PHYSICAL BASIS OF THE PULSATION THEORY OF VARIABLE STARS[1]

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It is not the purpose of the present paper to criticize the entire literature dealing with the theory of stellar variability (this would require a paper of much greater size) but to give a brief account of the physical content of the theory as it exists up to 1963. At the same time, special emphasis has been placed on those aspects of the problem which are of interest to the author and seem most worthy of attention. Thus, the paper does not pretend at any kind of full coverage of the literature. It should be noted that very useful surveys of many investigations on the theory of stellar variability may be found in papers by Rosseland (1), Ledoux & Walraven (2), Ledoux (3), and Ledoux & Whitney (4).

I. Brief Historical Survey of the Development of the Pulsation Theory of Stellar Variability

The explanation of the variability of stars by their pulsations (free oscillations) was first put forward by Ritter in a series of publications (5); it was also suggested by Umoff in 1896 during the defense of Belopolsky's thesis, in connection with the latter's discovery of the periodic Doppler shift of spectral lines in the atmosphere of δ Cephei.

The discovery of the temperature variations of Cepheids provided a new corroboration of the pulsation hypothesis, and made it impossible to explain the Cepheids as binary systems. Afterwards, in 1913, the pulsation hypothesis was formulated in fully definite form by Plummer, and in 1914 by Shapley.

In 1918–1926 a series of publications by Eddington (6, 7, 8) laid the foundations of the theory of conservative (adiabatic) free radial oscillations of gaseous spheres. In the same papers [especially in (8)], Eddington showed that free oscillations of stars must quickly decay [see also (9, 11)] and therefore that there must be in pulsating stars a continuously operating mechanism that transforms thermal energy into the mechanical energy of pulsation. In order not to violate the second law of thermodynamics, this mechanism must operate on the same principle as all other thermodynamic heat machines. For the case of small oscillations, Eddington (7, 8) derived a suitable formula for the magnitude of the dissipation of mechanical energy of oscillation W, arising as a result of the working of a thermodynamic machine, which transforms heat energy into mechanical energy:

$$W = -\int_M \oint \frac{\delta T}{T}\, dQ\, dm \qquad 1.$$

[1] The survey of literature pertaining to this review was concluded at the end of 1962.

Here $\delta T/T$ is the relative change of temperature at time t in the element of mass dM, and dQ is the quantity of heat put into the element of mass dM in the time dt. The first integral is taken over all of the elements of the "working body," whereas the cyclic integral is taken over the time of the oscillatory cycle.

If the dissipation of mechanical energy $W>0$, then the mechanical oscillations decay; if $W<0$ (the case of negative dissipation), the oscillations are amplified. In the latter case, thermal energy dQ brought into the system is transformed into mechanical energy of oscillation.

It is obvious from Equation 1 that the introduction of heat $dQ>0$ is especially favorable at the moment when the temperature of the system is increasing (e.g., because of its compression), i.e., when $\delta T/T>0$. An ordinary Diesel engine (and also every other thermodynamic engine) operates according to just such a principle. For the correct operation of a Diesel engine, the flash of fuel (and therefore the liberation of energy in the chamber) should occur at the moment of compression of the "working body," when $\delta T/T>0$.

Eddington [see (8, pp. 200–3)] points out two possible mechanisms, applicable to stellar conditions, for the excitation of stellar oscillations from the point of view of the thermodynamic formula Equation 1. The first mechanism is such that the energy generation, for example, by nuclear processes in the center of the star, becomes more intensive during the phase of stellar contraction and the corresponding increase of temperature, but is less intensive during the phase of stellar expansion. Therefore, as a result of the enhanced liberation of nuclear energy ($dQ>0$) during the contraction phase, the gas pressure increases; but during the expansion phase the pressure decreases, which creates a situation similar to that in an ordinary Diesel engine operation.

We must note at once that subsequent work by Cowling (12) [see also (10, 11, 13)] showed that this mechanism is inefficient under stellar conditions. Make the usual assumption that the amplitude distribution of nonadiabatic free oscillations of a real star along its radius is similar to that for the adiabatic free oscillations of the model of this star [nonadiabatic free oscillations, with such an amplitude distribution, which we have labeled "ordinary" (14; see also 15, 16), will really exist]. Then, as a consequence of the effect (well known for adiabatic stellar models) of nonhomology of the oscillation, which consists in an increase of the relative amplitude of free oscillations $\delta r/r$ from the center of the star to the periphery by a factor of from 10 to 10^6 (depending on the degree of concentration of the mass of the star to its center), nuclear reactions become ineffective for the excitation of oscillations. In particular, because of the relatively small amplitude of the central pulsations, the negative dissipation of the energy of the star's oscillation, derived from nuclear reactions occurring at its center, appears to be insufficient to compensate for the positive dissipation that occurs during the transfer of energy in the peripheral portion of the star, which has a considerably larger amplitude of oscillation. Thus, if the rate of nuclear energy generation is determined by the expression $\epsilon = \epsilon_0 \rho T^{\alpha}$, then for the value of nonhomology which is appropriate to stellar models with a likely degree of concentration of mass to the center and an index $\gamma = (d \ln P/d \ln \rho)_{ad} \approx 5/3$, oscillational instability may arise only for values of α of the order of hundreds or even thousands (12), whereas the contemporary theory of thermonuclear reactions in stars gives values for α of not more than several dozens. Further, nonadiabatic free oscillations of a star, which we have called "extraordinary" (14) (because for such oscillations the distribution of the relative amplitude of oscillation $\delta r/r$ along the radius has nothing in common with the distribution that occurs in the case of adiabatic free oscillations), and for which $\delta r/r$ has an especially sharp and great maximum in the center of the star, can also not be excited by nuclear energy reactions (14), just because of the great sharpness of this maximum.

The second possible mechanism, which, according to Eddington, is "fantastic in an ordinary engine but not necessarily so in the star," consists in a special "valving" operation of the medium when the energy flux passes through it. Eddington describes this mechanism in the following way: "Suppose that the cylinder of the engine leaks heat and the leakage is made good by a steady supply of heat. The ordinary method of setting the engine going is to vary the supply of heat, increasing it during compression and diminishing it during expansion. That is the first alternative we considered. But it would come to the same thing if we varied the leak, stopping the leak during compression and increasing it during expansion. To apply this method we must make the star more heat-tight when compressed than when expanded; in other words, the opacity must increase with compression." [See (8, p. 202).]

We must note that the second mechanism as formulated by Eddington also cannot be realized in the star as a mechanism for exciting oscillations in a star. From the above quotation, Eddington evidently visualized the whole star as stopping the leakage of energy in the contraction phrase, because the increase of the opacity coefficient in the entire volume retards the flow of energy. Conversely the opacity would decrease and the leakage increase during expansion. In reality, this effect cannot be realized over the entire star but only in the layer where the critical ionization of He^{+} occurs. Therefore, the appearance of negative dissipation by no means requires that the "opacity must increase with compression," "stopping the leak during compression and increasing it during expansion." We shall illustrate our statement by the following example. For the sake of simplicity, we suppose that the density of the energy flux H is given by $H = H_0 T^{\theta}$ where T is the temperature and θ is some effective index. For illustrative purposes, let us consider that both the temperature and oscillation amplitudes do not vary along the stellar radius. Then the condition for the occurrence of negative dissipation in the layer with thickness dr [i.e., $dQ = (\delta H/\delta r)dr > 0$ during contraction] will be not the negative sign of the effective index θ (as would follow from Eddington's formulation taking into account the temperature increase at the moment of contraction), but the negative sign of the derivative: $\delta\theta/\delta r < 0$ (15).

