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GRAVITATIONAL SPHERES OF THE MAJOR PLANETS, MOON AND SUN

G. A. Chebotarev

Institute of Theoretical Astronomy, Academy of Sciences, USSR

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The numerical values of the radii of the gravitational spheres of the planets, moon, and sun (Tables 2, 4, 6, 7, 8) are summarized on the basis of a unified system of astronomical constants.

1. Equations of Motion

We will consider the limited three-body problem: the sun, a planet, and a satellite. We denote by x, y, z the heliocentric coordinates of the satellite, by x_1, y_1, z_1 the heliocentric coordinates of the planet, and by m its mass. We then will write differential equations of the motion of the satellite in a heliocentric system of coordinates:

$$\begin{aligned}\frac{d^2x}{dt^2} + k^2\frac{x}{r^3} &= k^2m\left(\frac{x_1 - x}{\Delta^3} - \frac{x_1}{r_1^3}\right),\\ \frac{d^2y}{dt^2} + k^2\frac{y}{r^3} &= k^2m\left(\frac{y_1 - y}{\Delta^3} - \frac{y_1}{r_1^3}\right),\\ \frac{d^2z}{dt^2} + k^2\frac{z}{r^3} &= k^2m\left(\frac{z_1 - z}{\Delta^3} - \frac{z_1}{r_1^3}\right),\end{aligned} \quad (1)$$

where Δ is the distance between the satellite and the planet, which we will consider as the perturbing body. The value Δ is determined using the formula

$$\Delta^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2. \quad (2)$$

We now will write the equations of heliocentric motion of the planet:

$$\begin{aligned}\frac{d^2x_1}{dt^2} + k^2(1 + m)\frac{x_1}{r_1^3} &= 0,\\ \frac{d^2y_1}{dt^2} + k^2(1 + m)\frac{y_1}{r_1^3} &= 0,\\ \frac{d^2z_1}{dt^2} + k^2(1 + m)\frac{z_1}{r_1^3} &= 0.\end{aligned} \quad (3)$$

We will use the center of the planet as the origin of a coordinate system whose axes are parallel to the axes of the heliocentric system, and the corresponding coordinates of the satellite are denoted by ξ, η, and ζ. Then

$$\xi = x - x_1, \quad \eta = y - y_1, \quad \zeta = z - z_1. \quad (4)$$

After subtracting the corresponding equations (3) from equations (1) we obtain planet-centered equations of motion of the satellite in the following form:

$$\begin{aligned}\frac{d^2\xi}{dt^2} + k^2m\frac{\xi}{\Delta^3} &= k^2\left(\frac{x_1}{r_1^3} - \frac{x}{r^3}\right),\\ \frac{d^2\eta}{dt^2} + k^2m\frac{\eta}{\Delta^3} &= k^2\left(\frac{y_1}{r_1^3} - \frac{y}{r^3}\right),\\ \frac{d^2\zeta}{dt^2} + k^2m\frac{\zeta}{\Delta^3} &= k^2\left(\frac{z_1}{r_1^3} - \frac{z}{r^3}\right).\end{aligned} \quad (5)$$

We denote by R the acceleration which the sun imparts to a satellite in a case when the sun is assumed to be the central body and F is used to denote the perturbing acceleration caused by the attraction of the planet. Equations (1) show that

$$R = \frac{k^2}{r^2}; \quad (6)$$

$$F = k^2m\left[\left(\frac{x_1 - x}{\Delta^3} - \frac{x_1}{r_1^3}\right)^2 + \left(\frac{y_1 - y}{\Delta^3} - \frac{y_1}{r_1^3}\right)^2 + \left(\frac{z_1 - z}{\Delta^3} - \frac{z_1}{r_1^3}\right)^2\right]^{1/2}. \quad (7)$$

