Articles

INTERNAL STRUCTURE OF HYDROGEN PLANETS

- USSR -

[Following is a translation of an article by A.A.
Abrikosov, in the Russian-language periodical
Voprosy Kosmogonii (Problems of Cosmogony),
Vol III, Moscow, 1954, pages 11-19.]

We know that hydrogen plays an exceptional role in
the structure of celestial bodies. In particular, from
the data of observations it follows that hydrogen is the
principal component part of some planets of the solar
system, for example of Jupiter and Saturn. For this reason
it is of interest to determine the equation of state for
hydrogen, and the basic characteristics of hydrogen pla-
nets which are derived from that equation. Work of this
nature was first undertaken by Ramsey [1], who utilized
the equation of state for hydrogen which is derived in
reference [2]. However, the calculations presented in
this reference are based on entirely arbitrary and some-
times erroneous assumptions. Therefore we have investigated
this problem anew, by relying upon entirely different pre-
mises.

1. The Equation of State of Hydrogen at High Pressures.

A detailed derivation of the equation of state of
hydrogen will be presented in another communication. Here
we will limit ourselves to a presentation of the principal
results. The equation of state was calculated for two mo-
difications of hydrogen: the molecular and the atomic,
after which the conditions of phase transition were deter-
mined.

In reference [2], in computing the molecular modi-
fication it was assumed that the molecules which interact
with one another as a whole are packed in a face-centered
cubic lattice. Energy of interaction was determined from

data on temperature dependence of the second virial coeffi-
cient. These assumptions are not valid at high densities,
when distances between the molecules are of the order of
molecular dimensions. Moreover, the effected determinations
of the second virial coefficient relate to temperatures
which are not sufficiently high to permit a satisfactory
determination of the forces of repulsion between the mole-
cules, which play the greatest role at high pressures.

We made the assumption that the energy of molecular
lattice is composed of energies of interaction of individual
atoms. At the same time it was assumed that the molecules in
the lattice continue to remain in a singlet state. A veri-
fication of this assumption has shown that it is correct
within 1%. The lattice was considered to be hexagonal with
the axes of the molecules coinciding in direction with the
hexagonal axis. Such a form of the lattice reflects the fact
that at high densities the molecules cease to rotate; in
addition, a lattice of this form is simple as well as quite
densely packed. The three lattice parameters (side of tri-
angle within the layer, distance between layers, diameter
of molecule) were selected on the basis of the condition
of minimal energy with the given atomic volume. Energies
of interaction between atoms were determined essentially
by the method of Wang (minimum in effective nuclear charge).
The energy of interaction between atoms within a single
molecule was particularized thereafter by the method of
Hilleraas [3]. To the energy value thus obtained was added
the energy of zero-point vibrations, consisting of energy
of vibrations of atoms in a single molecule and the energy
of vibrations of molecules in the lattice. The latter was
calculated approximately on the basis of a quasi-isotropic
model. The data so obtained are applicable with densities
exceeding 0.226 g/cm^3. At lower densities there come into
play the Van der Waals' forces which we have not taken into
account, and furthermore there is possible a transition to
a rotary modification.

In reference [1] the atomic lattice is calculated
by using the Wigner-Zeitz method, the employment of which
reduces this problem to a modification of the problem con-
cerning an isolated atom. The applicability of this method
at high pressures is questionable. We, on the other hand,
have calculated the energy of metallic hydrogen as the
diagonal matrix element of Hamiltonian, taken by means of
electronic function which constitutes antisymmetrized
product of modulated plane waves of the form:

$$\psi_{k_e}(r) = e^{ik_e r} \sum_n e^{-\alpha |r - r_n|}$$

Here the summation is effected for all atoms in the lattice, k_e fill Fermi sphere, and parameter α is determined from minimum energy value. The lattice was assumed to be cubic. A face-centered lattice was found to be most advantageous. To the thus determined energy was added the energy of zero-point vibrations which was calculated by using a quasi-isotropic model, like in the case of the molecular lattice. It should be noted that this energy constitutes a fairly substantial portion of total energy: about 20% in the molecular lattice, and about 8% in the atomic lattice.

