1964PrCmg...3....8A

INTERNAL STRUCTURE OF HYDROGEN PLANETS

- USSR -

[Following is a translation of an article by A.A. Abrikosov, in the Russian-language periodical Voprosy Kosmogonii (Problems of Cosmogony), Vol III, Moscow, 1954, pages 11-19.]

We know that hydrogen plays an exceptional role in the structure of celestial bodies. In particular, from the data of observations it follows that hydrogen is the principal component part of some planets of the solar system, for example of Jupiter and Saturn. For this reason it is of interest to determine the equation of state for hydrogen, and the basic characteristics of hydrogen planets which are derived from that equation. Work of this nature was first undertaken by Ramsey [1], who utilized the equation of state for hydrogen which is derived in reference [2]. However, the calculations presented in this reference are based on entirely arbitrary and sometimes erroneous assumptions. Therefore we have investigated this problem anew, by relying upon entirely different premises.

1. The Equation of State of Hydrogen at High Pressures.

A detailed derivation of the equation of state of hydrogen will be presented in another communication. Here we will limit ourselves to a presentation of the principal results. The equation of state was calculated for two modifications of hydrogen: the molecular and the atomic, after which the conditions of phase transition were determined.

In reference [2], in computing the molecular modification it was assumed that the molecules which interact with one another as a whole are packed in a face-centered cubic lattice. Energy of interaction was determined from

Provided by the NASA Astrophysics Data System

1964PrCmg...3....8A

data on temperature dependence of the second virial coefficient. These assumptions are not valid at high densities, when distances between the molecules are of the order of molecular dimensions. Moreover, the effected determinations of the second virial coefficient relate to temperatures which are not sufficiently high to permit a satisfactory determination of the forces of repulsion between the molecules, which play the greatest role at high pressures.

We made the assumption that the energy of molecular lattice is composed of energies of interaction of individual atoms. At the same time it was assumed that the molecules in the lattice continue to remain in a singlet state. A verification of this assumption has shown that it is correct within 1%. The lattice was considered to be hexagonal with the axes of the molecules coinciding in direction with the hexagonal axis. Such a form of the lattice reflects the fact that at high densities the molecules cease to rotate; in addition, a lattice of this form is simple as well as quite densely packed. The three lattice parameters (side of triangle within the layer, distance between layers, diameter of molecule) were selected on the basis of the condition of minimal energy with the given atomic volume. Energies of interaction between atoms were determined essentially by the method of Wang (minimum in effective nuclear charge). The energy of interaction between atoms within a single molecule was particularized thereafter by the method of Hilleraas [3]. To the energy value thus obtained was added the energy of zero-point vibrations, consisting of energy of vibrations of atoms in a single molecule and the energy of vibrations of molecules in the lattice. The latter was calculated approximately on the basis of a quasi-isotropic model. The data so obtained are applicable with densities exceeding 0.226 g/cm^3. At lower densities there come into play the Van der Waals' forces which we have not taken into account, and furthermore there is possible a transition to a rotary modification.

In reference [1] the atomic lattice is calculated by using the Wigner-Zeitz method, the employment of which reduces this problem to a modification of the problem concerning an isolated atom. The applicability of this method at high pressures is questionable. We, on the other hand, have calculated the energy of metallic hydrogen as the diagonal matrix element of Hamiltonian, taken by means of electronic function which constitutes antisymmetrized product of modulated plane waves of the form:

Provided by the NASA Astrophysics Data System

1964PrCmg...3....8A

$$\psi_{k_e}(r) = e^{ik_e r} \sum_n e^{-\alpha |r - r_n|}$$

Here the summation is effected for all atoms in the lattice, k_e fill Fermi sphere, and parameter α is determined from minimum energy value. The lattice was assumed to be cubic. A face-centered lattice was found to be most advantageous. To the thus determined energy was added the energy of zero-point vibrations which was calculated by using a quasi-isotropic model, like in the case of the molecular lattice. It should be noted that this energy constitutes a fairly substantial portion of total energy: about 20% in the molecular lattice, and about 8% in the atomic lattice.

