IRREVERSIBILITY IN SMALL STELLAR DYNAMICAL SYSTEMS

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Received February 10, 1964

ABSTRACT

Irreversibility in stellar dynamical systems was studied by means of numerical experiments using n-body calculations carried out in a computer for n up to 32. In the experiments, two similar systems evolve simultaneously and the separation of their representative points in phase space was observed to grow exponentially with the time variable because of the effects of encounters. The time constant of this exponential growth defines a relaxation time different from the usual relaxation time of stellar dynamics. Certain properties of the calculation and comparisons of the observed values of this relaxation time with values expected on the basis of binary collisions led to the conclusion that n-body systems with inverse-square law forces behave as tightly coupled systems. The time constant is surprisingly short—about $4\tau/n$, where τ is the mean time between binary collisions of one particle. Hard spheres or r^{-6} potentials do not show the collective behavior of inverse-square law systems. Errors in calculations of n-body systems grow exponentially with the time through this mechanism, and may therefore invalidate the results of n-body calculations.

I. INTRODUCTION

As a physical system composed of a finite number of particles evolves, the point representing it in phase space moves through that space. The evolution of a second physical system, initially differing very little from the first, might also be represented by a trajectory in phase space. The two trajectories start from points very close to each other, but may be expected to move apart as the system evolves. Each of the trajectories is expected to be time-reversible, this being the microscopic time reversibility of statistical mechanics. However, the divergence of the two trajectories from each other is a manifestation of the macroscopic irreversibility of these systems. The trajectories cannot cross, but if they are initially near each other, the rate of divergence yields information on the rate of entropy production in the systems. It is expected that the distance between representative points will increase exponentially with the time, although no proof of this conjecture seems to exist. Further, there seem to be no theoretical estimates of the rates at which the representative phase points move away from each other.

These questions may be studied using the techniques of experimental arithmetic, with n-body calculations as the specific tool. As with any experimental arrangement, the experimenter using numerical methods must demonstrate the applicability and validity of his results. The work reported here started as a check on the validity of numerical methods in the n-body problem of stellar dynamics, but wider implications soon became apparent.

With the advent of fast electronic computers, considerable interest in the n-body problem has been shown since the pioneering work of Pasta and Ulam.[1] Recently, von Hoerner (1960, 1963) has conducted some very nice numerical experiments in a stellar dynamical context, and Aarseth (1963) and Kinman and Sherman (1963) have made further studies. The present work makes no pretense of dealing with the large numbers of particles treated in the work of Aarseth and that of Kinman and Sherman, but rather addresses itself to questions of reversibility in these numerical experiments.

On being checked under time reversal, n-body systems with inverse-square law forces were found to fail. The origin of the irreversibility was traced to the amplifying

[1] Pasta and Ulam (1953) studied inverse-square forces in two dimensions as part of their program. The writer is indebted to Professor John Pasta for a brief description of the work.

effect of close encounters by following the motion of a 4-particle system in detail. The terminal points after time reversal lay on an arc around the location of the first close collision, as shown in Figure 1, suggesting that the turning angle in that collision was incorrectly calculated as the system returned. This effect may be analyzed in terms of binary collisions to give

$$\langle \delta\Theta_2^2 \rangle = f \langle \delta\Theta_1^2 \rangle \,, \tag{1}$$

where $\delta\Theta_1$ and $\delta\Theta_2$ are the angular "errors" on leaving the first and second collisions and f is obtained by averaging over the relevant physical properties of the system: particle density, force law of the interaction, angle between the planes of scattering, and so on. The derivation is most easily carried out in the center of mass of the binary encounter, and shows that the integrals of the motion are constant to first order, which accounts for the conservation reported by von Hoerner (1960, 1963) and others (Aarseth 1963). A relation like that of equation (1) leads to exponential growth or decrease of the "errors," the mean square angular "error" being multiplied by f at each binary encounter. The e-folding time will be given by $(\tau/\ln f)$, where τ is the mean time between collisions of one particle in the system. This argument is independent of the specific force law assumed, and should hold for any potential, as well as for the $1/r$ potential of stellar dynamics.

