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On the Elliptic Restricted Problem of Three Bodies

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Allowing nonzero eccentricity for Jupiter's orbit, the differential equations of motion of the restricted problem are presented. The simple form of the equations is obtained by using the true anomaly of the primaries as the independent variable and by introducing a special set of dimensionless variables describing the position of the third body. The short- and long-time effe ts of the eccentricity of the orbits of the primaries are discussed in connection with the generalized Jacobi integral.

The similarity between the equations for the circular and for the elliptic case permits the introduction of regularizing transformations following methods applicable for the circular case. The major effect of the eccentricity is that the regularized equations are in the form of integro-differential equations.

INTRODUCTION

THE problem of three bodies represents one of the few cases in celestial mechanics where the applicability of the general case is less than that of a special version. The restricted problem of three bodies in its original form specifies the motion of two of the three bodies on circular orbits and *restricts* the effect of the third body on the primaries. In spite of the simplifications introduced by the above restrictions the "problème restreint" still belongs to the class that Birkhoff calls nonintegrable dynamical systems. Nevertheless, the now so frequently quoted "bon mot" applies; we know much more about the restricted problem than about the general problem. Furthermore, the restricted problem describes actual situations in celestial mechanics with a tolerable first approximation. The major fault of the approximation introduced by the restricted formulation is its questionable ability to treat the long-time behavior of practically important dynamical systems in celestial mechanics. The principal reason for this is that significant effects might be expected because of the eccentricity of the orbits of the primaries. Introducing elliptic orbits for Jupiter's motion (elliptic orbits for the primaries) generalizes the original restricted problem and significantly improves its applicability. The *elliptic restricted* problem (also called *pseudo restricted* problem) will not possess the only known integral of the circular problem, at least not in its usual and extremely useful sense. The elliptic problem has been discussed in considerable detail by Ovenden and Roy (1960), Kopal (1956, 1963), Huang (1963) and recently it was applied to the problem of the Kirkwood gaps by Brouwer (1963). The introduction of the special variables used in the present paper was suggested by an almost one hundred year old and erroneous paper by Scheibner (1866) and by the well-known Lagrangian equilateral solution of the general problem of three bodies. Such variables have also been indicated by Brown and Shook (1933) in connection with the problem of planetary perturbations.

The study of the long-time behavior of dynamical systems usually requires the establishment of a singularity-free set of differential equations so that Cauchy's theorem may be applicable. Such sets of equations for the original (circular) restricted problem were established a long time ago (Thiele 1895) and several possible regularizing transformations have been proposed since then (Birkhoff 1915; Arenstorf 1963; Deprit 1963).

Our purpose here is to extend the applicability of the restricted problem to the study of the long-time behavior of important systems in celestial mechanics along two lines. Firstly we will give the set of equations applicable to the eccentric case and secondly we will show its regularization.

EQUATIONS OF MOTION FOR $e\neq 0$

In this section we show that the equations of motion for the case of nonzero eccentricity can be written in a *form* identical with the form of the equations applicable to the circular case. The circular restricted problem is represented by

$$\begin{aligned}\ddot{x}-2\dot{y}&=\Omega_x,\\ \ddot{y}+2\dot{x}&=\Omega_y,\end{aligned}\tag{1}$$

where dots denote derivatives with respect to the dimensionless time, x and y are the synodic Cartesian rectangular dimensionless coordinates of the third body in a uniformly rotating system (mean motion=1), subscripts denote partial derivatives and

$$\Omega(r_1,r_2)=\tfrac{1}{2}[(1-\mu)r_1^2+\mu r_2^2]+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\tag{2}$$

with

$$\begin{aligned}r_1^2(x,y)&=(x-\mu)^2+y^2,\\ r_2^2(x,y)&=(x-\mu+1)^2+y^2.\end{aligned}\tag{3}$$

The above set of equations (1), (2), and (3) contains only one parameter which is related to the mass ratio of the primaries according to $\mu=m_2/(m_1+m_2)$, $m_2\leqslant m_1$.