In this case the onset of negative dissipation of the energy of oscillation of the *whole* star is possible only when the amplitude of oscillation of the layer in which the dissipation is negative is sufficiently great in comparison with the amplitude of the layers in which the dissipation is positive. For a peripheral layer, in which second ionization of He is occurring, this condition is realized if the stellar oscillations are sufficiently nonhomologous, i.e., if the amplitude $\delta r/r$ increases greatly from the center of the star to its periphery. It has been shown (16) that, as the nonhomology of the stellar oscillations increases, the efficiency of Eddington's second mechanism also increases and approaches "saturation" at the degree of nonhomology that corresponds to a central mass concentration for the polytrope with index $n=3$.

Eddington (17) next takes into consideration the zone of critical ionization of hydrogen. He expresses confidence in the existence of a connection between (*a*) the phase shift between oscillations of brightness and radial velocity and (*b*) the period-luminosity relation, and he believes that both depend on the existence of a peripheral zone of partially ionized hydrogen. Yet he rather positively inclines to the opinion that the pulsations of Cepheids are not likely to be caused by the ionization zone but rather "by sub-atomic stimulation," and that "the hypothesis of negative dissipation is not likely to be advocated except as a last resource" (he means here the negative dissipation created by the second or "valve" mechanism). He further states that in general "the steady supply of energy ϵ_0 at high temperature in the interior could be converted into mechanical energy of pulsation if the constitution of the star provided a suitable 'valve mechanism.' But, unless the conditions are widely different from what we suppose, there appears to be no mechanism which would regulate the flow of heat in the way required." The fact that only those stars are pulsating which occupy a definite narrow band in the *H-R* diagram Eddington explains by the fact that "pulsation is associated with a transient decrease of dissipation" (in this case he means a minimum of *positive* dissipation) rather than by a transient increase in the liberation of sub-atomic energy, since the latter "would not give the existing sequence of Cepheid variables, in which the stars of short period have much higher internal temperatures than those of long period."

Moreover, arguing the inevitability of the conclusion concerning the connection between the phase delay and the existence of a critical ionization zone in which the index θ, defined by the ratio $\delta T/T = \theta(\delta\rho/\rho)$, becomes considerably less than unity, Eddington takes the position that the second mechanism (i.e., the "valve mechanism") cannot be realized in a star. He shows that the two assumptions: (*a*) the index θ is constant throughout; and (*b*) maximum radiation occurs a quarter period after "minimum radius," when taken together, inevitably lead to the existence of negative dissipation caused by the "valve mechanism." Hence it follows "by reductio ad absurdum" that one of the two assumptions must be rejected, since according to Eddington the "valve mechanism" cannot be realized in stars. Since (*b*) is based on observation, Eddington concludes that (*a*) must be rejected, i.e., there must be a layer in which the index abruptly falls, and this is the zone of critical ionization.

In 1948, the author (18) pointed out that nuclear reactions are ineffective as the cause of Cepheid oscillations both because of the effect of non-homology and because of the very large amplitudes of the auto-oscillations excited by the energy of nuclear reactions (18, 19). He considered instead that Cepheids are auto-oscillating systems and examined whether stellar auto-oscillations could be maintained by a peripheral zone of critical ionization. Owing to the strongly nonadiabatic character of the oscillations, the hydrogen ionization zone cannot excite the auto-oscillations of giants. Instead, the zone of He^+ critical ionization was considered to be the cause of stellar auto-oscillations (9, 18). In later investigations (15, 16, 20–26, 27–32) it was shown, in contrast to Eddington's assertion about the improbability (or at least the small probability) of the excitation of stellar oscillations by means of a valve mechanism in the critical ionization zone, that if the content of helium in the envelopes of variable stars is about 15 per cent by number of atoms, it is possible to explain both the onset of auto-oscillations and the main features of all basic types of variable stars.

In 1958–1960, Cox and Whitney (33, 36) put forward similar ideas concerning the nature of stellar variability. However, it should be noted that both the method of solution of the equations of nonadiabatic oscillations and a number of devices used in the papers of Cox and Whitney seem to the author to be incorrect (29, 37).

Let us enumerate the observed facts which are now explained by the theory of the maintenance of stellar auto-oscillations by the He^+ ionization zone.

(*a*) The very fact of the existence of variable stars is explained, and the cause of the oscillations is given (9, 20–22, 24, 26).

(*b*) An explanation is given of the phase shifts between the oscillations of brightness and those of stellar radius, characteristic of Cepheids, of long-period variables of the RR Herculis and *o* Ceti types and of short-period variables of the β Canis Majoris type (9, 20, 21, 23–27, 30).

(*c*) The connection between the amplitudes of the luminosity and radius oscillations is explained for the types of variable stars enumerated in (*b*) above (20, 21, 23–27).

(*d*) An interpretation is given of the irregular changes in brightness and of a number of peculiarities which are characteristic of some types of semi-regular variables, namely, for variables of the AF Cygni and RS Cancri types (27).

(*e*) A number of peculiarities of the location of variable stars on the *H-R* diagram are interpreted. In particular, it is explained why variability is peculiar only to supergiants but not, for example, to stars of the main sequence [with the exception of that part of the main sequence which is close to the giant branch, some members of which are known to be short-period Cepheids (25)]. Also explained is why in moving along the "great sequence" from the spectra of type A to those of type M, the first variable stars that appear are Cepheids (i.e., variables from which the brightness maximum comes about one fourth of a period later than the epoch of maximum stellar contraction), followed by semiregular variables and long-period variables of the RR Herculis type (i.e., variables for which maximum brightness coincides with the epoch of maximum stellar contraction) and, finally, long-period variables of the *o* Ceti type (i.e., variables for which maximum brightness comes about one fourth of a period earlier than the epoch of maximum stellar contraction) (21, 24, 27, 30).

(*f*) The order of magnitude of the observed sizes of the amplitudes of the auto-oscillations of variable stars is explained (20, 24, 26).

(*g*) The observed value of the asymmetry of the radial velocity curve is explained (38).

(*h*) For the classical Cepheids (calculations have not yet been carried out for other types of variability) one obtains the period-luminosity relationship with the zero-point according to Baade (22–26, 30) and with a dispersion of about 1^m (28).

(*i*) The masses of the classical Cepheids obtained from the theory are about 1.6 times smaller than they should be according to the period-luminosity relationship for stars of the main sequence (22–24, 30).

II. Statement of the Problem of Stellar Auto-Oscillations

The basic equations[2] describing nonadiabatic stellar oscillations are a mechanical equation:

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{1}{\rho}\,\text{grad}\,P - \mathbf{g} \qquad 2.$$

and a thermal equation:

$$\frac{dS}{dt} = \frac{1}{\rho T}\,\text{div}\,(K\,\text{grad}\,T) + \frac{q}{T} \qquad 3.$$

Here $\mathbf{r}$ is the distance of the element considered from the center of the star; ρ is the density of the element; T is the temperature; P is the total pressure of gas and radiation; $\mathbf{g}$ is the acceleration of gravity; S is the specific entropy; q is the rate of energy liberation per unit mass; K is the coefficient of thermal conductivity (in the case of radiative thermal conductivity

$$K = \frac{4}{3}\,\frac{caT^3}{\kappa\rho}$$

where κ is Rosseland's mean coefficient of absorption). The solution of the nonlinear differential Equations 2 and 3, if it could be obtained, would

[2] The relativistic equations of motion are described by Thomas (39). In view of the smallness of the relativistic corrections, there is no need at present to apply the relativistic equations to the study of variable stars.

answer all questions pertaining to the theory of stellar variability: Does the given stellar model pulsate or not? What is the form or amplitude of the auto-oscillations, etc.? Till now, although the solution of the nonlinear system, Equations 2 and 3, is quite possible with the help of high-speed electronic computing machines, only those equations have been considered which have been obtained from Equations 2 and 3 as a result of linearization (33, 36, 40–45). It must be noted that the consideration of linearized equations is quite sufficient for the explanation of the very important question of whether a system is oscillationally unstable, i.e., whether or not auto-oscillations may be excited.