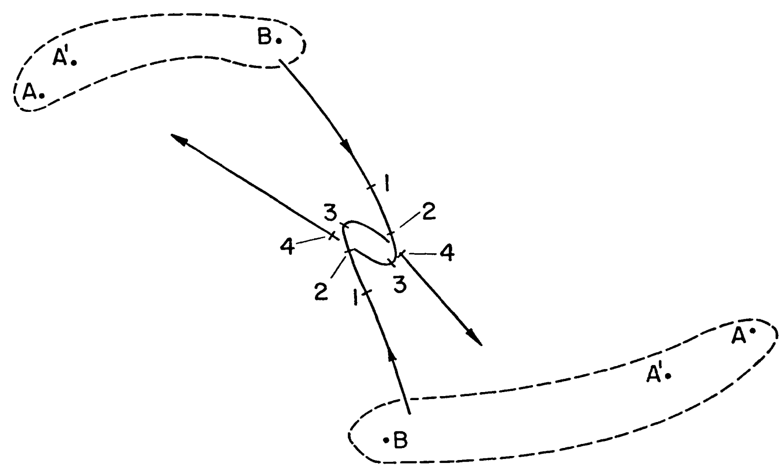

FIG. 1.—Particle orbits projected from configuration space for the two participants in a close collision. The particles start near B and proceed in the direction of the arrows. The numbered points on the orbits indicate the positions of the two particles at corresponding instants. The letters A, A', and B denote positions to which the particles returned after being time-reversed at different stages of the calculation.

For inverse-square law forces, the factor f may be quite large, as it involves an integral which diverges logarithmically at both limits of the impact parameter. Cutoffs must then be introduced, corresponding, as usual, to the mean interparticle distance at the upper limit, and to a distance of sensible departure from the inverse-square law of force at the lower limit. A number on the order of 10 results from elementary evaluations of f for a stellar dynamical system. The difficulties of definition of a binary encounter for this force law complicate the discussion.

The characteristic time of any such exponential process is a relaxation time. Normally, several different relaxation times may be defined for a physical system, each corresponding to a different kind of process, and this relaxation time is different from the usual relaxation time defined in stellar dynamics (cf. Chandrasekhar 1960). Relaxation phenomena of this kind may easily escape detection. The first integrals are well conserved throughout the calculation, and the resulting systems appear plausible.

II. DETAILS OF THE CALCULATION

a) *Integration*

The n-body problem of stellar dynamics was formulated according to the recipes of von Hoerner (1960). In this formulation, the integration is carried out for n particles of equal mass in Cartesian coordinates. Variable time steps based on the closest pair of particles are used to retain the accuracy of integration for close encounters while permitting coarser steps to be used when all the particles are well separated. The same time step is used for all particles. A second-order predictor-corrector integration method is used in which an attempt is made to retain reasonable accuracy without requiring re-evaluation of the forces more than once per integration step. The ten first integrals of the equations of motion were used as controls, and were constant to within the same limits as von Hoerner reported (1960).

b) *Initial Conditions*

The starting configuration was selected using random-number methods. While it would be possible to choose these numbers to build a plausible cluster at the outset, this was not done. Instead, the particles lay randomly inside a cube of unit edge in configuration space with the centroid on the origin, or, for cases where it was desired to normalize the total energy, the cube was magnified uniformly until the potential energy was -2. The initial velocities were similarly selected, but a system with zero angular momentum was obtained by using the inverse of the inertia tensor and the a priori angular momentum to compute the angular velocity that would reduce the final angular momentum to zero. Finally, the magnitudes of the velocities were adjusted to satisfy the virial theorem. A configuration with zero angular momentum permits reasonable estimates of the kinetic energy of random motions; if the system has angular momentum the partitioning of kinetic energy between random motions and rotation will change with the moment of inertia as the system evolves.

The system as it exists in the machine at any time may also be used as a more realistic starting condition. Such initial conditions were sometimes used.

c) *Time-Reversal Calculation*

The properties of the system under time reversal were calculated by letting the system evolve through some number of integration steps, then switching the sign of the time step and letting the system run backward until the total time variable reached zero. On the return to zero, the changes in corresponding velocities and positions were calculated and listed, as was the value of the time variable at the turnaround. The original set of initial conditions could then be re-established, and the system permitted to integrate with positive time steps for some new number of integration steps, and so on.

d) *Parallel Calculations*

The time-reversal method proved to be expensive to operate on a computer because the machine expends most of its efforts integrating over time intervals it has already covered. This difficulty may be circumvented by running parallel calculations. Parallel calculations provide a useful and sensitive tool for exploring irreversible phenomena in n-body calculations.