The elliptic restricted problem is described by

$$\begin{aligned}\xi''-2\eta'&=\omega_\xi,\\ \eta''+2\xi'&=\omega_\eta,\end{aligned}\tag{1a}$$

where primes denote derivatives with respect to the true anomaly (f) of Jupiter (viz. of the primaries), ξ and η are the synodic Cartesian rectangular dimensionless coordinates of the third body in a nonuniformly rotating system (mean motion$=1$), and

$$\omega(\rho_1,\rho_2)=(1+e\cos f)^{-1}\Omega(\rho_1,\rho_2) \quad (2a)$$

with

$$\rho_1(\xi,\eta)=r_1(\xi,\eta),\quad \rho_2(\xi,\eta)=r_2(\xi,\eta). \quad (3a)$$

The (x,y) dimensionless coordinates in the circular restricted problem are obtained by dividing the dimensional position coordinates (X,Y) by the fixed distance between the primaries. The (ξ,η) coordinates, on the other hand, are obtained by dividing the dimensional position coordinates of the elliptic problem (ξ^*,η^*) by the variable distance between the primaries. Therefore

$$\zeta=\frac{1+e\cos f}{a(1-e^2)}\zeta^*, \quad (4)$$

where $\zeta=\xi+i\eta$, $\zeta^*=\xi^*+i\eta^*$, $i^2=-1$, a is the semimajor axis of the elliptic orbit of Jupiter with respect to the sun, and e is the common eccentricity of the primaries.

The derivation of Eqs. (1) is well known and only for the sake of similarity will it be briefly mentioned. The dimensional differential equations of the circular restricted problem are

$$\frac{d^2X}{d\bar{t}^2}-2n\frac{dY}{d\bar{t}}=\frac{\partial F}{\partial X},\quad \frac{d^2Y}{d\bar{t}^2}+2n\frac{dX}{d\bar{t}}=\frac{\partial F}{\partial Y}, \quad (1c)$$

where X and Y are the dimensional Cartesian rectangular coordinates of the third body, in a uniformly rotating system (mean motion$=n$), $\bar{t}$ is the dimensional time, and

$$F=\tfrac{1}{2}n^2(X^2+Y^2)+k^2\left(\frac{m_1}{R_1}+\frac{m_2}{R_2}\right) \quad (2c)$$

with

$$R_1^2=(X-X_1)^2+Y^2,\quad R_2^2=(X-X_2)^2+Y^2. \quad (3c)$$

In Eq. (2c) m_1 and m_2 are the masses of the primaries; m_1 having the coordinates $(X_1,0)$ with $X_1>0$ and m_2 is located at $(X_2,0)$ with $X_2<0$. The distance between the primaries is $l=X_1-X_2$ and $k^2(m_1+m_2)=n^2l^3$. The origin of the (X,Y) coordinate system is at the center of mass of the primaries.

Introducing $t=n\bar{t}$ for the dimensionless time, $x=X/l$ and $y=Y/l$ for the nondimensional coordinates, $r_1=R_1/l$ and $r_2=R_2/l$ for the dimensionless distances, and finally $\mu=m_2/(m_1+m_2)$ and $1-\mu$ for the nondimensional masses of the primaries, we obtain Eqs. (1) together with the appropriate definitions for r_1, r_2 and Ω.