The transition point was determined from intersection of μ (p) curves, where μ is chemical potential and p is pressure. Its principal characteristics are: $p = 2.4 \cdot 10^6$ atm, density jump from $\rho = 0.621$ g/cm^3 (molecular) to $\rho = 1.12$ g/cm^3 (atomic). The complete equation of state can be represented by the following correlation:

$$\text{with } p < 2.4 \cdot 10^6 \text{ atm}$$

$$p = 2.37 \exp(-5.88 \rho^{-1/3}) \cdot 10^9 \text{ atm}$$

$$\text{with } p > 2.4 \cdot 10^6 \text{ atm}$$

$$p = (1.06 \rho^{5/3} - 0.887 \rho^{4/3}) \cdot 10^7 \text{ atm}$$

(ρ = density in g/cm^3). For the molecular modification the above-shown correlation is interpolational and holds for transition point up to $\rho \approx 0.2$ g/cm^3 within 3%.

2. Characteristics of Hydrogen Planets.

The equation of gravitational equilibrium of a sphere consisting of matter having a specified equation of state may be written in the form:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\mu}{dr}\right) = -4\pi m_1^2 G v^{-1}, \qquad (1)$$

where r is radius, μ -- chemical potential, G -- gravitational constant, v -- atomic volume, m_1 -- mass of the

Provided by the NASA Astrophysics Data System

1964PrCmg...3....8A

forming the planet. At the center the derivative $d\mu/dr$ vanishes. When atomic units are used for μ and v, it is convenient to use for r the units

$$r_0 = \sqrt{\frac{1}{4\pi G}\frac{e}{m_1}}\, r_k = 1{,}66 \cdot 10^9 \text{ см} \qquad (2)$$

(where r_k is Bohr's radius). On introducing the new variable $r\mu = \xi$, we get in terms of the new units:

$$\frac{d^2\xi}{dr^2} = -\frac{r}{v}. \qquad (3)$$

The boundary of the planet is determined by the condition p = 0 or, since energy of sublimation of solid hydrogen is very low (0.00017) we can use the condition: $\mu = \xi = 0$. The equation is integrated numerically from the initial point r = 0, $\xi = 0$, $\xi'(0) = \mu(0)$, $v(0) = v[\mu(0)]$ to the point of $\xi = 0$. The radius corresponding to this point is the radius of the planet. The mass of the planet can be determined from equation (1) in the following manner:

$$M = 4\pi m_1 \int_0^R \frac{r^2}{v}\, dr = \frac{1}{m_1 G} R^2 \frac{d\mu}{dr}\Big|_R;$$

where R is the radius of the planet.

By using the new units for radius and μ, and by introducing the variable ξ, we get:

$$M = R\frac{d\xi}{dr}\Big|_{r=R}, \qquad (4)$$

if the mass is expressed in M_0 units, which we will use hereinafter:

$$M_0 = 4\pi m_1 \frac{r_0^3}{r_k^3} = 6{,}48 \cdot 10^{29} \text{ г}. \qquad (5)$$

For the calculations it is necessary to have the graph of function $\mu(v)$. This graph is shown in Fig.1. For volumes greater than 50, for which we have no reliable

