# PERTURBATIONS OF AN EXPANDING UNIVERSE

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## ABSTRACT

Perturbations of a spatially homogeneous isotropic universe are investigated in terms of small variations of the curvature. It is found that rotational perturbations die away. Density perturbations grow relatively to the background, but galaxies cannot be formed by the growth of perturbations that were initially small. In the steady-state universe small rotational and density perturbations die away.

The behavior of gravitational radiation in expanding universes is also investigated. Its "energy density" decreases at the same rate as that of electromagnetic radiation, although its active gravitational effect is only half as great. If a small amount of viscosity is present, gravitational radiation will be completely absorbed in the steady-state universe but not in an evolutionary universe.

## I. INTRODUCTION

Perturbations of a spatially homogeneous and isotropic universe have been investigated in a Newtonian model by Bonnor (1957), in a Newtonian approximation to a relativistic model by Irvine (1965), and relativistically by Lifshitz (1946) and Lifshitz and Khalatnikov (1963). Lifshitz' method was to consider small variations of the metric tensor. This has the disadvantage that the metric tensor is not a physically significant quantity. That is, one cannot directly measure it but only its second derivatives. It is thus not always obvious what the physical interpretation of a given perturbation of the metric is. Indeed it need have no physical interpretation at all, but merely correspond to a coordinate transformation. Instead it seems preferable to employ a method which considers small variations of the physically significant quantity, the curvature. This method has an additional advantage in the discussion of the behavior of gravitational radiation in an expanding universe, since it includes the interaction between the gravitational radiation and the matter. This interaction was not present in the approximations mentioned above.

## II. NOTATION

Space time is represented as a four-dimensional Riemannian space with metric tensor $g_{ab}$ of signature $+2$. Covariant differentiation is indicated by a semicolon, and covariant differentiation along a world line by a prime. Square brackets around indices indicate antisymmetrization; round brackets, symmetrization. The conventions for the Riemann and Ricci tensors are

$$v_{a;[bc]} = 2R^{p}{}_{acb}v_p \,, \qquad R_{ab} = R_a{}^p{}_{bp} \,.$$

Also $\eta_{abcd}$ is the alternating tensor. Units are such that $k$, the gravitational constant, and $c$, the speed of light, are 1.

## III. THE FIELD EQUATIONS

We assume the Einstein field equation

$$R_{ab} - \tfrac{1}{2}g_{ab}R = -T_{ab} \,,$$

where $T_{ab}$ is the energy-momentum tensor of matter. We will assume that the matter consists of a perfect fluid. Then,

$$T_{ab} = \mu u_a u_b + p h_{ab}$$

where $\mu$ is the density, $p$ is the pressure, $u_a$ is the velocity of the fluid, $u_a u^a = -1$, and $h_{ab} = g_{ab} + u_a u_b$ is the projection operator into the hyperplane orthogonal to $u_a$:

$$h_{ab} u^b = 0 .$$

We decompose the gradient of the velocity vector $u_a$ as

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3} h_{ab} \theta - u'_a u_b ,$$

where $u'_a = u_{a;b} u^b$ is the acceleration, $\theta = u_a{}^{;a}$ is the expansion, $\sigma_{ab} = u_{(c;d)} h^c{}_a h^d{}_b - \frac{1}{3} h_{ab} \theta$ is the shear, and $\omega_{ab} = u_{[c;d]} h^c{}_a h^d{}_b$ is the rotation of the flow lines $u_a$. We define the rotation vector $\omega_a$ as

$$\omega_a = \tfrac{1}{2} \eta_{abcd} \omega^{cd} u^b .$$

We may decompose the Riemann tensor $R_{abcd}$ into the Ricci tensor $R_{ab}$ and the Weyl tensor $C_{abcd}$:

$$R_{abcd} = C_{abcd} - g_{a[d} R_{c]b} - g_{b[c} R_{d]a} - R/3 g_{a[c} g_{d]b} ,$$

$$C_{abcd} = C_{[ab][cd]} ,$$

$$C^a{}_{bca} = 0 = C_{a[bcd]} .$$

$C_{abcd}$ is that part of the curvature that is not determined locally by the matter. It may thus be taken as representing the free gravitational field (Jordan, Ehlers, and Kundt 1960). We may decompose it into its "electric" and "magnetic" components.