The derivation of Eqs. (1a) starts again with the dimensional description of the elliptic problem, viz:

$$\frac{d^2\xi^*}{d\bar{t}^2}-2\frac{df}{d\bar{t}}\frac{d\eta^*}{d\bar{t}}=-k^2m_1\frac{\xi^*-\xi_1^*}{P_1^3}-k^2m_2\frac{\xi^*-\xi_2^*}{P_2^3}+\eta^*\frac{d^2f}{d\bar{t}^2}+\xi^*\left(\frac{df}{d\bar{t}}\right)^2,$$

$$\frac{d^2\eta^*}{d\bar{t}^2}+2\frac{df}{d\bar{t}}\frac{d\xi^*}{d\bar{t}}=-k^2m_1\frac{\eta^*}{P_1^3}-k^2m_2\frac{\eta^*}{P_2^3}-\xi^*\frac{d^2f}{d\bar{t}^2}+\eta^*\left(\frac{df}{d\bar{t}}\right)^2, \quad (1d)$$

where ξ^* and η^* are the dimensional coordinates of the third body in the elliptic problem, corresponding to the (X,Y) dimensional coordinates in the circular problem, P_1 and P_2 correspond to R_1 and R_2, and the other symbols have been defined earlier. The origin of the system (ξ^*,η^*) is at the mass center of the primaries which now move on ellipses. The masses are located on the ξ^* axis whose rotation is now not uniform and the ξ_1^*, ξ_2^* abscissas of these masses are not constant. In fact

$$\xi_1^*=\frac{p_1}{1+e\cos f},\quad \xi_2^*=\frac{-p_2}{1+e\cos f}, \quad (5)$$

with

$$\frac{p_1}{p_2}=\frac{a_1}{a_2}=\frac{m_2}{m_1},$$

where a_1 and a_2 are the semimajor axes of the elliptic orbits of m_1 and m_2 described around their center of mass.

The terms on the left side of Eqs. (1d) are the total accelerations and the Coriolis effects. The first and second terms on the right side represent the gravitational forces; the last two terms are the centrifugal force (radial acceleration) and the force occuring because the system is not rotating with a uniform angular velocity (acceleration normal to the radius vector).

The equations of motion for the circular problem (1c), (2c) are obtained from Eq. (1d) by making the appropriate substitutions: $f=n\bar{t}$, $\xi^*=X$, $\eta^*=Y$, etc. The distances between the primaries and the third body are completely similar to R_1 and R_2 and they can be written as

$$P_1^2=(\xi^*-\xi_1^*)^2+\eta^{*2},\quad P_2^2=(\xi^*-\xi_2^*)^2+\eta^{*2}. \quad (3d)$$

Equations (1a) can now be obtained from Eqs. (1d) by using the true anomaly as the independent variable instead of the time $\bar{t}$ and by introducing dimensionless variables. The transformation of the independent variable is given by

$$\frac{d}{d\bar{t}}=\frac{df}{d\bar{t}}\frac{d}{df}.$$

We note that the following four orbits all have the same eccentricity: the orbit of m_1 relative to m_2, the orbit of m_2 relative to m_1, the orbit of m_1 with respect to the center of mass and the orbit of m_2 with respect to the center of mass (cf. Danby 1962). The semimajor axes are, of course, different, in fact $a_1=a\mu$ and $a_2=a(1-\mu)$, where a is the semimajor axis of the ellipse described by the relative motion.

The dimensionless variables are the (ξ^*,η^*) variables divided by the variable distance between the primaries, viz.:

$$\xi=\frac{\xi^*(1+e\cos f)}{a(1-e^2)},\qquad \eta=\frac{\eta^*(1+e\cos f)}{a(1-e^2)}. \tag{6}$$

The masses of the primaries are made dimensionless as in the case of the circular problem and the location of the primaries becomes fixed since we make ξ_1^* and ξ_2^* dimensionless by division by their variable distance, i.e.

