THE ASTROPHYSICAL JOURNAL

VOLUME 142 NOVEMBER 15, 1965 NUMBER 4

THE BLACK-BODY RADIATION CONTENT OF THE UNIVERSE AND THE FORMATION OF GALAXIES*

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Received March 8, 1965; revised June 1, 1965

ABSTRACT

A critical factor in the formation of galaxies may be the presence of a black-body radiation content of the Universe. An important property of this radiation is that it would serve to prevent the formation of gravitationally bound systems, whether galaxies or stars, until the Universe has expanded to a critical epoch. There is good reason to expect the presence of black-body radiation in an evolutionary cosmology, and it may be possible to observe such radiation directly.

Assuming that the Universe is expanding and evolving, very likely most scientists would agree on the over-all picture for the evolution of the Universe. At a remote time in the past the Universe contained only dense gaseous material, with neither stars nor galaxies. As the Universe expanded from this state the material became organized into galaxies and clusters of galaxies, and the material within galaxies passed through the generations of stars. Now a central question is what were the physical processes, and what were the physical parameters and conditions that determined how galaxies formed, with the observed distributions of mass and size, and the observed tendency for galaxies to be distributed in clusters.

An approach to the problem may be based on the following important property of an evolutionary cosmology (Dicke, Peebles, Roll, and Wilkinson 1965).

1. Assuming that the Universe expanded from a sufficiently highly contracted phase, the early Universe would have been opaque to radiation. As a result the radiation field would have achieved thermal equilibrium with the matter—the Universe would have been filled with black-body radiation. This fireball radiation suffers the cosmological redshift, so that it is very much cooled by the expansion of the Universe, but it retains its thermal, black-body character.

It may be possible to observe this fireball radiation directly by means of a microwave radiometer. Recently, Penzias and Wilson (1965) have reported that at 7-cm wavelength there appears to be isotropic background radiation with intensity equivalent to $3.5° \pm 1°$ K. Further measurements at other wavelengths are necessary to establish that this radiation has a black-body spectrum, as expected for the cooled fireball from the big bang. The purpose of this article is to show that, if the Universe does contain black-body radiation of this general amount, this radiation must have had an important effect on the early evolution of matter leading to galaxy formation.

* This research was supported in part by the National Science Foundation and by the Office of Naval Research of the United States Navy.

It is important to distinguish this possible thermal radiation from the integrated background radiation due to the galaxies. Eddington (1926) estimated that the starlight in our own Galaxy amounts to 3° K. However, this is an effective temperature, such that σT^4 is the total starlight radiation energy flux. The radiation intensity spectrum is quite different from the black-body radiation considered here.

The role of the thermal, fireball radiation in the formation of galaxies is summarized in the following remarks.

2. The expansion of the Universe adiabatically decompresses and cools its contents. So long as the temperature exceeds about 4000° K, the material in the Universe is ionized and is sufficiently opaque to radiation that material and radiation would remain in thermal equilibrium. The temperature T varies with the mean density of matter ρ according to the formula

$$T \propto \rho^{1/3} . \tag{1}$$

3. In the expanding Universe any sufficiently large-scale perturbation to a homogeneous mass distribution grows more pronounced with time, eventually tending to form a gravitationally bound system. This is the familiar Jeans gravitational instability (Gamow 1948).

TABLE 1

FORMATION OF PROTOGALAXIES AND CLUSTERS OF GALAXIES

Present mean density of matter (gm/cm³)	2×10^{-29}	2×10^{-29}	7×10^{-31}	7×10^{-31}
Present black-body radiation temperature (° K)	3	0 3	3	0.3
Present age of the Universe (yr)	7×10^{9}	7×10^{9}	1×10^{10}	1×10^{10}
Formation of gravitationally bound gas clouds:				
Time of formation (yr)	1×10^{5}	3×10^{3}	7×10^{5}	2×10^{4}
Temperature (° K)	4000	5000	4000	4500
Radius (pc)	10	0 2	40	1
Density within cloud (protons/cm³)	3×10^{4}	5×10^{7}	1×10^{3}	1×10^{6}
Mass of cloud ($M_\odot$)	2×10^{6}	6×10^{4}	1×10^{7}	3×10^{5}
Formation of galaxies:				
Minimum mass of galaxy ($M_\odot$)	2×10^{6}	6×10^{4}	1×10^{7}	3×10^{5}
Protogalaxy mass ($M_\odot$):				
$n=0$	10^{9}	10^{9}	10^{9}	10^{9}
$n=1$	10^{10}	10^{10}	10^{10}	10^{10}
Maximum mass of a cluster of galaxies ($M_\odot$):				
$n=0$	10^{13}	10^{13}	10^{12}	10^{10}
$n=1$	10^{16}	10^{17}	10^{14}	10^{13}