$$E_{ab} = -C_{apbq} u^p u^q ,$$

$$H_{ab} = -\tfrac{1}{2} C_a{}^{pqr} \eta_{qrbs} u_p u^s ,$$

$$C_{ab}{}^{cd} = 8 u_{[a} E_{b]}{}^{[c} u^{d]} - 4 \delta^{[c}_{[a} E^{d]}_{b]} - 2 \eta_{abpq} u^p H^{q[c} u^{d]} - 2 \eta^{cdrs} u_r H_{s[a} u_{b]} ,$$

$$E_{ab} = E_{(ab)} , \qquad H_{ab} = H_{(ab)} , \qquad E_a{}^a = H_a{}^a = 0 , \qquad E_{ab} u^b = H_{ab} u^b = 0 .$$

$E_{ab}$ and $H_{ab}$ each have five independent components. We regard the Bianchi identities,

$$R_{ab[cd;e]} = 0$$

as field equations for the free gravitational field. Then

$$C_{abcd}{}^{;d} = -R_{c[b;a]} + \tfrac{1}{6} g_{c[b} R_{;a]}$$

(Kundt and Trümper 1962). Using the decompositions given above, we may write these in a form analogous to the Maxwell equations:

$$h_a{}^b E_{bc;d} h^{cd} + 3 H_{ab} \omega^b - \eta_{abcd} u^b \sigma^c{}_e H^{de} = \tfrac{1}{3} h_a{}^b \mu_{;b} , \tag{1}$$

$$h_a{}^b H_{bc;d} h^{cd} - 3 E_{ab} \omega^b - \eta_{abcd} u^b \sigma^c{}_e E^{de} = (\mu + p) \omega_a , \tag{2}$$

$$\begin{aligned} \perp E'_{ab} + h_{(a}{}^f \eta_{b)cde} u^c H_f{}^{d;e} + E_{ab} \theta - E^c{}_{(a} \omega_{b)c} - E^c{}_{(a} \sigma_{b)c} - \eta_{acde} \eta_{bpqr} u^c u^p \sigma^{dq} E^{er} \\ + 2 H^d{}_{(a} \eta_{b)cde} u^c u'^e = -\tfrac{1}{2} (\mu + p) \sigma_{ab} , \end{aligned} \tag{3}$$

$$\begin{aligned} \perp H'_{ab} - h_{(a}{}^f \eta_{b)cde} u^c E_f{}^{d;e} + H_{ab} \theta - H^c{}_{(a} \omega_{b)c} - H^c{}_{(a} \sigma_{b)c} \\ - \eta_{acde} \eta_{bpqr} u^c u^p \sigma^{dq} H^{er} + 2 H^d{}_{(a} \eta_{b)cde} u^c u'^e = 0 , \end{aligned} \tag{4}$$

where $\perp$ indicates projection by $h_{ab}$ orthogonal to $u_a$ (cf. Trümper 1964).

The contracted Bianchi identities give

$$(R_{ab} - \tfrac{1}{2}g_{ab}R)^{;b} = -T_{ab}{}^{;b} = 0 , \tag{5}$$

$$\mu' + (\mu + p)\theta = 0 ,$$
$$(\mu + p)u'_a + p_{;b}h^b{}_a = 0 . \tag{6}$$

The definition of the Riemann tensor is

$$u_{a;[bc]} = 2R_{apbc}u^p .$$

Using the decompositions as above we may obtain what may be regarded as "equations of motion."

$$\theta' = 2\omega^2 - 2\sigma^2 - \tfrac{1}{3}\theta^2 + u'_a{}^{;a} - \tfrac{1}{2}(\mu + 3p) , \tag{7}$$

$$\perp\omega'_{ab} = -\tfrac{2}{3}\omega_{ab}\theta + 2\sigma_{c[a}\omega_{b]}{}^c + u'_{[p;q]}h^p{}_a h^q{}_b , \tag{8}$$

$$\perp\sigma'_{ab} = E_{ab} - \omega_{ac}\omega^c{}_b - \sigma_{ac}\sigma^c{}_b - \tfrac{2}{3}\sigma_{ab}\theta$$
$$- \tfrac{1}{3}h_{ab}(2\omega^2 - 2\sigma^2 + u_c{}^{;c}) + u'_a u'_b + u'_{(p;q)}h^p{}_a h^q{}_b , \tag{9}$$

where

$$2\omega^2 = \omega_{ab}\omega^{ab} , \qquad 2\sigma^2 = \sigma_{ab}\sigma^{ab} .$$