$$\xi_1=\frac{\xi_1^*(1+e\cos f)}{a(1-e^2)}=\frac{p_1}{a(1-e^2)}=\frac{a_1}{a}=\mu$$

and similarly

$$\xi_2=\mu-1.$$

With the above remarks Eqs. (1a) can be obtained from Eqs. (1d); nevertheless, previous computation of a few characteristic terms will facilitate matters. The first term of the first of Eqs. (1d) becomes

$$\frac{d}{d\bar{t}}\left(r\frac{d\xi}{d\bar{t}}+\frac{dr}{d\bar{t}}\xi\right)=r\frac{d^2\xi}{d\bar{t}^2}+2\frac{d\xi}{d\bar{t}}\frac{dr}{d\bar{t}}+\frac{d^2r}{d\bar{t}^2}\xi$$

or

$$r\left[\frac{d\xi}{df}\frac{d^2f}{d\bar{t}^2}+\frac{d^2\xi}{df^2}\left(\frac{df}{d\bar{t}}\right)^2\right]+2\frac{dr}{d\bar{t}}\frac{d\xi}{df}\frac{df}{d\bar{t}}+\frac{d^2r}{d\bar{t}^2}\xi,$$

where the variable distance between the primaries is denoted by

$$r=\frac{a(1-e^2)}{1+e\cos f}.$$

Transforming similarly the second term in the same equation (1d) and collecting terms we obtain

$$\left(\frac{d^2\xi}{df^2}-2\frac{d\eta}{df}\right)r\left(\frac{df}{d\bar{t}}\right)^2+\xi\left[\frac{d^2r}{d\bar{t}^2}-r\left(\frac{df}{d\bar{t}}\right)^2\right]$$
$$+\left(\frac{d\xi}{df}-\eta\right)\left(r\frac{d^2f}{d\bar{t}^2}+2\frac{dr}{d\bar{t}}\frac{df}{d\bar{t}}\right)=-k^2m_1\frac{\xi-\xi_1}{r^3\rho_1^3}-k^2m_2\frac{\xi-\xi_2}{r^3\rho_2^3}.$$

Division by $r(df/d\bar{t})^2$ and utilization of the properties of elliptic motion of the primaries, viz.:

$$\frac{d^2r}{d\bar{t}^2}-r\left(\frac{df}{d\bar{t}}\right)^2=-\frac{r^2}{a(1-e^2)}\left(\frac{df}{d\bar{t}}\right)^2,$$

$$r^2\frac{d^2f}{d\bar{t}^2}+2\frac{dr}{d\bar{t}}\frac{df}{d\bar{t}}=0,$$

and

$$\left(r^2\frac{df}{d\bar{t}}\right)^2=a(1-e^2)k^2(m_1+m_2)$$

gives Eqs. (1a), the desired result.

We conclude this part by the remark that the form of the equations of the restricted problem is invariant when the eccentric case is considered. This seems to be a new justification of Birkhoff's idea according to which one of the fundamental problems of celestial mechanics is expressed by equations of the general form

$$\ddot{z}+2i\lambda(x,y)\dot{z}=F_x+iF_y. \tag{7}$$

The invariant property of this equation when regularizing transformations are performed is well known and will be discussed later in this paper.

THE JACOBI INTEGRAL

The circular restricted problem in a rotating coordinate system has the property that its Hamiltonian does not depend explicitly on the time; therefore the problem possesses an integral. This is obtained by multiplying the first of Eqs. (1) by $\dot{x}$, the second by $\dot{y}$ adding the resulting equations, and integrating with respect to the independent (time) variable. The result is known as the Jacobi integral:

$$\dot{x}^2+\dot{y}^2=2\Omega(x,y)-C, \tag{8}$$

where C is the constant of integration.