We now denote by R_1 the acceleration which the planet imparts to the satellite in the case when the planet is assumed to be the central body. F_1 is used to denote the perturbing acceleration caused by the solar attraction. Then Eq. (5) gives

$$R_1 = \frac{k^2m}{\Delta^2}; \quad (8)$$

$$F_1 = k^2\left[\left(\frac{x_1}{r_1^3} - \frac{x}{r^3}\right)^2 + \left(\frac{y_1}{r_1^3} - \frac{y}{r^3}\right)^2 + \left(\frac{z_1}{r_1^3} - \frac{z}{r^3}\right)^2\right]^{1/2}. \quad (9)$$

We assume

$$\frac{x_1\xi + y_1\eta + z_1\zeta}{r_1\Delta} = \cos\varphi \quad (10)$$

and

$$\frac{\Delta}{r_1} = u. \tag{11}$$

The angle φ determines the angle between the directions from the center of the planet to the satellite and to the sun.

We will transform the expressions for F and F_1. Since

$$F = k^2 m \left[\frac{1}{\Delta^4} + \frac{1}{r_1^4} + \frac{2(x_1\xi + y_1\eta + z_1\zeta)}{\Delta^3 r_1^3}\right]^{1/2}, \tag{12}$$

then, taking into account (10), we find

$$F = k^2 m \left[\frac{1}{\Delta^4} + \frac{1}{r_1^4} + \frac{2\cos\varphi}{\Delta^2 r_1^2}\right]^{1/2} \tag{13}$$

or, introducing the value u, we obtain

$$F = \frac{k^2 m}{\Delta^2}(1 + 2u^2\cos\varphi + u^4)^{1/2}. \tag{14}$$

The expression for F_1 is transformed in a similar manner. Since

$$F_1 = k^2 \left[\frac{1}{r_1^4} + \frac{1}{r^4} - \frac{2(xx_1 + yy_1 + zz_1)}{r^3 r_1^3}\right]^{1/2} \tag{15}$$

and since

$$xx_1 + yy_1 + zz_1$$
$$= x_1(x_1 + \xi) + y_1(y_1 + \eta) + z_1(z_1 + \zeta)$$
$$= r_1^2(1 + u\cos\varphi) \tag{16}$$

and

$$r^2 = (x_1 + \xi)^2 + (y_1 + \eta)^2 + (z_1 + \zeta)^2$$
$$= r_1^2(1 + 2u\cos\varphi + u^2), \tag{17}$$

then F_1 can be transformed to the form

$$F_1 = \frac{k^2 u}{\Delta^2(1 + 2u\cos\varphi + u^2)}\{1 + (1 + 2u\cos\varphi + u^2)^2$$
$$- 2(1 + u\cos\varphi)(1 + 2u\cos\varphi + u^2)^{1/2}\}^{1/2}. \tag{18}$$

Now, making use of the smallness of the value u, we transform expressions (6), (14), and (18) to the final form:

$$R = \frac{k^2}{r_1^2}; \tag{19}$$

$$F = R_1 = \frac{k^2 m}{\Delta^2}; \tag{20}$$

$$F_1 = \frac{k^2\Delta}{r_1^3}(1 + 3\cos^2\varphi)^{1/2}. \tag{21}$$

In this study we have used the values of the orbital elements and masses of the major planets cited in Table 1.

The following numerical relation was used for conversion from the astronomical unit to kilometers: 1 a.u. = 149.6 million kilometers.

2. Sphere of Activity of a Planet

"Sphere of activity" refers to that region of space in which it is feasible to assume a planet as the central body and the sun as the perturbing body when computing perturbations.