4. With the Jeans instability alone we encounter a dilemma of initial conditions. The time at which a bound system forms depends critically on the details of the density perturbation evaluated at some chosen, initial time, and to form the observed galaxies it would be necessary to postulate extremely special initial conditions.

5. This unsatisfactory situation is avoided if it is assumed that the Universe contains black-body radiation. It will be shown that the radiation would prevent density perturbations from growing larger than the mean density itself until the Universe has expanded to a critical epoch.

6. After this epoch the Jeans instability leads to the formation of gravitationally bound gas clouds of well-defined size and mass. The properties of the gas clouds are listed in Table 1, where each column corresponds to definite assumptions about the present mean density and radiation temperature of the Universe. Subsequent evolution of the matter is not influenced by the black-body radiation.

7. The motion of the gas clouds is subject to a gravitational instability. This results in the formation of bound systems of gas clouds, which should collapse to form more massive protogalaxies.

This discussion is based on conventional general relativity and the homogeneous isotropic cosmological models. All quantities below will be expressed only in proper units, as measured with ordinary measuring rods and clocks and balances.

To obtain the critical condition mentioned in the fifth remark for forming a gravitationally bound system, suppose a spherical mass of gas has just achieved equilibrium, tending neither to expand nor collapse. As the Universe expands, the mean black-body electromagnetic-radiation energy density is decreasing, so that radiation is tending to flow out of the system. This is opposed by Thompson scattering of the radiation by the free electrons, if the temperature is above 4000° K. Thus, the radiation temperature within the bound system will exceed the mean radiation temperature by the amount

$$\frac{\Delta T}{T} \sim \frac{\sigma R^2 \rho H}{c\, m_p}. \tag{2}$$

In this equation it will be recalled that all quantities are measured in proper units, where R is the radius of the bound system, σ is the Thompson scattering cross-section for an electron, ρ is the density of matter within the system, H is Hubble's constant, and m_p is the mass of a proton.

Suppose first that the system is so large that $\Delta T/T \gg 1$. In this case radiation would be trapped inside the bound system, and for equilibrium the gravitational force would have to balance the radiation pressure of this trapped radiation. Ignoring the gas pressure, and assuming for the moment that the system is not too large, the condition for equilibrium is

$$\tfrac{4}{3}\pi G\left(\rho + \frac{2\,bT^4}{c^2}\right) R\rho \sim \frac{bT^4}{3R}, \tag{3}$$

where ρ and T are the density of matter and the radiation temperature within the system, and b is the radiation energy density constant ($b = 7.6 \times 10^{-15}$ erg/cm^3 ° K^4). In situations of interest the mass density of matter will be substantially below that in radiation (Table 2) so that expression (3) reduces to

$$R \sim c/(G\rho)^{1/2}. \tag{4}$$

But Hubble's constant is

$$H = (8\pi G b T^4/3c^2)^{1/2} \tag{5}$$

and with the mass density ρ much less than the radiation density bT^4/c^2, we see that expression (4) implies

$$R \gg c/H. \tag{6}$$

This means that the system is larger than the visible Universe. Evidently the simple Newtonian approximation (3) would not apply if expression (6) were valid, but more important, we see that the system must be unstable against gravitational collapse, for the mass of the system satisfies $GM/Rc^2 > 1$. However, we do not believe that any appreciable part of the observable Universe has already collapsed. Thus, the system could not be large enough to contain the radiation pressure. Therefore, for equilibrium the system must be small enough to allow the radiation to escape ($\Delta T/T \ll 1$ in eq. [2])

$$R^2 \lesssim c m_p/\sigma\rho H. \tag{7}$$

If expression (7) is satisfied, the radiation will be very nearly uniformly distributed in space. Then assuming that the center of the gravitationally bound system is at rest in the comoving coordinate frame, and assuming that the (proper) size of the system is constant, the edge of the system will be moving with velocity HR relative to the comoving coordinate frame at that point. Since the radiation is moving with the comoving coordinate frame there will be a radiation drag force per electron at the edge of the system amounting to

$$F_r = \frac{\sigma b T^4 H R}{c} . \tag{8}$$

This is the radiation force for temperatures in the range $T \ll 10^{10}$ ° K (that is, nonrelativistic electrons) to $T \gtrsim 4000°$ K (free electrons).