We also obtain what may be regarded as equations of constraint.

$$\theta_{;b}h^b{}_a = \tfrac{3}{2}[(\omega^b{}_{c;b} + \sigma^b{}_{c;b})h^c{}_a - u'^b(\omega_{ab} + \sigma_{ab})] , \tag{10}$$

$$\omega_a{}^{;a} = 2\omega_a u'^a , \tag{11}$$

$$H_{ab} = -h^f{}_{(a}\eta_{b)cde}u^c[\omega_f{}^{d;e} + \sigma_f{}^{d;e}] . \tag{12}$$

We consider perturbations of a universe that in the undisturbed state is conformally flat, that is,

$$C_{abcd} = 0 .$$

By equations (1)–(3), this implies,

$$\sigma_{ab} = \omega_{ab} = 0 ,$$

$$h_a{}^b\mu_{;b} = 0 = \theta_{;b}h^b{}_a .$$

If we assume an equation of state of the form $p = p(\mu)$, then by equations (6) and (10),

$$p_{;b}h^b{}_a = 0 = u'_a .$$

This implies that the universe is spatially homogeneous and isotropic since there is no direction defined in the 3-space orthogonal to $u_a$.

In this universe we consider small perturbations of the motion of the fluid and of the Weyl tensor. We neglect products of small quantities and perform derivatives with respect to the undisturbed metric. Since all the quantities we are interested in, with the exception of the scalars $\mu$, $p$, and $\theta$, have unperturbed value zero, we avoid perturbations that merely represent coordinate transformation and have no physical significance.

To the first order, equations (1)–(4) and (7)–(9) are

$$E_{ab}{}^{;b} = \tfrac{1}{3}h_a{}^b\mu_{;b} , \tag{13}$$

$$H_{ab}{}^{;b} = (\mu + p)\omega_a , \tag{14}$$

$$E'_{ab} + E_{ab}\theta + h^f{}_{(a}\eta_{b)\,cde}u^c H_f{}^{d;e} = -\tfrac{1}{2}(\mu + p)\sigma_{ab} , \tag{15}$$

$$H'_{ab} + H_{ab}\theta - h^f{}_{(a}\eta_{b)\,cde}u^c E_f{}^{d;e} = 0 , \tag{16}$$

$$\theta' = -\tfrac{1}{3}\theta^2 + u'_a{}^{;a} - \tfrac{1}{2}(\mu + 3p) , \tag{17}$$

$$\omega'_{ab} = -\tfrac{2}{3}\omega_{ab}\theta + u'_{[p;q]}h^p{}_a h^q{}_b , \tag{18}$$

$$\sigma'_{ab} = E_{ab} - \tfrac{2}{3}\sigma_{ab}\theta - \tfrac{1}{3}h_{ab}u'_c{}^{;c} + u'_{(p;q)}h^p{}_a h^q{}_b . \tag{19}$$

From these we see that perturbations of rotation or of $E_{ab}$ or $H_{ab}$ do not produce perturbations of the expansion or the density. Nor do perturbations of $E_{ab}$ and $H_{ab}$ produce rotational perturbations.

## IV. THE UNDISTURBED METRIC

Since in the unperturbed state the rotation and acceleration are zero, $u_a$ must be a gradient:

$$u_a = \tau_{;a} ,$$

where $\tau$ measures the proper time along the world lines. As the surfaces $\tau$ = constant are homogeneous and isotropic they must be 3-surfaces of constant curvature. Therefore the metric can be written,

$$ds^2 = -d\tau^2 + \Omega^2 d\gamma^2$$

where $\Omega = \Omega(\tau)$, and $d\gamma^2$ is the line element of a space of zero or unit positive or negative cuıvature. We define $t$ by

$$\frac{dt}{d\tau} = \frac{1}{\Omega} .$$

Then

$$ds^2 = \Omega^2(-dt^2 + d\gamma^2) .$$

In this metric, $u_a = (-\Omega, 0, 0, 0)$,

$$\theta = \frac{3\Omega'}{\Omega} = \frac{3}{\Omega^2}\frac{d\Omega}{dt} .$$

Then, by equations (5) and (7),

$$\mu' = -(\mu + p)\,3\,\frac{\Omega'}{\Omega} , \tag{20}$$

$$3\,\frac{\Omega''}{\Omega} = -\tfrac{1}{2}(\mu + 3p) . \tag{21}$$

If we know the relation between $\mu$ and $p$, we may determine $\Omega$. We will consider the two extreme cases, $p = 0$ (dust) and $p = \mu/3$ (radiation). Any physical situation should lie between these.