However, the investigations cited do not give satisfactory consideration to the problem of linear nonadiabatic stellar oscillations. We have no opportunity to criticize these works here and only note that some authors (41, 42, 44, 45) have used incorrect boundary conditions and others (33, 36) have used a method of solution of the equations of nonadiabatic oscillations in which quantities of the same small order as those determined are neglected (29).

Among the solutions of the linearized system obtained from Equations 2 and 3, only those are of physical interest which describe so-called free nonadiabatic oscillations.

The problem of free nonadiabatic stellar oscillations is to obtain those solutions of the linearized system for which the energy of oscillations described by the solutions is finite. This requirement of finiteness of the energy of oscillations, which is due to peculiarities in the equations of the linearized system in the stellar center and in the periphery, turns out to be equivalent to the application of boundary conditions linking the solution and its first derivatives in the center and the periphery (14, 15).

For a star of finite radius, the requirement of the finiteness of the energy of oscillations is equivalent to that of the finiteness of the solution of the linearized Equations 2 and 3 in the center (the latter is also true for a star of infinite radius) and at the periphery (14). For the linearized problem the requirement of a finite solution is equivalent to the application of two boundary conditions in the center and two at the periphery. Thus, all in all, there are four boundary conditions [see e.g. (15)]. Since the system of differential equations resulting from the linearization of systems 2 and 3, and from the introduction of an exponential time-dependence of the form $e^{i\omega t}$ for all quantities entering into these equations, is a complex system of the fourth order, its general solution contains four arbitrary (complex) constants. Since the problem is homogeneous, one of these constants may be arbitrary (it corresponds to arbitrary initial amplitudes and phases of the oscillations). Four boundary conditions are sufficient for the determination of the three other constants and the eigenvalue ω (the complex frequency of oscillations). The boundary problem is considered to be solved if both the eigenvalue numbers ω_i and the eigenfunctions are found (both are complex in general).

It has been found that the boundary problem of linear, nonadiabatic, free stellar oscillations, owing to peculiarities in the equations, may possess two types of solutions, "ordinary" and "extraordinary" (14). (This circumstance distinguishes it, for example, from a problem such as that of the oscillations of a uniform string with fixed or free ends, where there are no "extraordinary" oscillations.) The solutions of the "ordinary" type are quasi-adiabatic in the center of a star and strongly nonadiabatic at the periphery; with increasing degree of adiabaticity of the model (e.g., by diminishing the equilibrium luminosity L_0 of the model to zero) they turn into free oscillations of the corresponding adiabatic model. The solutions of the extraordinary type are strongly nonadiabatic in the center as well as at the periphery: the relative amplitude $\delta r/r$ has a very high and sharp maximum in the center of the star; with increasing degree of adiabaticity of the model, the extraordinary oscillations do not transform into free oscillations of the adiabatic model (14).

It is shown further that only ordinary oscillations may be self-excited in a real star (the excitation mechanism in this case is the peripheral zone of He^+ critical ionization) (14).

In Woltjer's method (40), the solution of the equations of nonadiabatic oscillations is carried out by successive approximations; an adiabatic solution is used as the zeroth approximation. The method essentially assumes the proximity of both adiabatic and nonadiabatic oscillations (we note that such proximity is absent even for ordinary oscillations at the star's periphery); otherwise the rapid convergence may not only disappear, the solution may even diverge.

The present writer (18, 19, 20, 29, 46, 47) has proposed another method unrelated to successive approximations and suitable for any degree of nonadiabaticity of the oscillations (whether linear or nonlinear). The method is based on a discrete treatment, as follows: the star is divided into concentric layers, then a "discrete model" of the star is constructed in which each of the layers is assigned a constant temperature, density, opacity coefficient, etc.

This method makes it possible to examine, with almost equal simplicity, adiabatic as well as nonadiabatic linear and nonlinear stellar oscillations. Whitney & Ledoux (48) considered linear free adiabatic oscillations in order to explore the possibilities of the method. The method has further been applied to the study of the nonadiabatic linear oscillations of a stellar envelope (21–28, 30, 31, 32) and to that of its adiabatic nonlinear oscillations (38). Recently, V. I. Aleshin (unpublished) has employed an electronic computer to calculate the auto-oscillations of a multilayer discrete model maintained by the zone of critical ionization of He^+.

Increasing the number of layers of the discrete model naturally increases the accuracy of the method (just as diminishing the integration step increases the accuracy of ordinary methods of integrating differential equations). It must be emphasized, however, that if the number of the layers is small, high accuracy may be achieved by the so-called "principle of correspondence" (21, 29, 32), which may be described as follows. The selection of the parameters of a discrete model is to some extent arbitrary. For example, the temperature of a given layer in the discrete (static) model may be selected within the limits of the spatial variation of the temperature of the corresponding layer in the continuous model. Therefore, by the selection of these parameters, one may always obtain proximity (preferably as close as possible) of of the quasi-adabiatic linear oscillations of a discrete model to the quasi-adiabatic linear oscillations of the corresponding continuous model (the calculation of the latter by ordinary methods of numerical integration of differential equations is not very difficult, since it amounts to the solution of a boundary problem for a second-order differential equation). Since the aforementioned proximity of the two models (i.e., in both models, the relative amplitude of the linear adiabatic oscillations of the displacement $\delta r/r$, the relative amplitude of the linear quasi-adiabatic oscillations of the energy flow $\delta L/L$, and their changes along the stellar radius are close to one another) was obtained as a result of the selection of the parameters of the discrete model, one ought to expect (this is just "the principle of correspondence") that the nonadiabatic and nonlinear oscillations of both discrete and continuous models will be close to one another).[3]

Although we have no doubt of the correctness of "the principle of correspondence," it has not yet been verified.[4] Therefore the integration of the complete system of nonadiabatic equations by ordinary methods (for example, with the help of an electronic computer) should be welcomed in every possible way, although we do not expect that such an integration can alter the results (14, 30, 31, 32) obtained with the help of four- and five-layer discrete models of a stellar envelope by an amount exceeding the inaccuracy of our calculations, which is caused by inaccurate data on the structure of the envelope. In our view, the most important result to be expected from such an integration will be the verification of "the principle of correspondence" as a basis on which discrete models with a small number of layers (four-five)

[3] It must be noted that establishment of the proximity of the quasi-adiabatic oscillations of the discrete model to those of the continuous model is basically equivalent to the imposition of boundary conditions. If, in the continuous model, the boundary conditions (the regularity of the solution in the center and at the periphery) are satisfied by the choice of a proper value of ω^2, then in the corresponding discrete model, where the regularity is provided by the discreteness itself, and ω^2 is determined to agree with that of the given continuous model, the boundary conditions will be analogous to the requirement of proximity of the quasi-adiabatic linear oscillations of a discrete model to those of a continuous model, attained by the choice of layer widths and temperatures.