If the system satisfies Jeans's criterion for gravitational instability, pressure forces may be neglected, and equation (8) must be balanced by the gravitational force per proton,

$$F_g = \tfrac{4}{3}\pi G m_p R \left(\rho_b + \frac{2bT^4}{c^2}\right). \tag{9}$$

TABLE 2

CONDITION FOR A GRAVITATIONALLY BOUND SYSTEM*

Age of the Universe (yr)	7×10^5	5×10^3
Temperature (° K)	4×10^3	4×10^4
Mass density due to radiation (gm/cm³)	2×10^{-21}	2×10^{-17}
Mean mass density of matter:		
Gm/cm³	2×10^{-21}	2×10^{-18}
Protons/cm³	10^3	10^6
Matter density within a bound system (protons/cm³)	10^6	10^{12}
Maximum mass ($M_\odot$)	10^{14}	10^9

* Assumed present conditions $T = 3$ °K, $\rho = 7 \times 10^{-31}$ gm cm³.

Equating (8) and (9), the density ρ_b within the system divided by the mean density of matter in the Universe, ρ, is found to be

$$\rho_b/\rho = \frac{2bT^4}{\rho c^2}\left(\frac{3\sigma c H}{8\pi G m_p} - 1\right) = \frac{2bT^4}{\rho c^2}(0.067 H/H_f - 1), \tag{10}$$

where in the final term we have assumed the present value of Hubble's constant is $H_f = 3.2 \times 10^{-18}$ sec^{-1} (100 km/sec Mpc).

The ratio (10) of the density ρ_b required for equilibrium to the mean matter density ρ is given in Table 2 for reasonable values of the mass density and radiation temperature in the present Universe. Also shown is the maximum mass of a bound system, as given by equation (7). It is evident that until the plasma recombines ($T \sim 4000°$ K) a bound system could have formed in this cosmology only if there were extremely large density fluctuations.

One could imagine a situation in which a system with density greater than the mean density is expanding just slightly less rapidly than the general expansion, so that the radiation drag and gravitational forces just balance. However, it is important to notice that this is an unstable balance. The radiation drag force is spherically symmetric, while within an elliptical mass distribution the gravitational force is larger, for the most part, along the minor axes. Thus the system tends to fragment, eventually into pieces so small that pressure forces can disperse them.

The significance of these remarks is that while $\rho_b/\rho \gg 1$ (eq. [10]) the Universe is stable against the development of large matter-density perturbations. On the other hand, given a smooth density distribution, small perturbations to the matter distribution tend to grow with time. It is important to recognize the distinction between these two cases. We shall show that small density perturbations ($\delta\rho/\rho \ll 1$) are not appreciably affected by the radiation drag. When the density excursions have grown comparable to the mean density radiation drag becomes an important factor, serving to prevent density perturbations from growing larger until ρ_b/ρ approaches unity.

To discuss the behavior of small density perturbations we write the matter density $\bar{\rho}(x,t)$ as

$$\bar{\rho}(x,t) = \rho(t)[1 + D(x,t)] , \tag{11}$$

where ρ is the mean matter density. This first-order perturbation problem was first discussed by Lifshitz (1946). Introducing the simplifying assumption that the linear dimensions of any perturbation are small compared with the radius of the visible Universe, it is shown in the Appendix that if the perturbation is resolved into Fourier components

$$D = \boldsymbol{d}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{12}$$

such that the wavelength $\lambda = 2\pi/|\boldsymbol{k}|$ satisfies

$$\frac{1}{\lambda}\frac{d\lambda}{dt} = -H \tag{13}$$

(so that the wave is taking part in the general expansion) then the Fourier amplitude $d(t)$ satisfies

$$\frac{d^2\boldsymbol{d}}{dt^2} + \left(\frac{\sigma b T^4}{m_p c} + 2H\right)\frac{d\boldsymbol{d}}{dt} = \left(4\pi G\rho - \frac{8\pi^2 kT}{m_p \lambda^2}\right)\boldsymbol{d} . \tag{14}$$

This is valid if λ is small compared with the radius of the visible Universe.