### *The Case for $p = 0$*

By equation (20), $\mu = M/\Omega^3$, $M$ = const. Therefore,

$$\frac{3}{M}\frac{\Omega''}{\Omega} - \frac{1}{2\Omega^3} = 0 , \qquad \frac{3}{M}(\Omega')^2 - \frac{1}{\Omega} = E , \qquad E = \text{const} .$$

*a*) For $E > 0$:

$$\Omega = \frac{1}{2E}[\cosh\sqrt{(EM/3)}\,t - 1] , \qquad \tau = \frac{1}{2E}[\sqrt{(3/EM)}\sinh\sqrt{(EM/3)}\,t - t] .$$

*b*) For $E = 0$:

$$\Omega = \frac{M}{12} t^2, \qquad \tau = \frac{M}{36} t^3 .$$

*c*) For $E < 0$:

$$\Omega = \frac{-1}{2E}[1 - \cos\sqrt{(-EM/3)}\,t], \quad \tau = \frac{-1}{2E}[t - \sqrt{(-3/EM)}\sin\sqrt{(-EM/3)}\,t].$$

$E$ represents the energy (kinetic + potential) per unit mass. If it is non-negative the universe will expand indefinitely; otherwise it will eventually contract again.

By the Gauss-Codazzi equations $^*R$, the curvature of the hypersurface $\tau =$ const. is

$$^*R = 2(-\tfrac{1}{3}\theta^2 + \mu) = -2\frac{EM}{\Omega^2}.$$

If $E \neq 0$, we normalize $M$ as $M = 3/|E|$ .

*The Case for $p = \mu/3$*

$$\mu' = -4\frac{\Omega'}{\Omega}, \qquad 3\frac{\Omega''}{\Omega} = -\mu, \qquad \mu = \frac{M}{\Omega^4},$$

and therefore

$$\frac{3(\Omega')^2}{M} - \frac{1}{\Omega^2} = E .$$

*a*) For $E > 0$:

$$\Omega = \frac{1}{E}\sinh t, \qquad \tau = \frac{1}{E}(\cosh t - 1).$$

*b*) For $E = 0$:

$$\Omega = t, \qquad \tau = \tfrac{1}{2}t^2 .$$

*c*) For $E < 0$:

$$\Omega = -\frac{1}{E}\sin t, \qquad \tau = \frac{1}{E}(\cos t - 1).$$

## V. ROTATIONAL PERTURBATIONS

By equation (6)

$$u'_{[c:d]} h_a{}^c h_b{}^d = -\frac{\omega_{ab} p'}{\mu + p}.$$

Therefore

$$\omega'_{ab} = -\omega_{ab}\left(\tfrac{2}{3}\theta + \frac{p'}{\mu + p}\right).$$

For $p = 0$,

$$\omega = \frac{\omega_0}{\Omega^2}.$$

For $p = \mu/3$,

$$\omega' = -\omega\left(\tfrac{2}{3}\theta + \tfrac{1}{4}\frac{\mu'}{\mu}\right) = -\tfrac{1}{3}\omega\theta .$$

Therefore $\omega = \omega_0/\Omega$.

Thus rotation dies away as the universe expands. This is in fact a statement of the conservation of angular momentum in an expanding universe.