[4] The fact that the discrete model reduces to the continuous model when the number of layers increases without limit (more precisely when the thickness of the layers approaches zero) implies that "the principle of correspondence" is fulfilled when the number of layers is sufficiently great. However, its correctness does not depend (or at least depends weakly) on the number of layers, and only this circumstance must be proved.

may be applied to the calculation of nonadiabatic oscillations (30, 31, 32).[5]

It is not superfluous to point out in conclusion the following advantages of the discrete method in comparison with Woltjer's method (40) and with the ordinary methods of integrating differential equations.

(*a*) The convergence of the method is obvious: for a sufficiently large number of layers, the discrete model has quite the same oscillations as those of the continuous model, and the convergence does not depend on the degree of nonadiabaticity or nonlinearity of the models' oscillations.[6]

(*b*) The universality of the method is clear: it is plainly possible (as regards volume of computation) to consider adiabatic, nonadiabatic, nonlinear, and auto-oscillations.

(*c*) The method has high accuracy for a relatively small amount of computation.[7]

It should also be pointed out that there are factors limiting the applicability of the discrete model and that it cannot treat such phenomena as extraordinary oscillations and shock waves, which can only be handled in the continuous model (14).

There is no need here to describe the calculations of nonadiabatic stellar oscillations carried out with the help of the discrete model (21, 30, 31, 32) and we therefore go directly to an account of the physical picture of the origin and main peculiarities of the auto-oscillations of variable stars.

III. Physical Basis of the Theory of Stellar Variability

Calculations by the author in 1953 (unpublished) showed that a helium

[5] Ledoux and Whitney express doubt as to the accuracy of such calculations: "the model and the application of the method itself (for instance the whole critical layer is one of the discrete shells) are still very rough so that the results fail to be completely convincing" (3). In this connection we should again like to draw attention to the fact that the parameters of the discrete model (30, 31, 32) were selected in such a way that its quasi-adiabatic oscillations appeared close to those of the corresponding continuous model. Therefore, if "the principle of correspondence" is accepted as correct, the results of calculation with a four-layer model must have high accuracy [see also (32), where the critical layer is divided into two discrete layers and an attempt is made to estimate the accuracy of the calculations with four- and five-layer models, without appealing to the hypothetical principle of correspondence].

[6] The rapidity of convergence in Woltjer's method (if there is any) is not clear for those cases (real for the star's periphery) when the oscillations are strongly nonadiabatic. Moreover, it is surely incorrect to restrict Woltjer's method to the first approximation (29) as all authors have done up until now.

[7] Of course, any method of numerical integration of the equations of motion of a continuous model is nothing but the application of the discrete treatment. However, such application does not make use of the device that was introduced by physical intuition during the construction of the discrete model (the "principle of correspondence") and therefore, in contrast to the discrete treatment supplemented by the "principle of correspondence," yields good quantitative results only when the integration step is small.

concentration of 7 per cent by numbers of atoms is not sufficient for the excitation of auto-oscillations.

Since the efficiency of the zone of critical ionization of He^+ depends not only on the helium content but also on the structure of the stellar envelope (above all on the value of the ratio m_a/m_z, as discussed below), which has not yet been determined exactly, it is difficult now to estimate the lower limit of helium content in a stellar envelope which (at the optimal value of the parameter y_z, see below) is sufficient for the excitation of auto-oscillations. This lower limit of the helium concentration, which depends on the envelope structure, is apparently between 10 per cent and 15 per cent (by number of atoms).

When the helium content is above 15 per cent, the opacity coefficient of the hydrogen-helium mixture, the specific heat, and other thermodynamic parameters of the He^+ critical ionization zone quickly reach saturation. Therefore an increase of the helium content to more than 15 per cent has little influence on the efficiency of the He^+ critical ionization zone, in exciting oscillations. We note that the efficiency has a rather flat maximum at helium content $\sim$60 per cent.

Be that as it may, the problem of the helium content in the envelopes of variable stars assumes paramount importance. Its quantity in the atmospheres of variable stars has not yet been determined by direct analysis from curves of growth, because of the high ionization potential of helium and the relatively low temperatures of the atmospheres of variable stars. Hence we must judge the helium content in the latter indirectly by that in the atmospheres of the hot stars and the sun, for which it is about 15 per cent (49).

Recently (1957) direct proof has been obtained of the presence of helium in variable-star atmospheres. Kraft found the emission of the helium line λ 5876 behind the shock-wave front in the envelope of the variable W Vir. (50). Assuming that the level populations behind the shock-wave front are appropriate to those in thermodynamic equilibrium, Wallerstein concluded that the ratio of hydrogen to helium by number in W Vir. is between 3 and 10 (50).

Let us now proceed to set forth the physical basis of the theory of pulsational variability.

Calculations (9, 24, 30, 51, 52) have shown that the critical ionization zone of He^+ lies at a depth in the star corresponding to $T=35000°$–$55000°$. For the sake of simplicity, let the star be divided into the region a internal to the zone, the critical ionization zone of He^+ b, and the atmosphere c above the zone (Fig. 1).[8]

Let the star perform radial oscillations at the fundamental frequency.

[8] In the papers (27–32), the critical ionization zone of He^+, i.e., the zone b, was taken conditionally as that zone of the stellar static model in which $\gamma_1 - 1 = (d\ln T/d\ln \rho)_{ad} < 0.26$ (the helium content is 14.3%).

The dissipation of the mechanical energy of oscillations during a period will, for small oscillations, be given by Equation 1:

$$W = -\oint\int_a \frac{\delta T}{T} dQ - \oint\int_b \frac{\delta T}{T} dQ - \oint\int_c \frac{\delta T}{T} dQ \qquad 4.$$

where the cyclic integrals are taken over the cycle of oscillation, and the other integrals over the volumes of the regions a, b, and c; $\delta T/T$ is the relative temperature variation during the oscillations (sinusoidal in time), and dQ is the heat entering the corresponding regions a, b, or c during the time dt.

Using the mean value theorem, we may rewrite Equation 4 as follows:

$$\begin{aligned} W &= -\oint\left(\overline{\frac{\delta T}{T}}\right)_a \int_a dQ - \oint\left(\overline{\frac{\delta T}{T}}\right)_b \int_b dQ - \oint\left(\overline{\frac{\delta T}{T}}\right)_c \int_c dQ \\ &= +\oint\left(\overline{\frac{\delta T}{T}}\right)_a L_a dt - \oint\left(\overline{\frac{\delta T}{T}}\right)_b (L_a - L_b) dt - \oint\left(\overline{\frac{\delta T}{T}}\right)_c (L_b - L_c) dt \end{aligned} \qquad 5.$$