Provided by the NASA Astrophysics Data System

1964PrCmg...3....8A

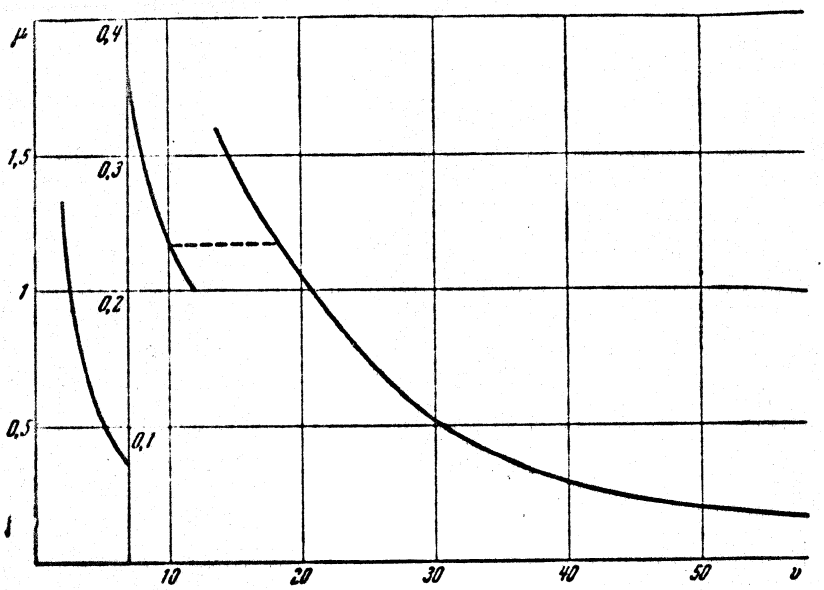

Fig.1

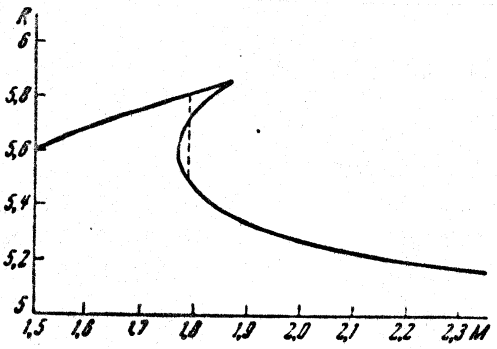

Fig.2

Provided by the NASA Astrophysics Data System

1964PrCmg...3....8A

data, we resort to extrapolation of our data to the minimum of energy at v = 131.5, E = 0.00017, which corresponds to the experimental data for solid hydrogen. This procedure is very inaccurate, but for volumes $v < 50$ with r = 0 the region of $v > 50$ ceases very rapidly to be significant, so that the inaccuracy of the curve of $\mu(v)$ has little effect on the results.

Numerical integration of the equation shows that in the region of planetary magnitudes, where the metal phase begins to appear, the R(M) curve has an S-shaped configuration (Fig.2), i.e., to the same amount of matter seem to correspond several values of the radius. This evidences that the metal phase appears not gradually, but all of a sudden, encompassing forthwith a substantial part of the planet. The value of the mass at which this sudden change occurs is determined from the condition of equality of the energy in the two states, or

$$\int_1^2 \frac{M\,dM}{R} = 0, \qquad (6)$$

where integration is effected along the R(M) curve, and points 1 and 2 correspond to one mass. Since over the S-shaped portion the variations of mass and radius are small in comparison with their magnitude, it is readily apparent that (6) approximates the condition of equality of areas formed by the curve on both sides of straight line M = const. Such an analysis yields a mass M = 1.786 at the point of sudden change, and radii $R_1 = 5.806$ (on the side of lower mass values) and $R_2 = 5.49$ (on the side of higher mass values). R_1 is the largest possible radius of a planet consisting of hydrogen. The radius R_x of the sphere which is formed from the metal phase is equal to 2.01, that is, it amounts to more than one-third of the radius of the planet. On further increase of the the mass of the planet the ratio R_x/R is rapidly approaching 1.

In Fig.3 is shown the dependence of the radius, expressed in r_0 units, on Brigg's logarithm of M (in M_0 units). The bottom curve represents the dependence of the radius of the metal phase R_x on lg M. The same data are shown in Fig.4 in the form of a $\bar{\rho}$ (lg M) graph. The average density $\bar{\rho}$ (in g/cm^3) is determined from R and M in accordance with the formula

$$\bar{\rho} = \frac{3m_1}{r_k^3}\frac{M}{R^3} = 33{,}9\frac{M}{R^3}\ \text{g/cm}^3 \qquad (7)$$