The transition point was determined from intersection of μ (p) curves, where μ is chemical potential and p is pressure. Its principal characteristics are: $p = 2.4 \cdot 10^6$ atm, density jump from $\rho = 0.621$ g/cm^3 (molecular) to $\rho = 1.12$ g/cm^3 (atomic). The complete equation of state can be represented by the following correlation:

with $p < 2.4 \cdot 10^6$ atm

$$p = 2.37 \exp(-5.88 \rho^{-1/3}) \cdot 10^9 \text{ atm}$$

with $p > 2.4 \cdot 10^6$ atm

$$p = (1.06 \rho^{5/3} - 0.887 \rho^{4/3}) \cdot 10^7 \text{ atm}$$

(ρ = density in g/cm^3). For the molecular modification the above-shown correlation is interpolational and holds for transition point up to $\rho \approx 0.2$ g/cm^3 within 3%.

2. Characteristics of Hydrogen Planets.

The equation of gravitational equilibrium of a sphere consisting of matter having a specified equation of state may be written in the form:

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\mu}{dr}\right) = -4\pi m_1^2 G v^{-1}, \tag{1}$$

where r is radius, μ -- chemical potential, G -- gravitational constant, v -- atomic volume, m_1 -- mass of the

forming the planet. At the center the derivative $d\mu/dr$ vanishes. When atomic units are used for μ and v, it is convenient to use for r the units

$$r_0 = \sqrt{\frac{1}{4\pi G}\frac{e}{m_1}}\, r_k = 1{,}66 \cdot 10^9 \text{ см} \qquad (2)$$

(where r_k is Bohr's radius). On introducing the new variable $r\mu = \xi$, we get in terms of the new units:

$$\frac{d^2\xi}{d\bar{r}^2} = -\frac{r}{v}. \qquad (3)$$

The boundary of the planet is determined by the condition $p = 0$ or, since energy of sublimation of solid hydrogen is very low (0.00017) we can use the condition: $\mu = \xi = 0$. The equation is integrated numerically from the initial point $r = 0$, $\xi = 0$, $\xi'(0) = \mu(0)$, $v(0) = v[\mu(0)]$ to the point of $\xi = 0$. The radius corresponding to this point is the radius of the planet. The mass of the planet can be determined from equation (1) in the following manner:

$$M = 4\pi m_1 \int_0^R \frac{r^2}{v}\, dr = \frac{1}{m_1 G} R^2 \frac{d\mu}{dr}\Big|_R;$$

where R is the radius of the planet.

By using the new units for radius and μ, and by introducing the variable ξ, we get:

$$M = R\frac{d\xi}{dr}\Big|_{r=R}, \qquad (4)$$

if the mass is expressed in M_0 units, which we will use hereinafter:

$$M_0 = 4\pi m_1 \frac{r_0^3}{r_k^3} = 6{,}48 \cdot 10^{20} \text{ г}. \qquad (5)$$

For the calculations it is necessary to have the graph of function $\mu(v)$. This graph is shown in Fig.1. For volumes greater than 50, for which we have no reliable
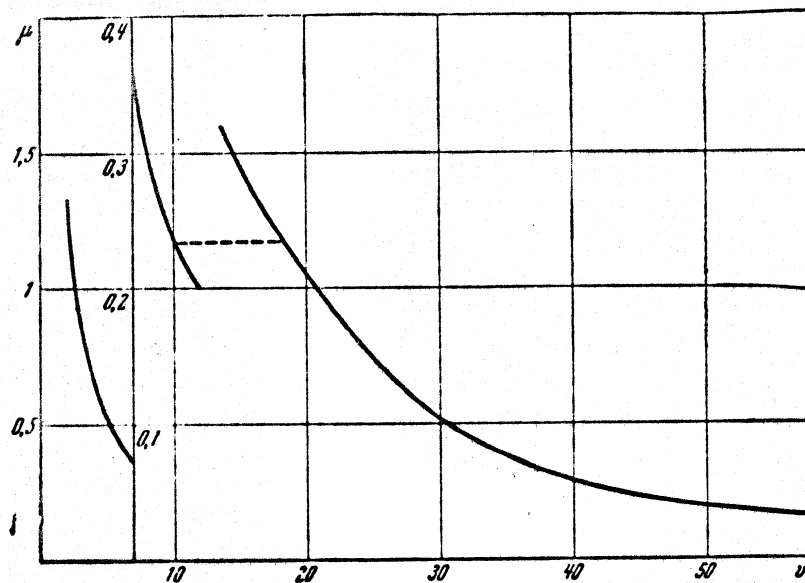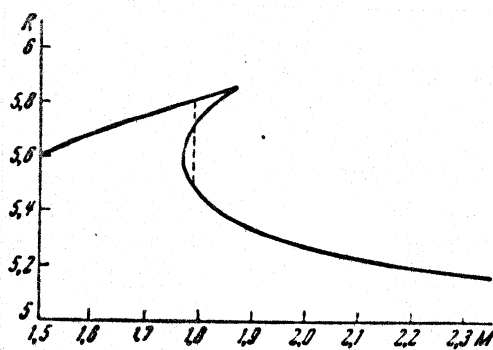
Fig.1