## VI. PERTURBATIONS OF DENSITY

For $p = 0$ we have the equations,

$$\mu' = -\mu\theta, \qquad \theta' = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}\mu .$$

These involve no spatial derivatives. Thus the behavior of one region is unaffected by the behavior of another. Perturbations will consist in some regions having slightly higher or lower values of $E$ than the average. If the universe as a whole has a value of $E$ greater than zero, a small perturbation will still have $E$ greater than zero and will continue to expand. It will not contract to form a galaxy. If the universe has a value of $E$ less than zero, a small perturbation can contract. However, it will only begin contracting at a time $\delta\tau$ earlier than the whole universe begins contracting, where

$$\frac{\delta\tau}{\tau_0}=\frac{\delta E}{E_0}.$$

Here $\tau_0$ is the time at which the whole universe begins contracting. There is only any real instability when $E = 0$. This case is of measure zero relative to all the possible values $E$ can have. However, this cannot really be used as an argument to dismiss it, as there might be some reason why the universe should have $E = 0$. For a region with energy $-\delta E$ in a universe with $E = 0$,

$$\Omega=\frac{1}{4\delta E}\left(t^2-\frac{t^4}{12}+\dots\right),\qquad \tau=\frac{1}{12\delta E}\left(t^3-\frac{t^5}{20}+\dots\right),$$

$$\mu=\frac{3}{\delta E\Omega^3}=\tfrac{4}{3}\tau^{-2}\left[1+\left(\frac{12}{5}\delta E\right)^{2/3}\tau^{2/3}+\dots\right].$$

For $E = 0$, $\mu = \frac{4}{3}\tau^{-2}$.

Thus the perturbation grows only as $\tau^{2/3}$. This is not fast enough to produce galaxies from statistical fluctuations even if these could occur. However, since an evolutionary universe has a particle horizon (Rindler 1956; Penrose 1964), different parts do not communicate in the early stages. This makes it even more difficult for statistical fluctuations to occur over a region until light had time to cross the region.

For $p = \mu/3$,

$$\mu'=-\tfrac{4}{3}\mu\theta\,,\qquad \theta'=-\tfrac{1}{3}\theta^2-\mu+u'_a{}^{;a}\,,\qquad u'_a=-\frac{h^b{}_a\mu_{;b}}{4\mu}.$$

As before, a perturbation cannot contract unless it has a negative value of $E$. The action of the pressure forces makes it still more difficult for it to contract. Eliminating $\theta$,

$$\mu\mu''-\tfrac{5}{4}(\mu')^2-\tfrac{4}{3}\mu^3+\tfrac{4}{3}\mu^2u'_a{}^{;a}=0\,,$$

$$u'_a{}^{;a}=u'_{a;b}h^{ab}+u'_au'^a=-\tfrac{1}{4}\,\frac{h^{ac}(h^b{}_a\mu_{;b})_{;c}}{\mu}$$

to our approximation; $h^{ac}\nabla_c h^b{}_a\nabla_b$ is the Laplacian in the hypersurface $\tau$ = constant. We represent the perturbation as a sum of eigenfunctions $S^{(n)}$ of this operator, where

$$S^{(n)}{}_{;c}u^c=0\,,\qquad h^{ac}(h^b{}_aS^{(n)}{}_{;b})_{;c}=-\frac{n^2}{\Omega^2}S^{(n)}.$$

These eigenfunctions will be hyperspherical and pseudohyperspherical harmonics in cases ($c$) and ($a$), respectively and plane waves in case ($b$). In case ($c$) $n$ will take only discrete values but in ($a$) and ($b$) it will take all positive values.

$$\mu=\mu_0\left[1+\sum_n B^{(n)}S^{(n)}\right],$$

where $\mu_0$ is the undisturbed density. Therefore

$$B''^{(n)}\mu_0 - \tfrac{1}{2}B'^{(n)}\mu'_0 - B^{(n)}\left(\tfrac{4}{3}\mu_0^2 - \frac{n^2}{3\Omega^2}\,\mu_0\right) = 0\,.$$

As long as $\mu_0 > n^2/4\Omega^2$, $B^{(n)}$ will grow. For $\mu_0 \gg n^2/4\Omega^2$,

$$B^{(n)} \simeq C\tau + D\tau^{-1}\,.$$

These perturbations grow for as long as light has not had time to travel a significant distance compared to the scale of the perturbation ($\sim\Omega/n$). Until that time, pressure forces cannot act to even out perturbations.

When $n^2/\Omega \gg \mu_0$,

$$\frac{d^2B^{(n)}}{dt^2} + \frac{dB^{(n)}}{dt}\,\frac{1}{\Omega}\,\frac{d\Omega}{dt} + \frac{n^2}{3}\,B^{(n)} = 0\,.$$

Therefore $B^{(n)} \simeq C\Omega^{-1/2}e^{i(n/\sqrt{3})t}$. We obtain sound waves whose amplitude decreases with time. These results confirm those obtained by Lifshitz and Khalatnikov (1963).