In the parallel calculation, a duplicate storage is set up representing a distinct n-body system in the computer memory. A very small perturbation is introduced into one component of a velocity or coordinate of one of the two systems. As the calculation proceeds, an integration step is carried out for one physical system, and then for the second, before proceeding to the next integration step. The two systems are kept together by using the same time step for both systems. Occasionally, the value of the time variable and the logarithm of the separation of the two points in phase space are printed:

$$\ln \Delta = \tfrac{1}{2} \ln \{ \Sigma [(x_2 - x_1)^2 + (u_2 - u_1)^2] \} . \quad (2)$$

The positions and velocities refer to corresponding particles of the first and second systems, and the summation extends over all particles. The argument of the logarithm is just the distance between two phase-points in the $6n$-dimensional γ-space.

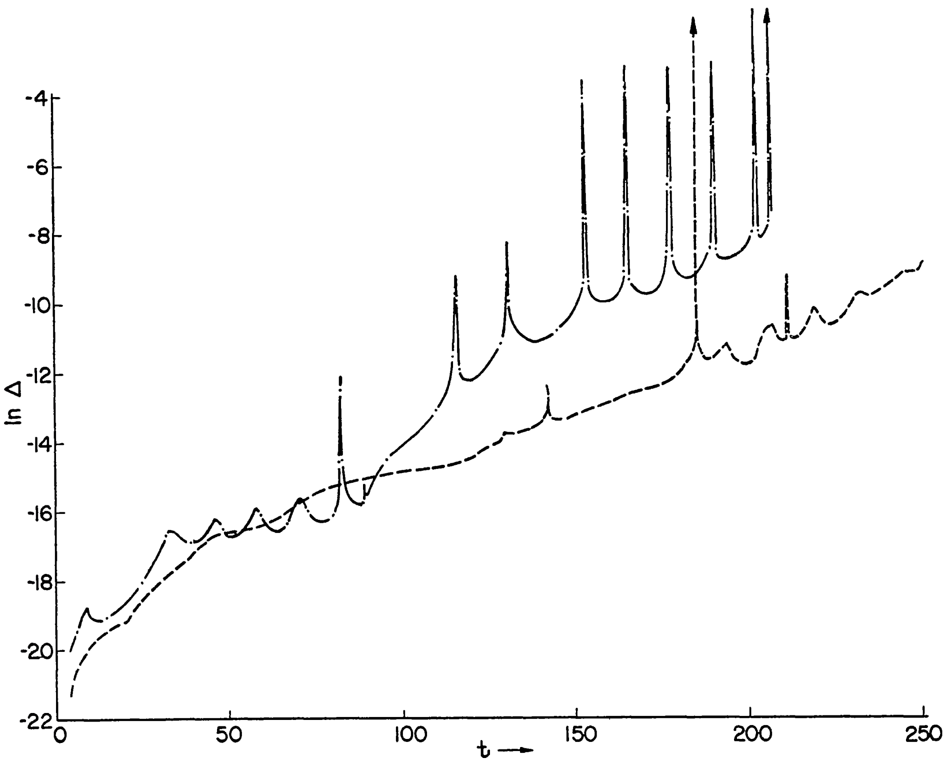

FIG. 2.—Tracks of the separation of phase points for 8-particle stellar dynamical systems. The dashed curve was based on the initial conditions of Section II, while the solid curve results from a different calculation starting from a system that had relaxed from the original atypical condition. The repeated equidistant spikes between $t = 150$ and $t = 220$ are formed by a single long-lived binary pair, which is destroyed in a close collision of one of its members with another star at $t = 225$.

The perturbation may be made much larger than the roundoff error in any one step, and may still be very small compared to the distance to a phase point representing the same system with two particle numbers permuted. The effect which develops is dominated by the perturbation, the cumulative effects of the numerous approximations in the numerical calculations remaining sensibly smaller because of the head start of the perturbation. The magnitude of the perturbation may be controlled much more easily in the parallel calculation than in the time-reversal study. Both the time-reversal and the parallel-calculation studies yield the same kind of information.