The concept of sphere of activity was introduced into astronomy by Laplace in connection with the study of the motion of comets at the time of their approach to the major planets. It is a common occurrence for comets to pass through the sphere of activity of Jupiter. The surface bounding the sphere of activity is determined by the condition

$$\frac{F}{R} = \frac{F_1}{R_1}. \tag{22}$$

Within the sphere of activity

$$\frac{F}{R} > \frac{F_1}{R_1}. \tag{23}$$

After substituting (19), (20), and (21) into Eq. (22), we find the following expression for the radius of the sphere of activity:

$$\Delta_1 = r_1\left(\frac{m^2}{\sqrt{1 + 3\cos^2\varphi}}\right)^{1/5}. \tag{24}$$

We obtained the equation in polar coordinates determining the surface of rotation which delimits the sphere of activity. The surface (24) in actuality differs little from a sphere. The ratio of the maximum and minimum values of the quantity Δ_1 is $\sqrt[5]{2} = 1.15$, with the semiminor axis directed from the center of the planet toward the sun. It is agreed that the sphere of activity of the planet is defined by the maximum radius of the surface of rotation, that is,

$$\Delta_1 = r_1 m^{2/5}. \tag{25}$$

If the comet in the course of its heliocentric motion is at a distance Δ_1 relative to the perturbing planet, at that time it enters the sphere of activity of the planet. The sphere of activity increases with the motion of the planet from perihelion to aphelion. Table 2 gives the values of the radius of the sphere of activity correspond-

TABLE 1. Orbital Elements and Masses of the Major Planets

	a	e	1 : m
Mercury	0.387099	0.205614	6000000
Venus	0.723332	0.006821	408000
Earth	1.000000	0.016751	329390
Mars	1.523688	0.093313	3093500
Jupiter	5.202803	0.048435	1047.355
Saturn	9.538843	0.055682	3501.6
Uranus	19.190978	0.047209	22869
Neptune	30.070672	0.008575	19314
Pluto	39.517738	0.247073	360000

TABLE 2. Radii of the Spheres of Activity of the Major Planets

	Δ_1 (min)	Δ_1 (max)	Δ_1 (min)	Δ_1 (max)
	in a.u.		in million km	
Mercury	0.00060	0.00091	0.090	0.136
Venus	0.00409	0.00415	0.612	0.621
Earth	0.00610	0.00631	0.913	0.944
Mars	0.00350	0.00422	0.524	0.631
Jupiter	0.30665	0.33786	45.87	50.54
Saturn	0.34428	0.38488	51.50	57.58
Uranus	0.32991	0.36261	49.35	54.25
Neptune	0.57551	0.58547	86.10	87.59
Pluto	0.17825	0.29523	26.67	44.17

TABLE 3. Ratio F_1/R_1

Mercury	0.088	Jupiter	0.50
Venus	0.150	Saturn	0.40
Earth	0.158	Uranus	0.26
Mars	0.100	Neptune	0.28
		Pluto	0.16

TABLE 4. Radii of Gravitational Spheres of the Major Planets

	Δ_2 (min)	Δ_2 (max)	Δ_2 (min)	Δ_2 (max)
	in a.u.		in million km	
Mercury	0.00013	0.00019	0.019	0.029
Venus	0.00112	0.00114	0.168	0.171
Earth	0.00171	0.00177	0.256	0.265
Mars	0.00078	0.00095	0.117	0.142
Jupiter	0.15298	0.16855	22.89	25.22
Saturn	0.15222	0.17017	22.77	25.46
Uranus	0.12091	0.13289	18.09	19.88
Neptune	0.21452	0.21823	32.09	32.65
Pluto	0.04959	0.08214	7.42	12.29

ing to the position of the planet at perihelion (minimal) and at aphelion (maximal).

It is of interest to estimate the maximum value of the ratio F_1/R_1 for points situated on the surface of the sphere of activity. After substituting (20) and (21), we find

$$F_1/R_1 = 2m^{1/5}. \tag{26}$$

Thus, this relation is dependent solely on the mass of the planet and is not dependent on its distance from the sun. The numerical values of the quantity (26) have been compared in Table 3.

3. Gravitational Sphere of the Planets

We now will discuss the gravitational sphere of a planet, by which we mean the region of space within which the attraction of the planet is greater than solar attraction.