From the foregoing we see that galaxies cannot form as the result of the growth of small perturbations. We may expect that other non-gravitational forces will have an effect smaller than pressure equal to one-third of the density and so will not cause relative perturbations to grow faster than $\tau$. To account for galaxies in an evolutionary universe, we must assume there were finite, non-statistical, initial inhomogeneities.

## VII. THE STEADY-STATE UNIVERSE

To obtain the steady-state universe we must add extra terms to the energy-momentum tensor. Hoyle and Narlikar (1964*a*) use

$$T_{ab} = \mu u_a u_b + p h_{ab} - C_a C_b + \tfrac{1}{2} g_{ab} C_d C^d\,,$$

where

$$C_a = C_{;a}\,, \qquad C_{;a}{}^{a} = -\,j_a{}^{;a}\,, \qquad j_a = (\mu + p)u_a\,.$$

Since

$$T_{ab}{}^{;b} = 0\,,$$

$$\mu' + (\mu + p)\theta + u^a C_a C_b{}^{;b} = 0$$

and

$$(\mu + p)u'_a + p_{;b}h^b{}_a - h^b{}_a C_b C_d{}^{;d} = 0\,. \tag{22}$$

There is a difficulty here, if we require that the "$C$" field should not produce acceleration or, in other words, that the matter created should have the same velocity as the matter already in existence. We must then have

$$h^b{}_a C_b = 0\,. \tag{23}$$

However, since $C$ is a scalar, this implies that the rotation of the medium is zero. On the other hand, if equation (23) does not hold, the equations are indeterminate (cf. Raychaudhuri and Bannerjee 1964). In order to have a determinate set of equations we will adopt equation (23) but drop the requirement that $C_a$ is the gradient of a scalar. The condition (23) is not very satisfactory, but it is difficult to think of one that is more satisfactory. Hoyle and Narlikar (1964*b*) seek to avoid this difficulty by taking a particle rather than a fluid picture. However, this has a serious drawback since it leads to infinite fields (Hawking 1965). From equation (17),

$$C_a = -\,u_a\left[1 - \frac{p'}{\mu' + p' + (\mu + p)\,\theta}\right].$$

Therefore,

$$C_{a};{}^{a}=-(\mu'+p')-(\mu+p)\theta=-\theta\left[1-\frac{p}{\mu'+p'+(\mu+p)\theta}\right]+\left[\frac{p'}{\mu'+p'+(\mu+p)\theta}\right]'.$$

For $\mu \gg p$, $\mu' + p' = \theta[1 - (\mu + p)]$. Therefore $\mu + p \to 1$. Thus, small perturbations of density die away. Moreover, equation (18) still holds, and therefore rotational perturbations also die away. Equation (19) now becomes

$$\theta' = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}(\mu + 3p) + 1 .$$

Therefore $\theta \to \sqrt{[3(\frac{1}{2} - p)]}$.

These results confirm those obtained by Hoyle and Narlikar (1963). We see therefore that galaxies cannot be formed in the steady-state universe by the growth of small perturbations. However, this does not exclude the possibility that there might by a self-perpetuating system of finite perturbations which could produce galaxies (Sciama 1955; Roxburgh and Saffman 1965).

### VIII. GRAVITATIONAL WAVES

We now consider perturbations of the Weyl tensor that do not arise from rotational or density perturbations, that is,

$$E_{ab};{}^{b} = H_{ab};{}^{b} = 0 .$$

Multiplying expression (15) by $u^c\nabla_c$ and expression (16) by $h^a{}_{(p}\eta_{q)}{}^{rbs}u_r\nabla_s$, we obtain, after reduction,

$$E''_{ab} - (E_{cd;e}h^c{}_f h^d{}_g h^e{}_k)_{;i}h^{ki}h^f{}_a h^g{}_b + \tfrac{7}{3}E'_{ab}\theta + E_{ab}[\theta' + \tfrac{4}{3}\theta^2 + \tfrac{1}{3}(\mu - 3p)] + \sigma_{ab}[\tfrac{1}{3}\theta(\mu + p) + \tfrac{1}{2}(\mu' + p')] = 0 . \tag{24}$$