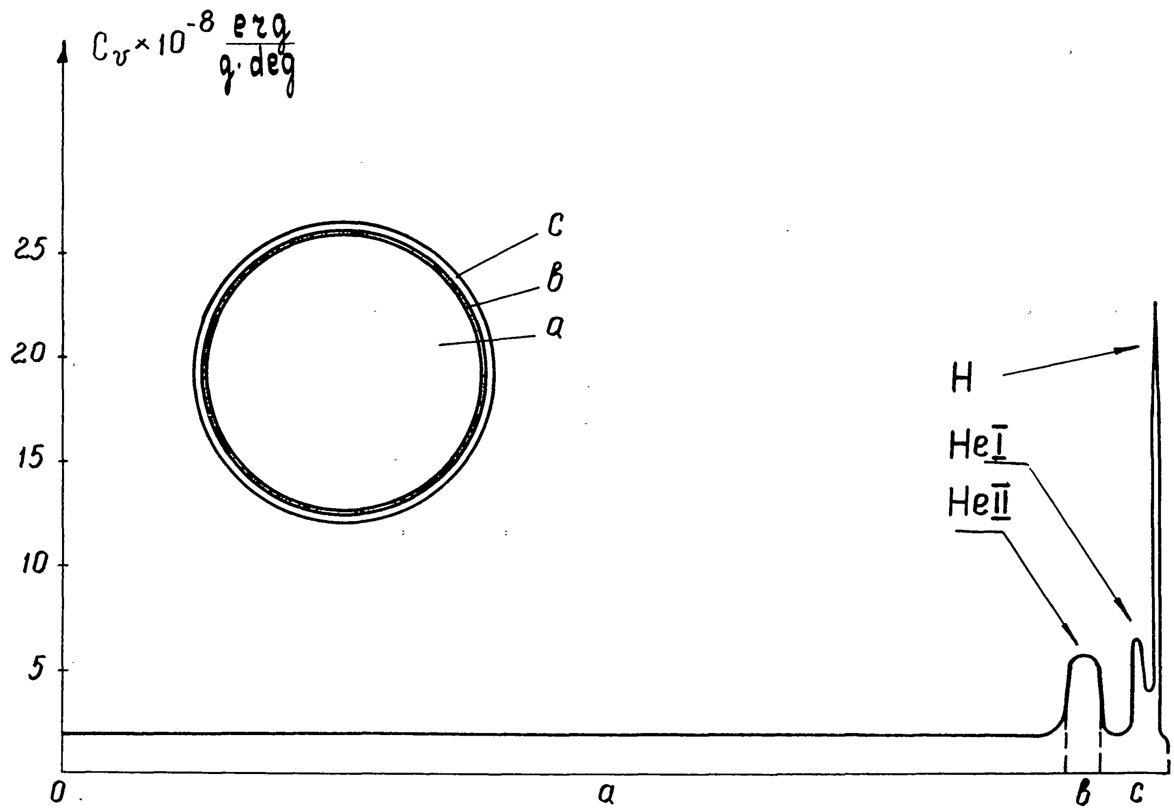

FIG. 1. Stellar regions a, b, c (correct relative scale). At the bottom is represented the variation of the specific heat at the constant volume c_v, and the disposition of critical ionization zones of He^+, He, and H along the stellar radius.

where

$$\left(\frac{\overline{\delta T}}{T}\right)_a; \quad \left(\frac{\overline{\delta T}}{T}\right)_b; \quad \left(\frac{\overline{\delta T}}{T}\right)_c$$

are the values averaged over the regions a, b, c, and L_a, L_b, L_c are the thermal fluxes flowing respectively through the external boundaries of the regions a, b, c.

The first term of Equation 5, representing the dissipation of the oscillation energy W_{int} of the internal part of the star, is (16)

$$W_{\text{int}} = \oint \left(\frac{\overline{\delta T}}{T}\right)_a L_a dt = \frac{1}{2} Z L_0 \left(\frac{\delta T}{T}\right)_{or} \left(\frac{\delta L}{L_0}\right)_a = \frac{1}{2} Z \left(\frac{\delta T}{T}\right)_{or} \delta L_a \qquad 6.$$

where $(\delta T/T)_{0r}$ and δL_a are the amplitudes of the quantities $(\delta T/T)_a$ and L_a at the internal boundary of the zone, and Z is the so-called factor of nonhomology of the stellar oscillations. For a model with a sufficiently high degree of mass concentration to the center (the mass concentration of the polytrope with index $n=3$ is already sufficient), $Z=0.537$ (16) with an accuracy of about 4 per cent.

It is clear from Equation 4 that if negative dissipation is to arise in region b, it is necessary that the ionized zone should absorb heat ($dQ_b>0$) during the compression of the star (and the zone) when $\delta T/T>0$, and lose it ($dQ_b<0$) during the expansion when $\delta T/T<0$.

The occurrence of negative dissipation during oscillations of the ionized zone, with radiative energy transfer,[9] is especially easy to understand from examination of the quasi-adiabatic approximation.[10] Since, in the critical ionization zone of He^+, the quanitity $\gamma_1-1=(d \ln T/d \ln \rho)_{\text{ad}}<0.26$ diminishes to values of $\gamma-1=0.22$–0.35 (9, 24, 30), then, in the course of the stellar oscillations, the temperature T changes insignificantly in comparison with the density, for $(\delta T/T)_{\text{ad}}=(\gamma_1-1)\ (\delta\rho/\rho)_{\text{ad}}$, and the variations of the density

[9] It has been shown (25) that in spite of the convective instability of the critical ionization zone of He^+ and the convection that it causes, in giants and supergiants radiative energy transfer predominates over convective transfer in this zone, and consequently the ionization zone of He^+ in giants and supergiants must be considered to be in radiative equilibrium. On the other hand, convective transfer predominates over radiative transfer in the ionization zones of dwarf stars, on account of the high density of their atmospheres. For this reason, dwarfs and subdwarfs cannot pulsate. [It has also been shown (25) that a zone with convective energy transfer cannot maintain stellar auto-oscillations.]

[10] The quasi-adiabatic approximation is that in which the value of the density of the radiation flux for the oscillation phase under consideration is determined from the stellar temperature and density distribution, which is derived on the assumption that the stellar oscillations are adiabatic.

primarily determine those of the opacity coefficient $\kappa = \kappa_0 \rho^m T^{-s}$ in the critical ionization zone of He^+ [if the content of helium atoms is 15 per cent in the zone of He^+, $m \approx 0.85$, $s \approx 1.0$ (53)]. Therefore, during compression, the opacity coefficient of zone b either increases, or diminishes, if at all, not so much as in region a. During expansion, the coefficient either diminishes, or increases, if at all, more weakly than in region a. Consequently, during star compression the radiation flux L_a entering zone b from the internal part of the star a and defined by the formula

$$L = -\frac{4caT^3}{3\kappa\rho}\frac{\partial T}{\partial r} \quad 7.$$

will either increase, or diminish, if at all, not so much as in region a. On the other hand, during star expansion it will either diminish or increase, if at all, not so much as in region a. Therefore, during the contraction phase, zone b will have to absorb the additional radiation flux emerging from region a, and during the expansion phase it radiates energy more intensively (as compared with the case when there is no zone at all). It follows from the calculations that this valve mechanism is preserved if the oscillations are not too nonadiabatic. As a result, negative dissipation is created by zone b.

It is evident that if negative dissipation arises in region b due to the absorption of the energy issuing from region a, then the loss of this energy must simultaneously lead to the onset of positive dissipation in region a.

Whether or not stellar oscillations are excited depends on which of the two absolute values turns out to be greater (for the present we neglect dissipation in region c as small; we shall see later that this assumption is true in the case of Cepheids and not in the case of long-period variables of the o Ceti type). It might seem from a first glance at Equation 4 that, since the value of the quantity $(\gamma_1 - 1)$ in the ionization zone b is small in comparison with that in the region a (where $\gamma - 1 \approx 2/3$), and consequently the temperature oscillations in the zone are relatively small, then taking into account that $dQ_a = -dQ_b$, we would find that zone b cannot create enough negative dissipation for the excitation of oscillations. As a matter of fact, this is not the case, owing to the effect of nonhomology of the oscillations, which causes a rise in the relative amplitude of the density oscillations $\delta\rho/\rho$ in zone b in comparison with that in region a, and a corresponding increase of $\delta T/T$. The effect of nonhomology is evident from the following estimates which, in contrast to the numerical computations (14, 29–32), are merely illustrative in character.