The surface bounding the gravitational sphere is determined by the condition

$$R_1 = R. \tag{27}$$

Within the gravitational sphere

$$R_1 > R. \tag{28}$$

After substituting (19) and (20) into Eq. (27), we obtain an expression for determination of the radius of the gravitational sphere of a planet in the form

$$\Delta_2 = r_1 m^{1/2}. \tag{29}$$

We assume for r_1 the minimum and maximum value of the planetary radius vector. Then, for the radius of the planetary gravitational sphere, we obtain the numerical values cited in Table 4.

It is interesting to note that of all the satellites of the major planets, only our moon (a = 0.384 million km) is at all times beyond the limits of the gravitational sphere of the planet. The outer satellites VIII and IX of Jupiter (both with retrograde motion) have orbits with semimajor axes of 23.5 and 23.7 million kilometers respectively, and therefore also are beyond the limits of the planetary gravitational sphere, but only when Jupiter is near perihelion or when the satellites themselves are near the aphelia of their orbits.

4. Hill Gravitational Sphere

The problem of libration points in the limited three-body problem has been discussed in detail in [1]. The libration point L_1 determines the maximum value of the radius of the closed region within which Hill's stable satellite motion is possible. In the case of larger radius values, the region of possible motions ceases to be closed and is connected to the region of possible motions around the sun. The distance Δ_3 of the libration point L_1 from a planet is defined by the following formula ([1], pp. 114-115):

$$\Delta_3 = a\left(\mu - \frac{1}{3}\mu^2 - \frac{1}{9}\mu^3\right), \tag{30}$$

where

$$\mu = \left(\frac{m}{3}\right)^{1/3}, \tag{31}$$

a small value for the planets of the solar system. The values μ have been compared in Table 5.

We will consider the Hill gravitational sphere to be a region of space with its center in the planet and with the radius Δ_3. The numerical value of the quantity Δ_3, derived using formula (30), is given in Table 6.

The surface of the Hill gravitational sphere can be considered as the theoretical boundary of the system of satellites of a particular planet. We will compute the ratio F_1/R_1 for points situated on the surface of the Hill sphere. After substituting the values F_1 and R_1 in formulas (20) and (21), we find that the maximum value of the quantity F_1/R_1 is

$$F_1/R_1 = {}^2/_3. \qquad (32)$$

The region of space bounded by a surface on which $R_1 = F_1$ has a radius equal to

$$\Delta_4 = r_1 \left(\frac{m}{\sqrt{1+3\cos^2\varphi}}\right)^{1/3}. \qquad (33)$$

This region differs little from a sphere, since the ratio of the maximum and minimum radii is $\sqrt[3]{2} = 1.26$. Assuming

$$\Delta_4 = am^{1/3}, \qquad (34)$$

we find that the radius of this gravitational sphere is approximately $\sqrt[3]{3} = 1.44$ times greater than the radius of the Hill sphere.

5. Lunar Gravitational Spheres

We will use formulas (25), (29), and (30) for determination of the gravitational spheres of the moon. In this case, it is necessary to consider two variants of the problem: a) moon – earth; b) moon – sun. The astronomical constants have the following numerical values:

Problem a	Problem b
a = 384,400 km	a = 149,600,000 km
e = 0.05490	e = 0.016,751
m = 1 : 81.375	m = 1 : 27,133,500

We will consider problem (a) first. Formula (25) gives for the radius of the lunar sphere of activity:

$\Delta_1(\text{min}) = 62{,}500$ km,
$\Delta_1(\text{max}) = 69{,}800$ km,
$\Delta_1(\text{mean}) = 66{,}150$ km.

The ratio F_1/R_1 on the surface of the sphere of activity is

$$F_1/R_1 = 2m^{1/5} = 0.83.$$

TABLE 5. $\mu = (m/3)^{1/3}$

Mercury	0.0038	Jupiter	0.0683
Venus	0.0093	Saturn	0.0457
Earth	0.0100	Uranus	0.0244
Mars	0.0048	Neptune	0.0258
		Pluto	0.0097

By using formula (29) we determine the radius of the lunar gravitational sphere:

$\Delta_2(\text{min}) = 40{,}000$ km,
$\Delta_2(\text{max}) = 45{,}000$ km,
$\Delta_2(\text{mean}) = 42{,}650$ km.