From equation (14), if

$$\lambda > \lambda_c \equiv \left(\frac{2\pi kT}{G\rho m_p}\right)^{1/2} , \tag{15}$$

$\boldsymbol{d}(t)$ grows with time. This is Jeans's criterion. For smaller wavelengths $\boldsymbol{d}(t)$ oscillates, and the disturbance is dissipated by the damping terms (in $d\boldsymbol{d}/dt$).

To estimate the rate of growth of the perturbation $\boldsymbol{d}$ when expression (15) is valid, consider a cosmologically flat Universe. Under the assumption that the particle mass density is greater than the mass density in radiation, the age of the model since start of expansion from infinite density is $t = (6\pi G\rho)^{-1/2}$, and $H = 2/(3t)$. Then, neglecting the pressure and radiation drag terms in (14) (i.e., setting $T = 0$), we find $\boldsymbol{d} \propto t^{2/3}$. This is the result obtained by Lifshitz (1946). Similarly, if the mass density in radiation dominates, $\boldsymbol{d} \propto t^{0\ 61}$.

These results apply once the electrons have become non-relativistic. We see that the density perturbations grow slowly, as a power of time. We believe it is reasonable to assume that there has been adequate time for the growth of perturbations on the scale of galaxies and clusters of galaxies. This would not have been the case if it were assumed that at the epoch when the electrons first became non-relativistic the Universe was strictly homogeneous, with the density perturbations only random (thermal) fluctuations. However, the assumption of an exactly symmetrical Universe at this epoch appears quite overidealized. In the following discussion we shall assume that the early Universe was sufficiently irregular that appreciable density perturbations did form due to the gravitational instability, but not so irregular that parts of the Universe have already suffered gravitational collapse. The general question of the homogeneity of the early Universe will be discussed in detail elsewhere.

When the density excursions have grown to a value comparable with the mean density, perturbations can develop very rapidly. To understand this, consider a region, roughly uniform and spherical, in which the density is twice the mean value. At time t_0 let the material in the region be expanding at the general rate, so the speed v of material in the patch a distance r from the center is given by

$$(v/r)^2(t = t_0) = \tfrac{8}{3}\pi G[\rho(t_0) + \rho_r(t_0)] , \tag{16}$$

where we have written separately the densities of matter, ρ, and radiation, ρ_r. If we neglect for a moment the radiation drag, the subsequent motion is given approximately by

$$(v/r)^2 = \frac{8\pi G}{3}[2\rho(t) + \rho_r(t) - \rho(t_0) r_0^2/r^2] . \tag{17}$$

Thus the patch stops expanding when $2\rho(t) + \rho_r(t) = \rho(t_0)r_0^2/r^2$. If the mass density in radiation does not exceed that in matter (Table 2) this is an expansion of a factor of 2 in radius. The patch would then collapse by a factor $\sim$2 in radius to a stable, bound system. However, it is important that this situation is very much altered if $\rho_b/\rho \gg 1$ (eq. [10]). In this case, we have shown that the patch must fragment and be dispersed.

Now we can draw the following general picture. In the initial very contracted Universe, there was a more or less uniform distribution of ionized hydrogen. That part of any density perturbation with wavelength satisfying expression (15) would grow with time, while the rest of the perturbation decayed. The resulting pattern of density fluctuations takes part in the general expansion of the Universe (eq. [13]). Notice from expressions (1) and (15) that the critical Jeans wavelength expands with the pattern. The fully developed pattern of density perturbations is characterized by a power spectrum cut off at the Jeans wavelength. Assuming that the power spectrum (contribution to the variance of $\bar{\rho}$ per wavenumber increment) does not increase with wavelength faster than λ^3 for $\lambda > \lambda_c$, the characteristic dimension of density fluctuations is the Jeans length λ_c. That is, the density perturbation at any point is correlated with the density perturbation at points a distance less than λ_c away, and uncorrelated with points much more distant than λ_c. Density excursions are roughly comparable to the mean density. Peaks in the density pattern tend toward rapid growth, only to break, because of the radiation drag, and return to the general level.