Assuming that the addition to the energy flux arising in region a during compression is either fully absorbed in zone b during compression (i.e., $L_b = 0$) or is shifted in phase by $\pi/2$ (which is energetically equivalent to complete absorption), and using Equation 6, we may write the following expression for the absolute value of the ratio of dissipation, W_b in zone b and W_a in region a:

$$\left|\frac{W_b}{W_a}\right| = \frac{\overline{\frac{1}{2}\left(\frac{\delta T}{T}\right)_{ob} L_0 \frac{\delta L_a}{L_0}}}{\frac{1}{2} Z\left(\frac{\delta T}{T}\right)_{or} L_0 \frac{\delta L_a}{L_0}} = \frac{\overline{(\gamma_1 - 1)_b \left(\frac{\delta\rho}{\rho}\right)_{ob}}}{\frac{2}{3} Z\left(\frac{\delta\rho}{\rho}\right)_{or}}$$

$$\cong \frac{\overline{(\gamma_1 - 1)}_b \frac{1}{\overline{(\gamma_2)}_b}\overline{\left(\gamma_2 \frac{\delta\rho}{\rho}\right)}_{ob}}{\frac{2}{3} Z \frac{1}{(\gamma_2)_r}\left(\gamma_2 \frac{\delta\rho}{\rho}\right)_r} \cong \frac{\overline{(\gamma_1 - 1)}_b (\gamma_2)_r}{\frac{2}{3} Z \overline{(\gamma_2)}_b}$$

In Equation 8, $\gamma_2 = (d \ln p/d \ln \rho)_{\rm ad}$ and it is assumed with high accuracy that $\gamma_2(\delta\rho/\rho) =$ const. along the stellar radius; more exactly, the relative amplitude of the pressure, $(\delta p/p)_{\rm ad} = \gamma_2(\delta\rho/\rho)_{\rm ad}$, varies along the stellar envelope nearly independently of the value of γ_2 in the envelope; an analogous approach was used by Eddington in the calculation of nonadiabatic oscillations (54).

For the excitation of oscillations, it is necessary that

$$\left|\frac{W_b}{W_a}\right| = \frac{\overline{(\gamma_1 - 1)}_b(\gamma_2)_r}{\frac{2}{3}Z\overline{(\gamma_2)}_b} > 1 \qquad 9.$$

Since, in conformity with conditions in the critical ionization zone of He^+, $\overline{(\gamma_1 - 1)}_b \approx 0.30$; $(\gamma_2)_r = 5/3$; $\overline{(\gamma_2)}_b \approx 1.3$ (9, 24, 29), then for homologous oscillations for which $Z = 1$, Equation 8 gives $W_b/W_a = 0.577 < 1$, i.e., the uniform density model cannot be excited by the critical ionization zone.

It is well known, however, that with increase of mass concentration to the center of the star, the factor Z rapidly diminishes and, within an accuracy of 4 per cent, becomes equal to 0.537 (16) for stars with central mass concentration exceeding the concentration of the polytrope of index $n = 3$, regardless of the internal structure of the star and, to some extent, of the opacity law. Therefore, oscillations of stellar models with sufficiently great central mass concentration may be excited by the critical ionization zone if other favorable conditions, of which we shall speak below, are realized. In such cases we obtain the following estimate, according to Equation 8: $|W_b/W_a| = 0.577/0.537 = 1.07 > 1$.

It is also evident from Equation 9 that when the values of $(\gamma_1 - 1)$ in the ionization zone are very small, the latter becomes ineffective for the excitation of oscillations in any model; this will be particularly true for the hydrogen ionization zone for which the value of the quantity $(\gamma_1 - 1)$ is typically very small.

It is apparent that if the oscillations in zone b are strongly nonadiabatic, then the zone cannot create sufficient negative dissipation, being incapable of any appreciable absorption of the radiation flow. For, in this case, the energy absorbed during the compression of the layer, having caused some nonadiabatic heating of the layer, would diminish the opacity coefficient of this layer: $\kappa=\kappa_0\rho^{0.85}T^{-0.65}$. Consequently, the layer becomes transparent to radiation and "releases" the accumulated (as assumed) energy, thus not creating negative dissipation.

It is necessary to estimate quantitatively the degree of nonadiabaticity of the star layer under consideration, which is capable of maintaining auto-oscillations of the helium zone. As the parameter for a sufficiently precise and rather simple determination of the degree of nonadiabaticity, we introduce the ratio of the nonadiabatic variation of the temperature $(\delta T/T)_{\text{nad}}$, caused by thermal influx to the layer during either its contraction or expansion and calculated on the assumption of quasi-adiabatic, infinitesimally small oscillations of the layer, to the adiabatic temperature variation $(\delta T/T)_{\text{ad}}$, simultaneously arising as a result of contraction or expansion:

$$y_z = \left(\frac{\delta T}{T}\right)_{\text{nad}} : \left(\frac{\delta T}{T}\right)_{\text{ad}} = \frac{\Delta\left(\frac{\delta L}{L_0}\right)_{\text{qsad}} L_0 P}{\pi \Delta M c_v T(\gamma_1 - 1)\left(\frac{\delta\rho}{\rho}\right)_{\text{ad}}} \qquad 10.$$

Here P is the period, ΔM is the mass of the layer, c_v is the specific heat of the layer at constant volume, and $(\gamma_1-1)(\delta\rho/\rho)_{\text{ad}}=(\delta T/T)_{\text{ad}}$.

Two remarks are pertinent relative to the determination of the parameter y_z (10), as suggested in (9). First, the true nonadiabatic temperature change during the oscillation of the star will be smaller than that calculated from the quasi-adiabatic approximation on account of the Le Chatelier-Braun principle. Second, the effective specific heat of the layer differs from that at constant volume c_v, since the layer changes its volume and performs mechanical work. However, these circumstances in no way reduce the importance of the parameter introduced above for the estimation of the nonadiabaticity of the oscillations of the given stellar layer, and in particular for comparison of the degree of nonadiabaticity both of the continuous models and of the discrete models that correspond to them. If the oscillations of the ionization zone b are strongly nonadiabatic ($y_z>2-2.5$), then, as has been explained above, and as calculations based on the discrete model show (21, 30, 31, 32), the zone is incapable of creating sufficient negative dissipation for the excitation of auto-oscillations.

In the other limiting case, when the oscillations of zone b are rather close to the quasi-adiabatic case ($y_z<0.5$), considerable negative dissipation occurs, which, however, is more than compensated by the positive dissipation occurring in the region c lying above the zone and by the positive dissipation in the quasi-adiabatic region a (the latter, in contrast to the dissipation in region c, remains nearly unchanged during the variation of the parameter y_z). The fact is that if, for a star of given structure, the zone of critical ionization of He^+ is located so deep that the oscillations of region c above it are also close to adiabatic, then, since the adiabatic index of the oscillations is

$\gamma_1 = 5/3$,[11] the positive dissipation of the oscillation energy is so great that auto-oscillations are impossible.

It appears from the above that the helium zone may turn out to be effective for the maintenance of stellar auto-oscillations only for a definite degree of nonadiabaticity of its oscillations. Computation shows that the appropriate range of values of the parameter y_z of the He^+ zone is between 0.5 and 2.5.