In empty space with a non-expanding congruence $u^a$, this reduces to the usual form of the linearized theory,

$$\Box^2 E_{ab} = 0 .$$

The second term in equation (24) is the Laplacian in the hypersurface $\tau =$ constant, acting on $E_{ab}$. We will write $E_{ab}$ as a sum of eigenfunctions of this operator:

$$E_{ab} = \Sigma A^{(n)} V_{ab}{}^{(n)} ,$$

where

$$V'_{ab}{}^{(n)} = 0 ,$$

$$(V^{(n)}{}_{cd;e}h^c{}_f h^d{}_g h^e{}_k)_{;i}h^{ki}h^f{}_a h^g{}_b = -\frac{n^2}{\Omega^2}V_{ab}{}^{(n)} ,$$

$$V^{(n)}{}_{ab};{}^{b} = 0 , \qquad V^{(n)}{}_a{}^a = 0 .$$

Then

$$E'_{ab} = \frac{1}{\Omega}\Sigma V_{ab}{}^{(n)}\frac{dA^{(n)}}{dt} ,$$

$$E''_{ab} = \frac{1}{\Omega^2}\Sigma V_{ab}{}^{(n)}\left(\frac{d^2A^{(n)}}{dt^2} - \frac{1}{\Omega}\frac{d\Omega}{dt}\frac{dA^{(n)}}{dt}\right).$$

Similarly,

$$\sigma_{ab} = \Sigma D^{(n)} V_{ab}{}^{(n)} .$$

Then, by expression (19),

$$\frac{dD^{(n)}}{dt} = \Omega A^{(n)} - 2D^{(n)}\frac{1}{\Omega}\frac{d\Omega}{dt}.$$

Substitution in equation (24) yields

$$\frac{d^2 A^{(n)}}{at^2} + \frac{6}{\Omega}\frac{d\Omega}{dt}\frac{dA^{(n)}}{dt} + A^{(n)}\left[n^2 + \frac{3}{\Omega}\frac{d^2\Omega}{dt^2} + \frac{6}{\Omega^2}\left(\frac{d\Omega}{dt}\right)^2 + \frac{\mu+3p}{3}\Omega^2\right]$$

$$+ D^{(n)}\left[(\mu+p)\frac{d\Omega}{dt} + \tfrac{1}{2}(\mu'+p')\Omega^2\right] = 0 .$$

We may differentiate again and substitute for $dD^{(n)}/dt$.

For $n \gg 1$ and $\Omega \gg 1/n^2$,

$$A^{(n)} \simeq \frac{1}{\Omega^3} e^{int},$$

so the gravitational field $E_{ab}$ decreases as $\Omega^{-1}$ and the "energy" $\frac{1}{2}(E_{ab}E^{ab} + H_{ab}H^{ab})$ as $\Omega^{-6}$. We might expect this, as the Bianchi identities may be written, to the linear approximation,

$$\Omega g^{ed}\frac{\partial}{\partial x^e}(\Omega^{-1}C_{abcd}) = J_{abc} .$$

Therefore, if the interaction with the matter could be neglected $C_{abcd}$ would be proportional to $\Omega$ and $E_{ab}$, $H_{ab}$ to $\Omega^{-1}$.

In the steady-state universe when $\mu$ and $\theta$ have reached their equilibrium values,

$$R_{ab} = (\tfrac{1}{2} + p)g_{ab} .$$

Thus

$$J_{abc} = R_{c[a;b]} - \tfrac{1}{6}g_{c[a}R_{;b]} = 0 .$$

Thus the interaction of the "$C$" field with gravitational radiation is equal and opposite to that of the matter. There is then no net interaction, and $E_{ab}$ and $H_{ab}$ decrease as $\Omega^{-1}$.

The "energy" $\frac{1}{2}(E_{ab}E^{ab} + H_{ab}H^{ab})$ depends on second derivatives of the metric. It is therefore proportional to the frequency squared times the energy as measured by the energy-momentum pseudotensor, in a local co-moving Cartesian coordinate system which depends only on first derivatives. Since the frequency will be inversely proportional to $\Omega$, the energy measured by the pseudotensor will be proportional to $\Omega^{-4}$ as for other rest-mass zero fields.