The same steps when performed with Eqs. (1a), the elliptic problem, result in

$$\xi'^2+\eta'^2=2\int(\omega_\xi d\xi+\omega_\eta d\eta). \tag{9}$$

The integrand is now not a total differential, since ω depends on the independent variable (f) explicitly. In fact

$$d\omega=\omega_\xi d\xi+\omega_\eta d\eta+\omega_f df; \tag{10}$$

therefore the integral can be written as

$$2\int(d\omega-\omega_f df)=2\omega-2\int\omega_f df-C. \tag{11}$$

Since $\omega(\xi,\eta,f)=(1+e\cos f)^{-1}\Omega(\xi,\eta)$, the integral can also be expressed as

$$\frac{2\Omega}{1+e\cos f}-2e\int\frac{\Omega\sin f}{(1+e\cos f)^2}df-C \tag{12}$$

and the form of the Jacobi integral, corresponding to Eq. (8) now becomes

$$\xi'^2+\eta'^2=2\omega(\xi,\eta,f)-C-2e\int_0^f\frac{\Omega\sin f}{(1+e\cos f)^2}df. \tag{13}$$

The left side and the first two terms on the right side show complete analogy to the Jacobi integral of the circular problem with the remark that zero velocity curves—omitting the last term in Eq. (13)—can be constructed for any fixed value of f. At $f=t=0$ the Hill curves of the elliptic problem are identical to the Hill curves of the circular problem, but in general the expression

$$2\omega-C=2\Omega(1+e\cos f)^{-1}-C \tag{14}$$

shows that—still neglecting the last term in Eq. (13)—the zero-velocity curves pulsate. That is, a fixed Hill curve will have a variable value of C attached to it. The variation during one revolution of the primaries is between $(1+e)^{-1}\Omega$ and $(1-e)^{-1}\Omega$ and it amounts to $2e\Omega$ (for small e). Considering the well-known sensitivity of the structure of the Hill curves regarding the Jacobi constant, the above-mentioned variation, even for small eccentricity, is significant. Equation (14) describes what in dynamics is known as the quasi-steady effect, since at every "instant" (i.e. at every value of f, or in other words, at every position of the nonuniformly rotating synodical system) the relation holds and it is meaningful. The true and essential "unsteady" effects appear with the inclusion of the last term in Eq. (13). The interpretation of this term and its evaluation is as follows.

Along an orbit for given f values the ξ and η coordinates of the third body are to be substituted in $\Omega(\xi,\eta)$ and the resulting expression,

$$2e\int_0^f\frac{\Omega(f)\sin f}{(1+e\cos f)^2}df$$

is to be evaluated. An expansion of this expression to the second order in the eccentricity gives

$$2e\int_0^f\Omega(f)\sin f\,df-2e^2\int_0^f\Omega(f)\sin 2f\,df.$$

This shows that the quasi-steady approach, by neglecting the integral in Eq. (13), errs in first order.

For orbits of brief duration, that is, when interest in an orbit is only from some initial point to a point which is reached in a time during which f changes little, the unsteady effect represented by the integral might be a small contribution. This is, of course, not the case if the orbit is near a singularity.

We close with the remark that the above-described situation is similar to the generalized Bernoulli theorem of time-dependent hydrodynamical problems. An integral of the Euler or of the Navier–Stokes equations leads to essentially the same phenomenological questions (cf. Truesdell 1950 and Szebehely 1950).

REGULARIZATION OF THE CIRCULAR PROBLEM

In this section the same procedure is followed as previously; first a short outline will be given of the regularization of the circular problem and then the same steps will be followed for the eccentric case.

Equations (1) can be written as

$$\ddot z+2i\dot z=\text{grad}_z\Omega, \tag{15}$$

where $z=x+iy$ and $\text{grad}_z\Omega=\Omega_x+i\Omega_y$.

Introducing the $z=g(w)$ transformation from the physical plane (z) to the transformed plane (w) and at the same time introducing a time transformation from the physical time (t) to a parameter (τ) which will serve as the independent variable in the (w) plane, we have

$$\dot z=\frac{dg}{dw}\frac{dw}{d\tau}\frac{d\tau}{dt} \tag{16}$$

and

$$\ddot z=\frac{dg}{dw}\frac{dw}{d\tau}\ddot\tau+\left[\frac{dg}{dw}\frac{d^2w}{d\tau^2}+\frac{d^2g}{dw^2}\left(\frac{dw}{d\tau}\right)^2\right]\dot\tau^2. \tag{17}$$

The right-hand side of Eq. (15) transforms into

$$(\overline{dg/dw})^{-1}\,\text{grad}_w\Omega, \tag{18}$$

where bar denotes conjugate, $w=u+iv$ and $\text{grad}_w\Omega=\Omega_u+i\Omega_v$.