The latter condition imposes definite limitations on the structure of stars that are capable of auto-oscillations.

It must be noted that besides this condition, one must also fulfill the "rough" condition demanding dominantly radiative transport of energy in the ionized zone (25). For this purpose, the acceleration of gravity in the stellar envelope must be sufficiently small, as observed in giants and supergiants; dwarf stars cannot pulsate (25).

Because the parameter y_z plays a central role in the whole theory—its value determines both the onset and type of pulsational variability—, one must know how this parameter and the fundamental parameters of the star are interrelated (i.e., the mass M, the radius R, the luminosity L, and the period P).

In order to form a general idea of the influence of these parameters upon the parameter y_z, we take as a stellar model a polytrope of index n and assume, for the sake of simplicity, that the degree of ionization of the gas does not depend on the pressure but is determined only by its temperature (14, 24).

As is well known, the temperature and density of polytropic spheres may be found with the aid of the relations:

$$T = T_c u, \qquad \rho = \rho_c u^n$$

where u is Emden's function of index n, and the central temperature and density are given by the formulae:

$$T_c = \frac{\eta G M R'}{R_{\text{gas}}(n+1)M'R} \tag{11}$$

$$\rho_c = \frac{1}{4\pi}\left(\frac{R'}{R}\right)^3 \frac{M}{M'} \tag{12}$$

In Equation 12, G is the gravitational constant, η is the molecular weight, R_{gas} is the gas constant, R' and M' are the radius and mass of the star in Emden's units, R and M are the radius and mass of the star in ordinary units.

Because of the assumption that the ionization is independent of pressure, the temperature of the ionization zone must be the same for all stellar models. Using Equation 11 we obtain

[11] The zones of critical ionization of He and H, for which $\gamma \approx 1$, are located in region c. However, the masses of these zones are considerably smaller than that of the layer with $\gamma_1 \approx 5/3$, which lies in region c between the zones of critical ionization of He^+ and He. Therefore, we may take the effective adiabatic index of the oscillations of region c to be $\gamma_1 \approx 5/3$.

$$T = T_c u \approx \frac{M}{R} u = \text{const.}$$

Thus, we have for the value of u in the zone:

$$u \sim \frac{R}{M} \qquad 13.$$

Taking into consideration the fact that the temperature gradient dT/dr is nearly constant in the stellar envelope, we find from Equations 11 to 13:

$$\frac{dT}{dr} \sim \frac{T_c}{R} \sim \frac{M}{R^2} \sim g \qquad 14.$$

$$\rho \sim \frac{M}{R^3} u^n \sim \frac{M}{R^3} \frac{R^n}{M^n} = M^{1-n} R^{n-3} \qquad 15.$$

Here ρ is the density of the ionization zone.

The thickness of the ionization zone is

$$d \sim \frac{1}{dT/dr} \sim \frac{R^2}{M} \sim \frac{1}{g}$$

From this and from Equation 15 we obtain for the mass ΔM of the ionization zone:

$$\Delta M = 4\pi R^2 \rho d \sim R^2 M^{1-n} R^{n-3} \frac{R^2}{M} = M^{-n} R^{n+1} \qquad 16.$$

Since the period $P \sim 1/\sqrt{M/R^3} = R^{3/2} M^{-1/2}$, then according to Equation 16 we obtain for y_z:

$$y_z \sim \frac{PL_0}{\Delta M} \sim R^{3/2} M^{-1/2} M^n R^{-(n+1)} L_0 = M^{n-1/2} R^{-n+5/2} T^4_{\text{eff}} \qquad 17.$$

Numerical calculations, according to static theory, of the structure of stellar envelopes with a composition of 85 per cent H and 15 per cent He show that the effective polytropic index of Cepheid envelopes is $n \approx 2.5$, whereas for long-period variables $n \approx 1.5$.

It is clear that in the case of oscillational instability, the increase of amplitude will not be unlimited, i.e., a certain finite amplitude of auto-oscillations will be established. Actually, as the amplitude increases, the positive dissipation arising in the internal region a increases approximately in proportion to the square of the amplitude (for oscillations of not very large amplitude), whereas the amount of the negative dissipation created by the zone of double ionization of helium is limited by the finite absorbing capacity of the zone and rapidly approaches "saturation" as the amplitude increases.

On the basis of these considerations, it was shown (22, 24) that the bolometric amplitude of auto-oscillations δL_c of the luminosity of Cepheids (and in general of stars with relatively small dissipation in region c may be estimated from the formula:

$$\delta L_c \approx 0.42 \frac{E_1}{P} \qquad 18.$$

where P is the period and $E_1 = 0.5\ (\chi_1 e \Delta M / m_{He})$ characterizes the energetic capacity of the second ionization zone of helium (χ is the second ionization potential of helium; e is the relative helium content according to mass, ΔM is the mass of the second ionization zone, m_{He} is the mass of the helium atom). We note that Equation 18 determines, so to speak, the maximum amplitude of the auto-oscillations, which is established when the structure of the stellar envelope is optimal for the excitation of auto-oscillations. The true amplitude of the auto-oscillations may be considerably lower or even equal to zero, as a result of the deviation of the envelope parameters (especially y_z) from the optimal values.

An estimate of the auto-oscillation amplitude of δ Cephei according to Equation 18 leads to the value $\delta L_c = 1.9 \cdot 10^{36}$ erg/sec, which is in agreement with the observed value $\delta L_c = 2.15 \cdot 10^{36}$ erg/sec (22, 24).

It is appropriate to say a few words here concerning the problem of asymmetry of the auto-oscillations, in particular, the problem of the asymmetry of the observed radial-velocity curve.

After integration of this curve, it may be noticed that the acceleration of the stellar surface is great near maximum compression and less near maximum expansion. Eddington (7, 8) naturally assumed that this asymmetry in the motion is caused by a variation in the elasticity of the gas, which increases as the star is compressed. However, the attempt to explain the asymmetry of the oscillations by means of a conservative (adiabatic) model of a star with uniform density, performing homologous oscillations, failed, since the observed asymmetry requires that the amplitude of the radial velocity variation be nearly ten times larger than the observed value (55). The same result was obtained in a number of papers (56–60) in which the asymmetry of the adiabatic oscillations in models with different degrees of mass concentration to the center was studied by the so-called Rosseland method (1, 61). However, the latter method is inadequate, since it reduces the problem of the oscillations of a given stellar model to equations which have no relation to the description of the oscillations of this model (19, 29, 38).

The failure of these attempts led some authors to the conclusion that consideration of the nonadiabaticity of the oscillations would probably explain the magnitude of the asymmetry of the oscillations. This is not so, however, because, in the computation of linear oscillations, taking account of the nonadiabaticity makes very little change in the phase and the amplitude of the displacement $\delta r/r$ (21, 30, 32).[12]

Lautman (62) managed to obtain curves of adiabatic oscillations that possess considerable asymmetry. But, in doing so, he solved the problem with incorrect boundary conditions, which exclude the reflected wave in the

[12] One cannot but agree, however, that the degree of nonadiabaticity of oscillations, without determining substantially the magnitude of the asymmetry, might cause a noticeable deviation in the values of the radial velocity obtained for one and the same degree of contraction but for different directions of motion of the stellar surface. [See the phase patterns for δ Cephei and η Aquilae, Fig. 46 of the survey by Ledoux & Whitney (4).]

linear approximation. If the boundary conditions are such that a standing wave is formed, then Lautman does not obtain any asymmetry.