When ρ_b/ρ (eq. [10]) approaches unity, at time t_c, patches of higher density, with dimensions of the order of λ_c, now are in a position to evolve toward gravitationally bound systems. If the mean density of matter is ρ_c at this time (t_c) the mass of one of the systems is of the order of

$$M_c \equiv \tfrac{4}{3}\pi\rho_c\lambda_c^3 . \tag{18}$$

Notice that

$$\frac{GM_c m_p}{2kT_c\lambda_c} = \frac{4\pi^2}{3} . \tag{19}$$

That is, the matter is formed at roughly an equilibrium configuration. This means that the bound systems have density and characteristic dimensions very roughly of the order of ρ_c and λ_c.

Assuming that the present thermal radiation temperature is 3° K (Penzias and Wilson 1965) and the present matter density is 7×10^{-31} gm cm^3 (Oort 1958) equation (18) implies a mass $M_c \sim 10^7\, M_\odot$, roughly the mass of a dwarf galaxy (see Table 1).

With much of the matter in the Universe now concentrated in these discrete clouds, there is at any point a gravitational field, of the general order of GM_c/L, where L is the mean distance between clouds. A cloud thus tends to move toward nearby, higher-density regions, leading to clustering of the clouds.

The development of this gravitational instability depends on the power spectrum of the density fluctuations at the time t_c of formation of the gas clouds. With a flat power spectrum, the total power (i.e., contribution to the variance of $\bar{\rho}$) in wavelengths greater than λ goes as λ^{-3}. However, we can find no reason to expect a characteristic random (flat) spectrum, so for the purpose of a brief discussion of the formation of bound clusters of gas clouds, we shall characterize the power spectrum $\boldsymbol{d}_k{}^2$ at time t_c as

$$\boldsymbol{d}_k{}^2 \propto \lambda^n , \tag{20}$$

where the index n would vanish for a flat spectrum, and for boundedness $n < 3$. This is the power spectrum per increment of wavenumber k.

In the cosmological models in Table 1 the clusters of gas clouds are forming at a time when the mass density in radiation approximately may be neglected. With this assumption, we consider first a cosmologically flat Universe, such that the acceleration parameter, $q = 4\pi G\rho/(3H^2)$ is equal to $\frac{1}{2}$.

Let A be a spherical region which is expanding with the general expansion of the Universe and such that, within the volume of A, there would be a mass M of matter on the average. Then the actual mass within A at time t_c is uncertain, by the amount

$$(\delta M/M)_c \sim (M/M_c)^{-0.5+n/6} . \tag{21}$$

The functional form of expression (21) is obtained using expression (20) by integrating the density perturbation over A and averaging the square of this integral, with the assumption of random phases of the $\boldsymbol{d}_k$. The normalization is obtained by noting that when $M \sim M_c$, $\delta M \sim M_c$.

As long as $\delta M/M \ll 1$, we have shown above that the uncertainty (21) grows as $t^{2/3}$. Now consider the epoch t such that

$$(t/t_c)^{2/3}(M/M_c)^{-0.5+n/6} = 1 . \tag{22}$$

At this time we can make the following assertions. Within spherical volumes equal to that of the region A the total mass varies by a factor ~ 2. Within much larger volumes the mass is nearly constant. There exist systems, of mass

$$M \sim M_c(t/t_c)^{4/(3-n)} , \tag{23}$$

which are in the process of forming bound systems. In smaller subsystems this process occurred earlier, and the subsequent collapse, or adjustment to equilibrium is already well advanced. This process should be pictured as a roughly continuous progression along a hierarchy in the departure from the general expansion.

We see immediately from equation (23) that in this cosmology the maximum mass of a cluster of galaxies in the present Universe is

$$M \lesssim M_c(t_f/t_c)^{4/(3-n)} , \tag{24}$$

where t_f is the present age of the Universe. Adopting the value of Hubble's constant $H_f = 100$ km/sec Mpc, and assuming a cosmologically flat space, $q = \frac{1}{2}$, we have

$$\rho_f = 2 \times 10^{-29} \text{ gm cm}^3 . \tag{25}$$

This is the first density used in Table 1. The maximum mass of a cluster of galaxies as given by equation (24) is given in Table 1 for two different values of the index n in equation (20). It is seen that this limit can be consistent with Abell's (1961) conclusion that galaxies are ordered or clustered up to a scale $\sim 10^{16}\ M_\odot$.