## IX. ABSORPTION OF GRAVITATIONAL WAVES

As we have seen, gravitational waves are not absorbed by a perfect fluid. Suppose, however, that there is a small amount of viscosity. We may represent this by the addition of a term $\lambda\sigma_{ab}$ to the energy-momentum tensor, where $\lambda$ is the coefficient of viscosity (Ehlers 1961).

Since

$$T_{ab}{}^{;b} = 0$$

we have

$$\mu' + (\mu + p)\theta - 2\lambda\sigma^2 = 0 \tag{25}$$

and

$$(\mu + p)u'_a + p_{;b}h^b{}_a + \lambda\sigma_{cb}{}^{;b}h^c{}_a = 0 . \tag{26}$$

Equations (15) and (16) become

$$E'_{ab} + E_{ab}\theta + h^f{}_{(a}\eta_{b)cde}u^c H_f{}^{d;e} = -\tfrac{1}{2}(\mu + p)\sigma_{ab} - \tfrac{1}{2}\lambda(E_{ab} - \tfrac{1}{3}\sigma_{ab}\theta) , \tag{27}$$

and

$$H'_{ab} + H_{ab}\theta - h^f{}_{(a}\eta_{b)\,cde}u^c E_f{}^{d;e} = -\tfrac{1}{2}\lambda H_{ab}\,. \tag{28}$$

The extra terms on the right of equations (27) and (28) are similar to conduction terms in Maxwell's equations and will cause the wave to decrease by a factor $e^{-(\lambda/2)\tau}$. Neglecting expansion for the moment, suppose we have a wave of the form

$$E_{ab} = {}_0E_{ab}e^{i\nu\tau}\,.$$

This will be absorbed in a characteristic time $2/\lambda$ independent of frequency. By expression (25) the rate of gain of rest-mass energy of the matter will be $2\lambda\sigma^2$ which by expression (19) will be $2\lambda_0 E^2\nu^{-2}$. Thus the available energy in the wave is $4_0E^2\nu^{-2}$. This confirms that the density of available energy of gravitational radiation will decrease as $\Omega^{-4}$ in an expanding universe. From this we see that gravitational radiation behaves in much the same way as other radiation fields. In the early stages of an evolutionary universe when the temperature was very high we might expect an equilibrium to be set up between black-body electromagnetic radiation and black-body gravitational radiation. Since they both have two polarizations, their energy densities should be equal. As the universe expanded they would both cool adiabatically at the same rate. As we know that the temperature of black-body extragalactic electromagnetic radiation is less than 5° K, the temperature of the black-body gravitational radiation must be also less than this which would be absolutely undetectable. Now the energy of gravitational radiation does not contribute to the ordinary energy-momentum tensor $T_{ab}$. Nevertheless it will have an active gravitational effect. By the expansion equation,

$$\theta' = -\tfrac{1}{3}\theta^2 - 2\sigma^2 - \tfrac{1}{2}(\mu + 3p)\,.$$

For incoherent gravitational radiation at frequency $\nu$,

$$\sigma^2 = {}_0E^2\nu^{-2}\,.$$

But the energy density of the radiation is $4_0E^2\nu^{-2}$. Therefore,

$$\theta' = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}\mu_G - \tfrac{1}{2}(\mu + 3p)\,,$$

where $\mu_G$ is the gravitational "energy" density. Thus gravitational radiation has an active attractive gravitational effect. It is interesting that this seems to be just half that of electromagnetic radiation.

Gravitational radiation emitted in an expanding universe will eventually be absorbed by other matter if $\int\lambda d\tau$ diverges. Clearly this is so for the steady-state universe since $\lambda$ will be constant. In evolutionary universes $\lambda$ will be a function of time. Now for a gas, $\lambda \propto T^{1/2}$, where $T$ is the temperature. For a monatomic gas, $T \propto \Omega^{-2}$; therefore the integral will diverge (just). However, the expression used for viscosity assumed that the mean free path of the atoms was small compared to the scale of the disturbance. Since the mean free path $\propto \mu^{-1} \propto \Omega^{-3}$ and the wavelength $\propto \Omega^{-1}$, the mean free path will eventually be greater than the wavelength and so the effective viscosity will decrease more rapidly than $\Omega^{-1}$. Thus complete absorption of gravitation radiation emitted will not occur.

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