4\pi} \int_0^{\pi} \sin \vartheta \, 2\pi \sin \vartheta \, d\vartheta = \frac{\pi}{4}. \tag{19}$$

Substituting $\overline{\sin\vartheta}$ and K_1, expressed through m_1/m using (16) and (17), into (18), we find

$$\frac{m_1}{m} = \frac{3-q}{2-q} \frac{160 \sin^2 \varepsilon}{3\pi^2 (1 + 1/2\theta)} \frac{K^2}{Gm^3 r}. \tag{20}$$

For numerical estimates it is convenient to introduce the velocity of rotation at the equator v_r and the Keplerian angular velocity v_s at the surface of the planet:

$$K = \frac{2}{5} \mu m r v_r, \qquad v_c = \sqrt{Gm/r}. \tag{21}$$

Then from (20) we obtain

$$\frac{m_1}{m} = \frac{3-q}{2-q} \frac{10}{3(1 + 1/2\theta)} \left(\frac{8\mu \sin \varepsilon}{5\pi} \frac{v_r}{v_c} \right)^2. \tag{22}$$

The masses of the largest bodies falling onto the planet, computed using this formula, on the assumption of a power function distribution for them with a value $q = 3/2$ (distribution by radii with the exponent $p = 3q - 2 = 2.5$) are given in the first column of Table 1.

For Uranus, in place of $\sin \varepsilon$ we used the ratio $\pi K_1/4K$, found on the assumption that the systematic component of the moment of Uranus K_0 corresponds to a period of rotation of 15 h (approximately the same as for Neptune).

When $\theta \geq 3$ the role of the parameter in (22) characterizing the relative velocities of bodies before encountering the planet is insignificant. We assumed $\theta = 3$. Change of q from $3/2$ to $5/3$ ($p = 3$) increases m_1/m by only $4/3$ times. Only when $q \rightarrow 2$ ($p \rightarrow 4$) does the result change appreciably. When $q = 2$ on the right-hand side of (22) in place of $(3-q)/(2-q)$ we have $\ln(m_1/m_0)$, where m_0 are the masses of the smallest particles in the used distribution. However, a distribution with $q = 2$, in which a large part of the mass is accounted for by large particles, apparently is unrealistic. Data on size distribution of asteroids [6, 7], meteors [8], and comets [9] indicate values p of about 2.6-3.4.

The second column of the table gives values m_1/m computed for $q = -\infty$, that is, for a case when all the falling bodies have identical masses. These values are smaller than the preceding values by a factor of three. Another limiting case can be considered, when the random component K_1 of the moment is imparted by only one body m_{11}, while all the remaining matter falling onto the planet imparts to it only regular rotation (K_0). Then

$$K \sin \varepsilon = \frac{\pi}{4} K_1 = \frac{\pi}{4} m_{11} \bar{l} v = \frac{\pi}{4} m_{11} \frac{2}{3} l_0 v \tag{23}$$

and

$$\frac{m_{11}}{m} = \frac{6\sqrt{2}\,\mu}{5\pi\sqrt{1 + 1/2\theta}} \frac{v_r}{v_c} \sin \varepsilon. \tag{24}$$

The values m_{11}/m are given in the last column of the table. They are less variable from planet to planet than the values m_1/m and do not involve the assumption of a specific form of the size distribution function for the bodies. The values m_{11}/m can be considered the upper limit for masses of bodies falling onto the planets.

TABLE 1

Planet	m_1/m		m_{11}/m	Planet	m_1/m		m_{11}/m
	$q = 3/2$	$q = -\infty$			$q = 3/2$	$q = -\infty$	
Earth	$8 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	$1 \cdot 10^{-2}$	Saturn	$3 \cdot 10^{-2}$	$1 \cdot 10^{-2}$	$6 \cdot 10^{-2}$
Mars	$2 \cdot 10^{-3}$	$6 \cdot 10^{-4}$	$1.3 \cdot 10^{-2}$	Uranus	$5 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	$8 \cdot 10^{-2}$
Jupiter	$3 \cdot 10^{-4}$	$9 \cdot 10^{-5}$	$5 \cdot 10^{-3}$	Neptune	$5 \cdot 10^{-3}$	$2 \cdot 10^{-3}$	$2 \cdot 10^{-2}$

These results of computations reveal that despite the absence of final data on the size distribution function for the bodies the masses of the largest bodies falling onto the planets during the course of their formation are determined quite reliably: there is not more than a threefold deviation in either direction. The masses of the largest bodies falling onto the earth were about 10^{-3} of the mass of the earth. As a result of the lunar tidal effect the earth's rotation is slowed, and although the inclination ε of the axis increases, the value $v_r \sin \varepsilon$ decreases [10]. If the moon was formed considerably closer to the earth than its present position, $v_r \sin \varepsilon$ in the past was considerably greater. Therefore, it is not impossible that the value m_1/m for the earth determined above should be increased by a factor of 2-3.

The retrograde rotation of Uranus can be attributed naturally to the relatively greater sizes of the bodies forming the planet. The masses of the largest bodies falling onto Uranus attained 0.05 of the planetary mass. The bodies in the zone of formation of Saturn also were of considerable size. The largest of these bodies were 0.03 of the planetary mass. Therefore, the rotation of Saturn differs little in its anomalous character from that of Uranus. The apparent cause of the anomalies is the flight of larger bodies from the zone of Jupiter into the zones of these planets. Jupiter grew considerably more rapidly and attained a critical mass earlier. Upon attaining this critical mass the gravitational scattering of bodies from its neighborhood began. It should be noted that the estimates of m_1/m for Jupiter and Saturn made above are in need of appreciable refinement because they do not take into account the accretions of gaseous hydrogen in the final stage of growth of these planets. However, such a refinement is possible only on the basis of the theory of growth of the major planets, which requires consideration of a number of poorly studied factors, and such a theory has not yet been developed.

LITERATURE CITED

1. V. S. Safronov, Izv. AN SSSR, Ser. geofiz., No. 7, 1 (1965).
2. V. S. Safronov, Voprosy kosmogonii, 7, 59, 121 (1960).
3. B. V. Gnedenko, Course in the Theory of Probability [in Russian] (Fizmatgiz, Moscow, 1962), p. 181.
4. A. V. Artem'ev and V. V. Radzievskii, Astron. zh., 42, 124 (1965) [Soviet Astronomy – AJ, Vol. 9, p. 96].
5. E. L. Ruskol, Astron. zh., 37, 690 (1960) [Soviet Astronomy – AJ, Vol. 4, p. 657].
6. C. O. R. Jashek, Observatory, 80, 119 (1960).
7. S. Piotrowski, Acta Astron. ser. a, 5, 115 (1954).
8. H. Brown, J. Geophys. Res., 65, 1679 (1960).
9. E. J. Öpik, Monthly Notices Roy Astron. Soc., 120, 404 (1960).
10. H. Gerstenkorn, Z. Astrophys., 36, 245 (1955).