Finally, using formula (30), we compute the radius of the Hill sphere:

$$\Delta_3 = 58{,}050 \text{ km}.$$

We note that the Hill sphere is smaller in dimensions than the sphere of activity, whereas, for the major planets on the other hand, the Hill sphere is larger in dimensions than the sphere of activity. It is easy to confirm that the Hill sphere is smaller in dimensions than the sphere of activity if $m > {}^1/_{243}$. If $m > {}^1/_9$, the Hill sphere is smaller than the gravitational sphere.

We now will consider problem (b). Formula (25) gives for the radius of the lunar sphere of activity:

$\Delta_1(\text{min}) = 156{,}400$ km,
$\Delta_1(\text{max}) = 161{,}700$ km,
$\Delta_1(\text{mean}) = 159{,}050$ km.

The radius of the gravitational sphere is calculated by using Eq. (29):

$\Delta_2(\text{min}) = 28{,}200$ km,
$\Delta_2(\text{max}) = 29{,}200$ km,
$\Delta_2(\text{mean}) = 28{,}700$ km.

Formula (30) gives the radius of the Hill sphere:

$$\Delta_3 = 344{,}800 \text{ km}.$$

By comparing problems (a) and (b) we see that for the radius of the lunar gravitational field we should assume a value corresponding to the moon – sun problem, while for the radius of the sphere of activity and the Hill sphere we should assume values corresponding to the moon – earth problem. The final results are compared in Table 7.

6. Solar Gravitational Sphere

If it is assumed that the entire mass of the Galaxy is concentrated at its center and the sun moves around the center of the Galaxy in an undisturbed orbit, formulas (25), (29), and (30) can be used for determination of the dimensions of the gravitational spheres of the sun. For the mass of the Galaxy we assume the value

TABLE 6. Radii of the Hill Sphere for the Major Planets

	Δ_3 in a.u.	Δ_3 in million km		Δ_3 in a.u.	Δ_3 in million km
Mercury	0.00148	0.221	Saturn	0.42881	64.15
Venus	0.00674	1.008	Uranus	0.46494	69.56
Earth	0.01001	1.497	Neptune	0.77035	115.24
Mars	0.00724	1.083	Pluto	0.38392	57.43
Jupiter	0.34697	51.91			

TABLE 7. Lunar Gravitational Spheres

	Radius, km	Comments
Δ_1(min)	62 500	Sphere of activity
Δ_1(max)	69 800	
Δ_2(min)	28 200	Gravitational sphere
Δ_2(max)	29 200	
Δ_3	58 050	Hill sphere

$$M = 1.3 \cdot 10^{11} \text{ solar masses.}$$

and for the radius of the galactic orbit of the sun we assume the value

$$a = 8 \cdot 10^3 \text{ pc} = 16.5 \cdot 10^8 \text{ a.u.}$$

The results of the computation have been given in Table 8.

As a comparison we point out that the distance from the sun to α Centauri is 1.3 parsecs and the orbit of Pluto has a radius of 40 a.u.

If it is agreed to assume a Hill sphere for the boundaries of the solar system, it is found that these boundaries extend to the nearest stars. However, stable motion of the planets (with direct motion) probably is possible only within the solar gravitational sphere. Only retrograde motions will be stable in the outer region of the solar system. Perturbations from the nearest stars already can be of importance at distances of the order of 30-50 thousand a.u. from the sun.

TABLE 8. Solar Gravitational Spheres

	Radius		Comments
	in pc	in a.u.	
Δ_1	0.29	60 000	Sphere of activity
Δ_2	0.022	4500	Gravitational sphere
Δ_3	1.1	230 000	Hill sphere

LITERATURE CITED

1. M. F. Subbotin, Course in Celestial Mechanics [in Russian] (1937), Vol. 2, Chapter VII.