III. RESULTS

Typical results of the parallel calculations are shown for systems 8 and 12 of particles with $G = m = -E = 1$ in Figures 2 and 3. The two tracks in each figure illustrate different initial conditions—the solid curve was obtained from a system that had evolved through a considerable time from the kind of starting condition described above.

The general character of these tracks shows a steady increase of $\ln\Delta$ with time, but has a series of spikes superposed. The steady linear trend of the minima between the spikes illustrates the expected exponential increase with time and the slope of this trend gives the relaxation time. The general trend may be followed from $\ln\Delta \approx -20$ to $\ln\Delta \approx 0$, the latter corresponding to about the mean particle separation or to the r.m.s. particle velocity. This means that the separation of the phase points has grown by a factor of about 10^8–10^9. Beyond this, the system will blow up. This long track illustrates the sensitivity of parallel calculations for studying irreversibility in n-body systems.

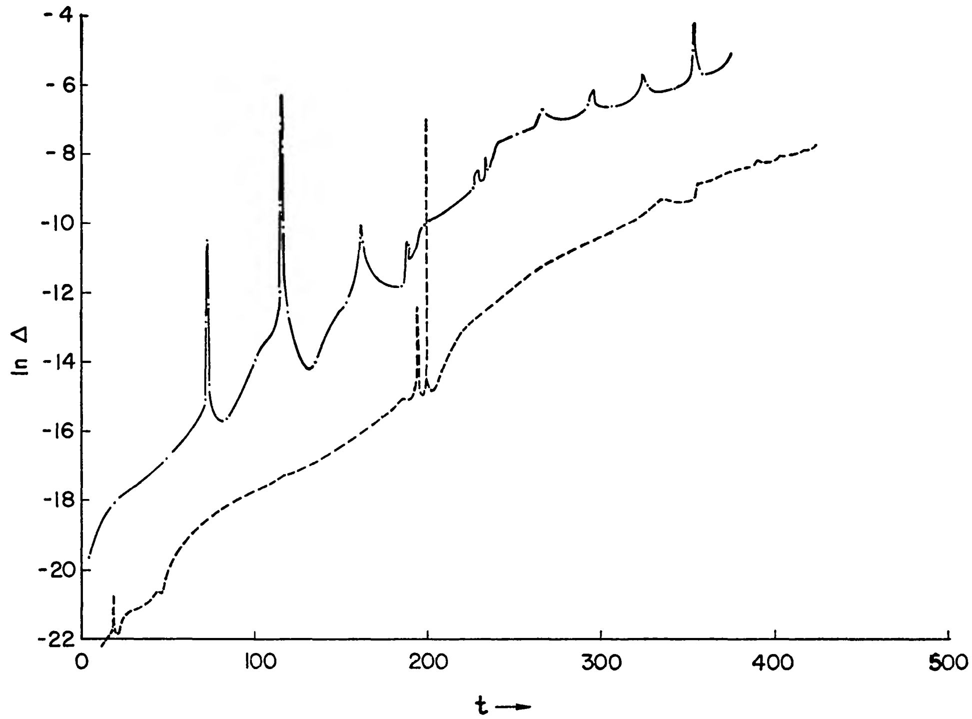

FIG. 3.—Twelve-particle system; the tracks are labeled as in Fig. 2

The spikes are of impressive amplitude—factors of 1000 and more over the prevailing level. The time intervals between successive points on the plots of Figures 2 and 3 (not shown) get very short in spikes, indicating that each spike is associated with a close collision somewhere in the system. The spike results because the two systems enter the close collision at slightly different phases. The recovery afterward indicates that the emergent particles are very close to each other—after the collision, the systems differ very little from each other.

A run with the time steps half as great as the normal (von Hoerner's $\mu = 16$ instead of the usual 8) showed that the character of these plots does not result from the approximations in the numerical integration. The resulting track in the $\ln\Delta$–t plane was identical with that for the normal integration step except that more detail was shown.

The calculation gives $\Delta = 0$ for all time if the perturbation is set to zero.