Fig.2

,data, we resort to extrapolation of our data to the minimum of energy at v = 131.5, E = 0.00017, which corresponds to the experimental data for solid hydrogen. This procedure is very inaccurate, but for volumes v < 50 with r = 0 the region of v > 50 ceases very rapidly to be significant, so that the inaccuracy of the curve of μ (v) has little effect on the results.

Numerical integration of the equation shows that in the region of planetary magnitudes, where the metal phase begins to appear, the R(M) curve has an S-shaped configuration (Fig.2), i.e., to the same amount of matter seem to correspond several values of the radius. This evidences that the metal phase appears not gradually, but all of a sudden, encompassing forthwith a substantial part of the planet. The value of the mass at which this sudden change occurs is determined from the condition of equality of the energy in the two states, or

$$\int_{1}^{2} \frac{M \, dM}{R} = 0, \qquad (6)$$

where integration is effected along the R(M) curve, and points 1 and 2 correspond to one mass. Since over the S-shaped portion the variations of mass and radius are small in comparison with their magnitude, it is readily apparent that (6) approximates the condition of equality of areas formed by the curve on both sides of straight line M = const. Such an analysis yields a mass M = 1.786 at the point of sudden change, and radii R_1 = 5.806 (on the side of lower mass values) and R_2 = 5.49 (on the side of higher mass values). R_1 is the largest possible radius of a planet consisting of hydrogen. The radius R_x of the sphere which is formed from the metal phase is equal to 2.01, that is, it amounts to more than one-third of the radius of the planet. On further increase of the the mass of the planet the ratio R_x/R is rapidly approaching 1.

In Fig.3 is shown the dependence of the radius, expressed in r_0 units, on Brigg's logarithm of M (in M_0 units). The bottom curve represents the dependence of the radius of the metal phase R_x on lg M. The same data are shown in Fig.4 in the form of a $\bar{\rho}$ (lg M) graph. The average density $\bar{\rho}$ (in g/cm^3) is determined from R and M in accordance with the formula

$$\bar{\rho} = \frac{3m_1}{r_h^3} \frac{M}{R^3} = 33,9 \frac{M}{R^3} \quad \text{g/cm}^3 \qquad (7)$$
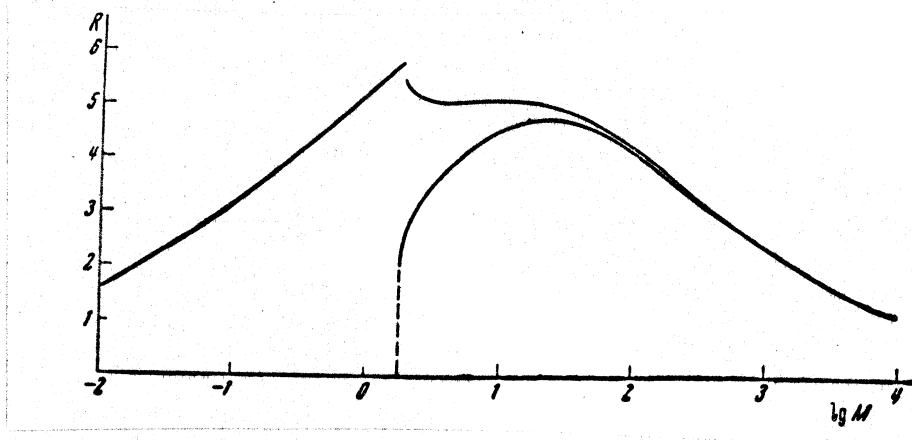
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Fig.3

On the same graph are plotted the points corresponding
to the planets Jupiter, Uranus and Neptune.