Although it appears unlikely that the present value of q is very much larger than $\frac{1}{2}$,

q could be nearly equal to zero (Sandage 1961). Therefore, we have considered also the assumption

$$\rho_f = 7 \times 10^{-31} \text{ gm cm}^3 . \tag{26}$$

This is the estimated mean density of matter in galaxies (Oort 1958). It is interesting that in this limit of small mass density (small q) the expanding Universe becomes very nearly stable against small density perturbations. This follows directly from equation (14). Thus, the growth of the most massive organized clusters of matter is cut off at a time, t_3, when $2q$ departs from unity, and the maximum mass of a cluster of galaxies is given by equation (24) with t_f replaced by the effective time t_3. The resulting maximum masses for this case are shown in Table 1. It is seen that for any of the conditions assumed in the table a reasonable value of n may be chosen to obtain a reasonable upper mass limit.

In this theory a galaxy of normal size would be formed by accretion of many of the original gas clouds. In the manner just described, gravitational instability can lead to the formation of a bound system of clouds of gas, including the original gas clouds and more massive subsystems formed earlier. The clouds within the system may collide, radiate much of the energy of the collision, and fall toward the center of the system, thus leading to the formation of a massive central nucleus. A cloud which passes close to the nucleus may be captured due to gas drag, or it may be torn apart by tidal stresses, and the remnants captured by the nucleus. The situation is further complicated because the various subsystems, or gas clouds, would be in various stages of evolution, with masses ranging from M_c up to a mass approaching that of the total system.

Without attempting to provide a reasonably complete description of this process, we shall show only that the accretion necessary for the formation of galaxies can take place. Consider a gravitationally bound system, with mass $M = NM_c$. At time t_c the material within the system would have occupied a region with dimension

$$R_1 \sim N^{1/3}\lambda_c \tag{27}$$

and from equation (22) the system would have stopped expanding at time

$$t_2 \sim t_1 N^{(3-n)/4}$$

when the system had grown to a size

$$R_2 \sim R_1(t_2/t_1)^{2/3} \sim \lambda_c N^{(5-n)/6} . \tag{28}$$

As before, in obtaining equation (28) we have assumed an expansion parameter $q \sim \frac{1}{2}$.

Now we simplify the problem by supposing that the system contains original gas clouds only, and ask whether the system will have collapsed, due to collisions, by the present time t_f. The mean velocity of gas clouds in the system is

$$v \sim \left(\frac{GM_c N}{\lambda_c N^{(5-n)/6}}\right)^{1/2} \sim \left(\frac{GM_c}{\lambda_c}\right)^{1/2} \tag{29}$$

nearly independent of N. The cross-section for collision is λ_c^2. Using equations (28) and (29), the largest mass M for which the system could have collapsed by the present time satisfies

$$1 \sim \lambda_c^2 \left(\frac{GM_c}{\lambda_c}\right)^{1/2} \frac{N}{\lambda_c^3 N^{(5-n)/2}} t_f$$

or

$$M \sim M_c(G\rho_c t_f^2)^{1/(3-n)} . \tag{30}$$

This rough mass estimate, listed in Table 1, is of the same general order as observed galaxy masses, $\sim 10^{11}\, M_\odot$. The significance of this result is that there is time for gas clouds within a fairly massive system to have collided with each other to form a protogalaxy.

APPENDIX

THE DENSITY PERTURBATION EQUATION

We consider small perturbations away from a homogeneous, isotropic cosmological model. The calculation is simplified by confining attention to perturbations with dimensions small compared with the radius of the visible Universe. In this approximation Newtonian gravity theory applies (Dicke, Callan, and Peebles 1964). Furthermore, we shall take the temperature of matter and radiation to be a function of (world) time only. This is quite adequate in the linear perturbation case for the characteristic dimensions of density perturbations considered here. The calculation is similar to that of Bonnor (1957), but some new effects due to the radiation are taken into account.