Provided by the NASA Astrophysics Data System

1964PrCmg...3....8A

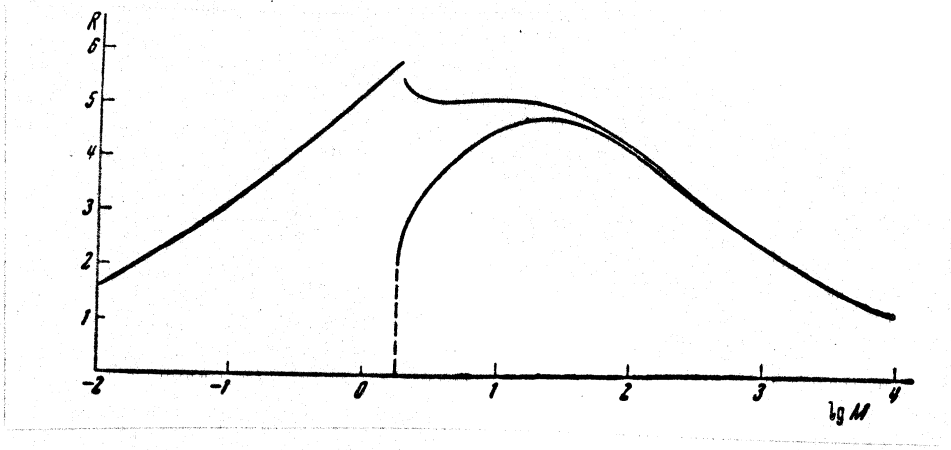

Fig.3

On the same graph are plotted the points corresponding to the planets Jupiter, Uranus and Neptune.

Integration of the equation of gravitational equilibrium also provides information on the distribution of density and pressure within the planet. In Fig.5 are shown the graphs of p(r) and ρ (r) before and after the sudden change -- the formation of the metal phase. The p is shown in million atmospheres, and the ρ in g/cm^3. Fig. 6 and 7 show the dependence of ρ / ρ_0 and p/p_0 on the radius, for different values of ρ_0 and p_0 -- the density and pressure at the center of the planet. Curve I -- $\rho_0 = 0.226$ g/cm^3, $p_0 = 0.151 \cdot 10^6$ atm; II -- $\rho_0 = 0.607$ g/cm^3, $p_0 = 2.31 \cdot 10^6$ atm; III -- $\rho_0 = 1.41$ g/cm^3, $p_0 = 4.80 \cdot 10^6$ atm; IV -- $\rho_0 = 7.53$ g/cm^3, $p_0 = 1.75 \cdot 10^8$ atm; V -- $\rho_0 = 1.13 \cdot 10^3$ g/cm^3, $p_0 = 1.188 \cdot 10^{12}$ atm; VI -- $\rho_0 = 1.13 \cdot 10^6$ g/cm^3, $p_0 = 1.1284 \cdot 10^{17}$ atm. These curves give an idea of all the possible types of distribution of density and pressure in the hydrogen planets. The last point (Curve VI) is located at the edge of relativistic Fermi gas at the center and the corresponding mass $M \approx 10^4$ is the limit of the applicability of the theory.

Provided by the NASA Astrophysics Data System

1964PrCmg...3....8A

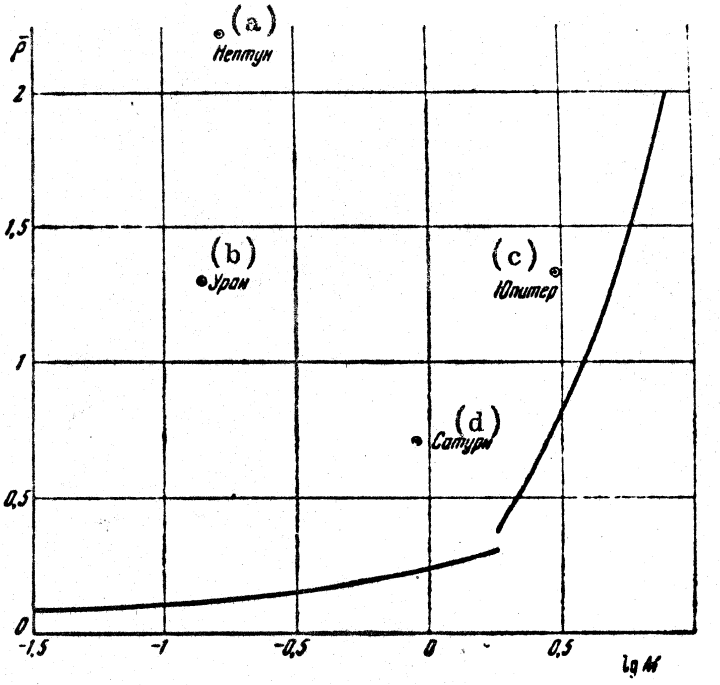

Fig.4

Legend: a) Neptune; b) Uranus; c) Jupiter; d) Saturn.