Integration of the equation of gravitational equi-
librium also provides information on the distribution of
density and pressure within the planet. In Fig.5 are shown
the graphs of $p(r)$ and ρ (r) before and after the sudden
change -- the formation of the metal phase. The p is shown
in million atmospheres, and the ρ in g/cm^3. Fig. 6 and 7
show the dependence of ρ /ρ_0 and p/p_0 on the radius, for
different values of $_0$ and p_0 -- the density and pressure
at the center of the planet. Curve I -- $\rho_0 = 0.226$ g/cm^3,
$p_0 = 0.151 \cdot 10^6$ atm; II -- $\rho_0 = 0.607$ g/cm^3, $p_0 = 2.31 \cdot 10^6$
atm; III -- $\rho_0 = 1.41$ g/cm^3, $p_0 = 4.80 \cdot 10^6$ atm; IV --
$\rho_0 = 7.53$ g/cm^3, $p_0 = 1.75 \cdot 10^8$ atm; V -- $\rho_0 = 1.13 \cdot 10^3$
g/cm^3, $p_0 = 1.188 \cdot 10^{12}$ atm; VI -- $\rho_0 = 1.13 \cdot 10^6$ g/cm^3,
$p_0 = 1.284 \cdot 10^{17}$ atm. These curves give an idea of all
the possible types of distribution of density and pressure
in the hydrogen planets. The last point (Curve VI) is
located at the edge of relativistic Fermi gas at the center
and the corresponding mass $M \approx 10^4$ is the limit of the
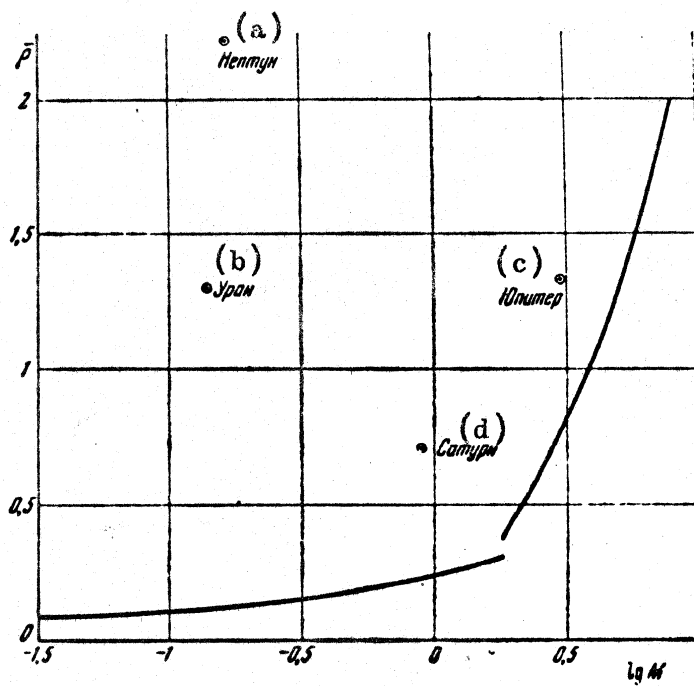applicability of the theory.