Performing the substitutions of Eqs. (16), (17), and (18) into Eq. (15) and rearranging terms we obtain

$$\frac{d^2w}{d\tau^2}+\frac{dw}{d\tau}(\dot\tau)^{-2}(\ddot\tau+2i\dot\tau)$$
$$=-\left(\frac{dw}{d\tau}\right)^2\frac{d^2g}{dw^2}\left(\frac{dg}{dw}\right)^{-1}+\left(\left|\frac{dg}{dw}\right|\dot\tau\right)^{-2}\text{grad}_w\Omega. \tag{19}$$

The Jacobi integral for the circular problem, Eq. (8) in complex notation is

$$|\dot z|^2=2\Omega-C=2U, \tag{20}$$

which, using Eq. (16) transforms into

$$\left|\frac{dw}{d\tau}\right|^2=2U\left(\left|\frac{dg}{dw}\right|\dot{\tau}\right)^{-2}, \tag{21}$$

where $U=\Omega-\frac{1}{2}C$.

The kinematic side of the transformation is

$$d\tau=|dg/dw|^{-2}dt, \tag{22}$$

which gives for $\ddot{\tau}$ occuring in Eq. (19)

$$\frac{d^2\tau}{dt^2}=-\left|\frac{dg}{dw}\right|^{-6}\left(\overline{\frac{dg}{dw}}\frac{d^2g}{dw^2}\frac{dw}{d\tau}+\frac{dg}{dw}\overline{\frac{d^2g}{dw^2}}\overline{\frac{dw}{d\tau}}\right). \tag{23}$$

Substituting (22) and (23) into Eq. (19) gives

$$\frac{d^2w}{d\tau^2}+2i\frac{dw}{d\tau}\left|\frac{dg}{dw}\right|^2=\left|\frac{dg}{dw}\right|^2\text{grad}_wU+\left|\frac{dw}{d\tau}\right|^2\overline{\frac{d^2g}{dw^2}}\left(\overline{\frac{dg}{dw}}\right)^{-1}, \tag{24}$$

while the Jacobi integral becomes

$$\left|\frac{dw}{d\tau}\right|^2=2\left|\frac{dg}{dw}\right|^2U. \tag{25}$$

The right side of the equation of motion (24), by means of the Jacobi integral, can be written as

$$\left|\frac{dg}{dw}\right|^2\text{grad}_wU+2U\overline{\frac{d^2g}{dw^2}}\frac{dg}{dw}=\text{grad}_w\left(U\left|\frac{dg}{dw}\right|^2\right);$$

therefore Eq. (24) becomes

$$\frac{d^2w}{d\tau^2}+2i\frac{dw}{d\tau}|g'|^2=\text{grad}_w(U|g'|^2), \tag{26}$$

where $g'=dg/dw$ is the derivative of the transformation function.

Equation (26) is the regularized representation of the restricted problem provided the proper $g(w)$ transformation is selected.

REGULARIZATION OF THE ELLIPTIC PROBLEM

Equations (1a) describing the elliptic problem can be written as

$$\zeta''+2i\zeta'=\text{grad}_\zeta\omega \tag{27}$$

and the Jacobi integral takes the form [see Eq. (13)]

$$|\zeta'|^2=2\omega-C-2e\int_0^f\frac{\Omega\sin f}{(1+e\cos f)^2}df. \tag{28}$$

This latter equation can also be written as

$$|\zeta'|^2=2(V-Z) \tag{29}$$

with

$$V=\omega-\tfrac{1}{2}C \tag{30}$$

and

$$Z=e\int_0^f\frac{\Omega\sin f}{(1+e\cos f)^2}df. \tag{31}$$

Note that Eq. (27) corresponds to Eq. (15), Eq. (29) to (20) and V to the previously used U function.