In completely ionized hydrogen the electron density is $\bar{\rho}(\boldsymbol{x},t)/m_p$, where m_p is the mass of a proton and $\bar{\rho}(\boldsymbol{x},t)$ is the mass density. If the matter is moving with velocity $\boldsymbol{u}$ relative to the comoving coordinate frame, radiation, isotropic in the comoving frame, exerts a volume force $\sigma b T^4 \rho \boldsymbol{u}/(m_p c)$. Therefore, the equations of motion for matter are

$$\bar{\rho}\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]=-\bar{\rho}\nabla\phi-\nabla p-\frac{\sigma b T^4}{m_p c}\bar{\rho}(\boldsymbol{v}-H\boldsymbol{r}). \qquad (31)$$

This equation is expressed in coordinates, approximately Minkowski, at rest relative to the comoving coordinate frame at the origin, $\boldsymbol{r}=0$. The Newtonian gravitational potential satisfies

$$\nabla^2\phi = 4\pi G(\bar{\rho}+2bT^4/c^2)\,. \qquad (32)$$

The factor of 2 in the radiation energy density follows from the linearized form of Einstein's field equations (Tolman 1934). The equation of state is

$$p = 2\bar{\rho}(\boldsymbol{x},t)kT/m_p \qquad (33)$$

and the continuity equation is

$$\frac{\partial\bar{\rho}}{\partial t}+\boldsymbol{\nabla}\cdot(\bar{\rho}\boldsymbol{v})=0\,. \qquad (34)$$

In a first-order perturbation calculation, we write

$$\boldsymbol{v} = \boldsymbol{r}H+\boldsymbol{u}(\boldsymbol{r},t)\,,\qquad \bar{\rho}=\rho(t)[1+D(\boldsymbol{r},t)]\,,\qquad \phi=\phi_0(t)+\psi(\boldsymbol{r},t)\,. \qquad (35)$$

The unperturbed variables satisfy

$$\rho \propto a^{-3}\,, \qquad (36)$$

where $a(t)$ is the usual expansion parameter, and

$$\phi_0 = \tfrac{2}{3}\pi G(\rho_0+2bT^4/c^2)r^2\,,$$
$$\ddot{a}/a = -\tfrac{4}{3}\pi G(\rho_0+2bT^4/c^2)\,. \qquad (37)$$

To first order in the perturbations (35), equations (31), (32), and (34) become

$$\frac{\partial \boldsymbol{u}}{\partial t}+H\left(\boldsymbol{u}+r\frac{\partial \boldsymbol{u}}{\partial r}\right)=-\boldsymbol{\nabla}\psi-\frac{2kT}{m_p}\boldsymbol{\nabla}D-\frac{\sigma b T^4}{m_p c}\boldsymbol{u}\,, \qquad (38)$$

$$\nabla^2\psi = 4\pi G\rho D\,, \tag{39}$$

$$\frac{\partial D}{\partial t} + \nabla\cdot\boldsymbol{u} + Hr\,\frac{\partial D}{\partial r} = 0\,. \tag{40}$$

On taking the divergence of equation (38), and using expressions (39) and (40) and the formula

$$\nabla\cdot(r\,\partial\boldsymbol{u}/\partial r) = r\,\frac{\partial}{\partial r}(\nabla\cdot\boldsymbol{u}) + \nabla\cdot\boldsymbol{u}\,, \tag{41}$$

we obtain

$$\left(\frac{\partial}{\partial t} + 2H + Hr\frac{\partial}{\partial r}\right)\left(\frac{\partial D}{\partial t} + Hr\,\frac{\partial D}{\partial r}\right) = 4\pi G\rho D + \frac{2kT}{m_p}\nabla^2 D - \frac{\sigma b T^4}{m_p c}\left(\frac{\partial D}{\partial t} + Hr\,\frac{\partial D}{\partial r}\right). \tag{42}$$

This equation is simplified by transforming to new, comoving coordinates,

$$y = r/a(t)\,. \tag{43}$$

Then assuming a plane wave, $D = d(t)e^{i\boldsymbol{k}_0\boldsymbol{y}}$, we obtain equation (14), where it should be noticed that the propagation vector $\boldsymbol{k}_0$ is constant in comoving coordinates, so that the proper wavelength is expanding with the general expansion of the Universe (eq. [13]).

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