Fig.4

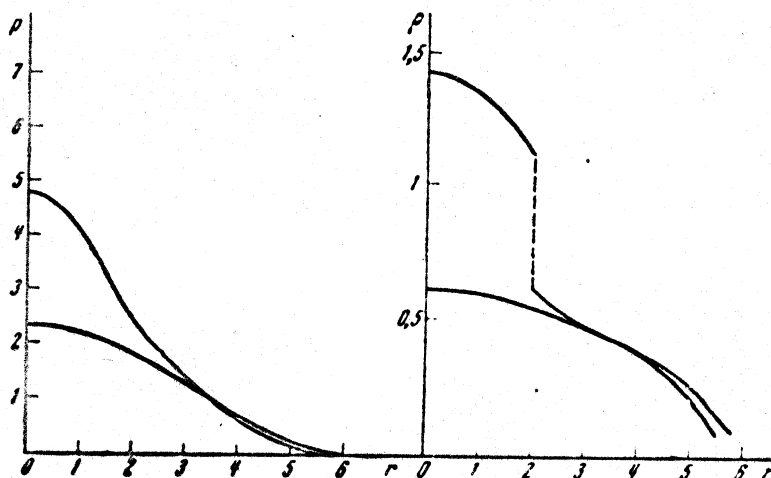Legend: a) Neptune; b) Uranus; c) Jupiter; d) Saturn.
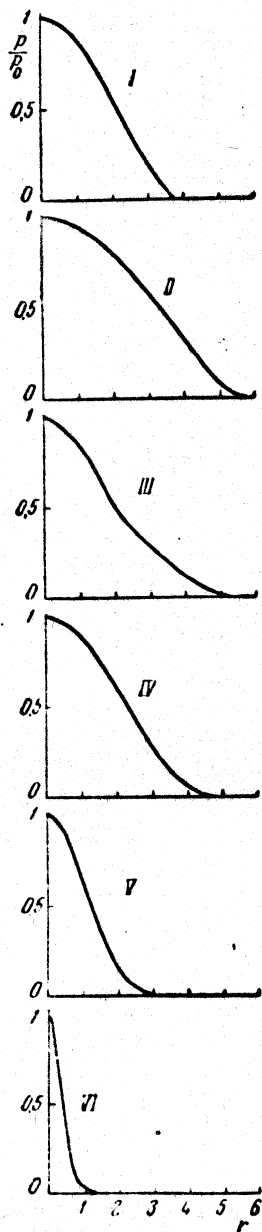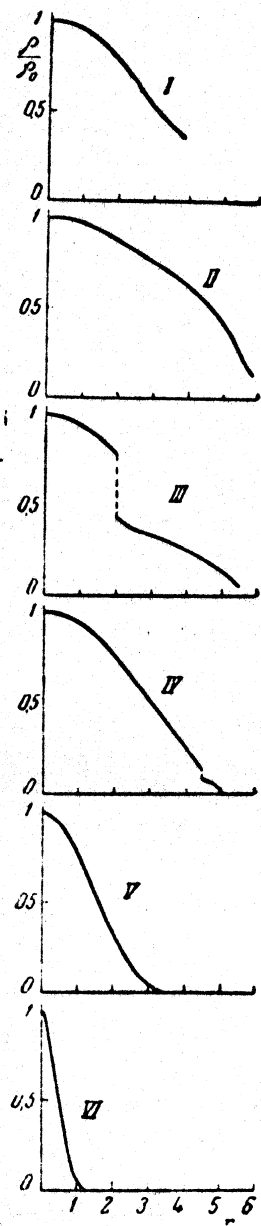

Fig.5

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Fig.6

Fig.7

A comparison of the results so obtained with the known data for the four planets, shows that none of them consists of pure hydrogen. From the comparison it is apparent that the planets contain also a heavier matter. In all probability it is helium, since in cosmic abundance it is next to hydrogen, and exceeds considerably all the other elements taken together. An attempt to determine the admixture of helium was made in reference [1], but the method employed by the author is such a rough approximation that it can not yield even qualitatively reliable results. The author does not know the equation of state of helium and utilizes most questionable data on specific volumes. Moreover, in reference [1] there is made the completely unsubstantiated, and probably erroneous, assumption that the concentration of helium is constant. We believe it more correct to assume that the helium is concentrated near the center of the planet as a result of diffusion, so that the planet contains an inner sphere of helium and an outer envelope of hydrogen. If we make this assumption, then, from the data on radii and masses of planets, using the equation of state for both substances -- hydrogen and helium -- it is possible to determine the amount of either in the planets. An estimate shows that in this manner it is possible to explain even the structure of Uranus and Neptune, which are much more dense than Jupiter and Saturn.

In conclusion I wish to thank Academician L.D. Landau, who prompted me to carry out the calculations described in the present paper, and has made a number of valuable suggestions in the course of the performance of this work.

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