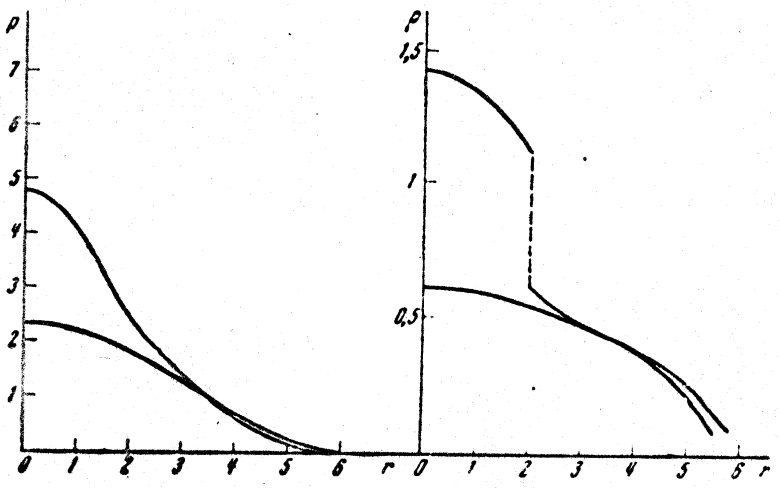

Fig.5

Provided by the NASA Astrophysics Data System

1964PrCmg...3....8A

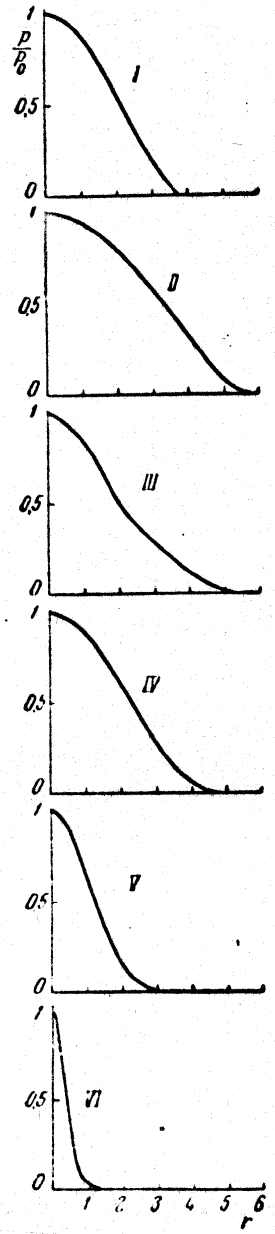

Fig.6

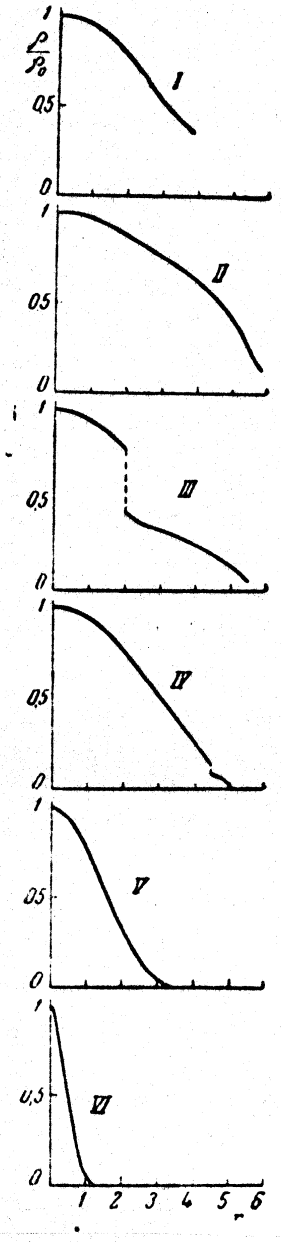

Fig.7

Provided by the NASA Astrophysics Data System