Some observations about the calculation are pertinent: (1) Spikes are associated with short time steps, indicating close encounters. However, no sequences of short time steps were observed which were not associated with spikes. (2) In checks designed specifically to test this point, it was established that a spike results whether or not the perturbation is introduced on a particle that participates in the first close collision. Furthermore, the relative height of the spike over the general level of the neighboring minima is about the same even if the first close encounter occurs fairly early in the integration. (3) The quantity $\tau d(\ln\Delta)/dt$ is not independent of the number of particles, as would be expected for error growth due to binary collisions, but instead the quantity $(1/n)\tau d(\ln\Delta)/dt$ is approximately independent of the number of particles, with a value of 0.2–0.3. In arriving at this conclusion, the mean time, τ, was calculated as $\langle s\rangle/\langle v\rangle$, where $\langle s\rangle$ is a mean interparticle distance and $\langle v\rangle$ is the r.m.s. particle velocity. This is essentially a statistical result—the individual values scatter rather widely.

These three observations indicate that a system behaves as if it were tightly coupled, and that $\ln\Delta$ grows much faster than predicted on the basis of binary collisions. Three or four different systems were studied for each number of particles, for $n = 4$, 8, 12, 16, 24, and 32. In the smaller systems, particle evaporation led to a noticeable change of slope.

IV. CONCLUSIONS

a) General

The principal conclusion to be drawn from these results is that the separation of the points representing two similar stellar systems in γ-space increases very rapidly—much more rapidly than can be explained on the basis of binary collisions as the amplifying mechanism. Co-operative phenomena predominate in determining the evolution of the system, to as high an order as has been explored so far (32 particles), but the dependence on particle number cannot continue to indefinitely large numbers of particles if the rate of separation of the systems is to remain finite. This result casts doubt on the arguments in stellar dynamics that start from consideration of binary encounters.

Similar calculations with repulsive r^{-6} potentials do not show this kind of collective effect—the collective effect results from the long range of inverse-square law forces. With the r^{-6} potential, interactions are essentially absent except when the particles collide, so the binary collision picture is valid. Tracks like Figures 2–3 show long horizontal sections between the spikes, indicating that the particles travel undisturbed with that potential. It approximates to the hard-sphere case. The general picture of an exponential growth also prevails for the r^{-6} potential. The system must be enclosed in a box because of the repulsive potential.

The increasing separation of representative phase points is equivalent to an increase of the phase volume accessible to the system. The entropy of the system, which is proportional to the logarithm of the volume of a $(6n - 10)$-dimensional sphere, increases linearly with the time. The appearance of co-operative phenomena is reminiscent of Mayer's theorem of statistical mechanics (Mayer 1960, 1961), according to which entropy is produced in microscopic systems as information flows into higher and higher order correlations. An expansion into a hierarchy of correlation functions is not likely to be useful for stellar dynamics because the system of equations cannot be closed at reasonable orders—the behavior of the system is determined by correlations of very high order. A measure of the rate at which information flows from single particles into aggregates of various sizes has not been devised. This might conceivably be studied by a parallel calculation using the n separations of corresponding points in the 6-dimensional μ-space.

b) Implications for n-Body Calculations

Because of the growth of computational errors n-body calculations in stellar dynamics are of limited usefulness. Each error—whether of the integration or of roundoff—

corresponds to a minute displacement of the representative phase point. The calculation does not proceed along a physical trajectory in the phase space, but is continually being shifted from one trajectory to another. As each phase point diverges exponentially from its neighbors, the phase point of the calculated system has an ever increasing volume accessible to it. The Liouville theorem is not applicable.

In general, the calculated system is not causally related to its previous conditions in the way that the physical system is. The calculated system is quasi-randomly somewhere in the exponentially growing sphere. If, in fact, all the phase space on the shell of the constraints is equally accessible to the system, the calculation will lead through a sequence of states permitting valid ensemble averages. It is not clear, however, that all of the phase space permitted by the constraints will be populated by this quasi-random sequence in the same way as by a physical system. Had the relaxation time of the phase-point separation turned out to be much longer than the usual relaxation time of stellar dynamics (cf. Chandrasekhar 1960; von Hoerner 1960), n-body calculations might still be used without fear that the calculated system tends toward states which are improbable for physical systems.

The mechanism causing the exponential growth of phase-point separation is physical —recourse to calculations of higher precision will not improve the reliability of n-body calculations.

These conclusions do not invalidate themselves since a perturbation much greater than those of calculational errors is used, and the divergence of two phase points is then followed only for short times.

It is a pleasure to thank the National Science Foundation for a grant which assisted this work. The calculations were carried out on the computer Maniac III at the University of Chicago; time was furnished by the AEC under Contract No. AT(11-1)-614.

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