The transformation now is from the ζ plane to the w plane according to $\zeta=g(w)$ and the independent variable of the ζ plane (the true anomaly, f) will be transformed into τ as before.

Computation of ζ' and ζ'' is similar to Eqs. (16) and (17), only t is changed into f. For instance,

$$\zeta'=\frac{d\zeta}{df}=\frac{dg}{dw}\frac{dw}{d\tau}\frac{d\tau}{df}.$$

Equation (19) for the elliptic case is obtained without difficulty by writing f in place of t, ω for Ω, and the dots are to be replaced by primes, since instead of $\dot{\tau}=d\tau/dt$, we have $\tau'=d\tau/df$.

The kinematics of the transformation is given, similarly to equation (22), by

$$d\tau=|dg/dw|^{-2}df, \tag{32}$$

which gives

$$\frac{d^2w}{d\tau^2}+2i\frac{dw}{d\tau}\left|\frac{dg}{dw}\right|^2=\left|\frac{dg}{dw}\right|^2\text{grad}_wV+\left|\frac{dw}{d\tau}\right|^2\overline{\frac{d^2g}{dw^2}}\left(\overline{\frac{dg}{dw}}\right)^{-1}, \tag{33}$$

corresponding to Eq. (24).

At this point the derivations for the circular and for the elliptic problems branch since the Jacobi integral is utilized. Equation (29) is transformed into

$$\left|\frac{dw}{d\tau}\right|^2=2\left|\frac{dg}{dw}\right|^2(V-Z), \tag{34}$$

by means of which the right side of Eq. (33) becomes

$$\left|\frac{dg}{dw}\right|^2\text{grad}_wV+2(V-Z)\overline{\frac{d^2g}{dw^2}}\frac{dg}{dw}=\text{grad}_w\left(V\left|\frac{dg}{dw}\right|^2\right)-2Z\overline{\frac{d^2g}{dw^2}}\frac{dg}{dw}.$$

Therefore Eq. (33) can be written as

$$\frac{d^2w}{d\tau^2}+2i\frac{dw}{d\tau}|g'|^2=\text{grad}_w(V|g'|^2)-2Zg'\bar{g}'', \quad (35)$$

where $g'=dg/dw$ as before.

Note that Eq. (35) and the corresponding Eq. (26) for the circular case show the only essential difference in the appearance of the last term in Eq. (35). The V function which replaces U is defined by Eq. (30) as

$$V=\Omega(1+e\cos f)^{-1}-\tfrac{1}{2}C \quad (36)$$

and since $e<1$, the $g(w)$ transformation which regularized Ω and U for the circular case will also regularize ω and V for the elliptic problem.

The explicit form of the last term in Eq. (35) is

$$T=2Zg'\bar{g}''=2eg'\bar{g}''\int_0^f \frac{\sin f}{(1+e\cos f)^2}\Omega df$$

or, by introducing the new independent variable τ by Eq. (32), we obtain

$$T=2g'\bar{g}''e\int_{\tau(0)}^{\tau} \frac{\sin f(\tau)}{[1+e\cos f(\tau)]^2}(\Omega|g'|^2)d\tau. \quad (37)$$

The integral occurring in this expression is convergent since firstly the $\Omega|g'|^2$ term is regular at the singularities because $g(w)$ regularized Eq. (26). Secondly, the $f(\tau)$ relation is given by

$$f(\tau)=\int_{\tau(0)}^{\tau}|g'|^2d\tau;$$

therefore $f(\tau)$ is defined and finite at the singularities. In fact, at the singularities $|g'|=0$ in order to regularize Ω; therefore $g'=0$ and $T=0$ at the collisions.

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