Aleshin (38) was the first to show, through calculation of nonlinear adiabatic oscillations of a discrete five-layer model of a stellar envelope, that for the observed value of the half-amplitude of the oscillations (about five per cent of the stellar radius), the asymmetry will be of the order of that observed only when the nonhomology of the oscillations is sufficiently large [for example, if it is the same as that in modern models of supergiant stars (63)].

To understand why considerable asymmetry may arise only when the nonhomology is sufficiently large, let us consider the following idealized limiting case. Let the oscillations be such that a sphere of radius r, passing through the external layers of the star, is motionless while a sphere of radius $r_2 > r_1$ performs oscillations with amplitude $\delta r_2 / r_2 \approx 0.05$ (in both cases we deal with Lagrangian coordinates). Then, despite the smallness of the amplitude, the degree of compression of the gas between the spheres $r = r_1$ and $r = r_2$, and consequently the asymmetry of the oscillations, may be very large only if $r_2 - r_1$ is sufficiently small (i.e., if the nonhomology of the oscillations is sufficiently large).

Aleshin (unpublished) also carried out calculations of the auto-oscillations of several discrete stellar models. As a result, he obtained not only the asymmetries of the brightness and radial velocity oscillations that are demanded by the observations, but also, for some models, various kinds of humps on the descending or ascending branches of the curve (similar to those observed, for example, in the variable η Aquilae).

Let us proceed now to an account of the physical picture of the formation of phase shifts between the oscillations of brightness and radius of variable stars. The explanation of amplitude-phase interrelations between the oscillations of brightness and radius of variable stars has till now been a stumbling block in the pulsation theory.

As was noted above and substantiated in detail in (25), the transport of energy in the zone of second helium ionization is by radiation and not by convection, i.e., it is determined by Equation 7. By logarithmic differentiation, we obtain

$$\frac{\delta L}{L_0} = -1.85\frac{\delta\rho}{\rho} + 3.65\frac{\delta T}{T} + 2\frac{\delta r}{r} + \frac{\delta(\partial T/\partial r)}{\partial T/\partial r} \qquad 19.$$

where we have assumed that if the chemical composition of the star is 15 per cent He and 85 per cent H (by numbers of atoms), the opacity coefficient of the absorbing part of the He^+ ionization zone may be approximated by the formula $\kappa = \kappa_0 \cdot \rho^{0.85} T^{-0.65}$, according to the computations of Lyast (53).

To understand how the phase lag in the oscillations of the radiation flux arises during its passage through the ionized zone, we examine the oscillations both of the star and of the zone by means of the vector diagram in Figure 2, at first in the quasi-adiabatic approximation.

In conformity with the fact that the adiabatic index defined by the

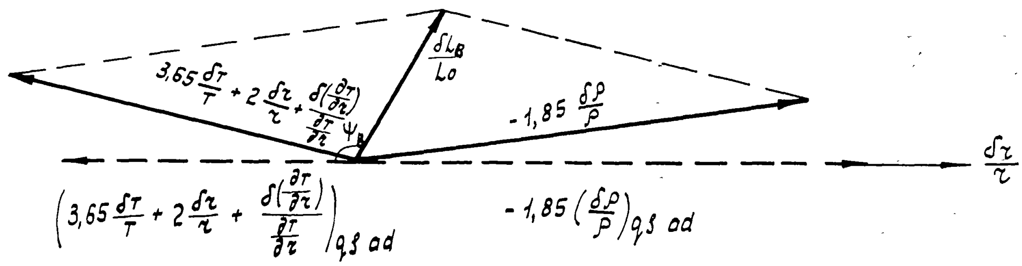

FIG. 2. The vector diagram illustrating the occurrence in the ionized zone (i.e. in region b) of the phase shift $\psi b \gtrsim (\pi/2)$ between the oscillations of the relative displacement $\delta r/r$ and those of the relative change of the radiation flux issuing from the zone $\delta L_b/L_0$.

relationship $(\delta T/T)_{\text{qsad}} = (\gamma_1 - 1)(\delta\rho/\rho)_{\text{qsad}}$ becomes equal to $\gamma_1 = 1.25 - 1.35$, the term $-1.85(\delta\rho/\rho)_{\text{qsad}}$ (which is represented in the diagram by a dotted line, Fig. 2) may exceed in absolute value the sum of the terms $[3.65(\delta T/T) + 2(\delta r/r) + \delta(\partial T/\partial r)/(\partial t/\partial r)]_{\text{qsad}}$ (also represented in the diagram by a dotted line; it must be emphasized that this sum is determined by its first term which considerably exceeds the other two terms).

In taking into account the nonadiabaticity of the oscillations, these vectors change and occupy the positions represented by the continuous lines, as illustrated in Figure 2; the geometric sum of these vectors $\delta L_b/L_0$ represents the vector of the relative oscillations of the flux density issuing from the ionized zone b (the absolute value of the vector represents the relative amplitude of the oscillations in the radiation flux when issuing from the zone, and the angle ψ_z is the phase lag of the epoch of the maximum radiation flux when issuing from the zone, relative to the epoch of maximum contraction of the star). One may see, from the diagram in Figure 2, that the phase shift of the oscillations in the radiation flux, when issuing from the zone, is $\psi_1 \gtrsim \pi/2$, and that the amplitude of the oscillations at this point is several times less than its value before entering the zone, for there

$$\left| 3.65\frac{\delta T}{T} + 2\frac{\delta r}{r} + \frac{\delta(\partial T/\partial r)}{\partial T/\partial r} \right| \gg \left| 1.85\frac{\delta\rho}{\rho} \right|$$

Thus, the ionized zone damps the amplitude of the radiation flux passing through it, making it several times smaller.

The fact that, in the case of nonadiabatic oscillations, the vectors $[3.65(\delta T/T) + 2(\delta r/r) + \delta(\partial T/\partial r)/(\partial T/\partial r)]$ and $1.85(\delta\rho/\rho)$ will occupy just the positions indicated in the diagram in Figure 2 is confirmed both by computations (21, 30, 32) and by the following considerations. As regards the quantity $3.65(\delta T/T) + 2(\delta r/r) + \delta(\partial T/\partial r)/(\partial T/\partial r)$, we set forth the following reasoning. This quantity may be approximated with high accuracy by its first term $3.65(\delta T/T)$. We separate it into its adiabatic and nonadiabatic parts:

$$3.65\frac{\delta T}{T} = 3.65\left(\frac{\delta T}{T}\right)_{\text{ad}} + 3.65\left(\frac{\delta T}{T}\right)_{\text{nad}}$$

where $(\delta T/T)_{\text{nad}}$ is caused by the absorption (or emission) of the radiation flux by the zone. If the displacement of the oscillations of the internal boundary of the zone is expressed by $\delta r/r = A_0 \sin \omega t$, then the time dependence of the oscillations of the radiation flux at the internal boundary of the ionized zone, where the oscillations may be regarded as quasi-adiabatic and where $\gamma_1 \approx 5/3$, will be of the form $\delta L_a/L = -B_0 \sin \omega t$ (i.e., at the moment of maximum stellar contraction, the flux becomes maximal and the phase shift $\psi = 0$). At the external boundary of the zone

$$\frac{\delta L_b}{L_0} = -B_1 \sin(\omega t - \psi_z), \quad \text{where} \quad B_1 \ll B_0 \qquad (A_0, B_0, B_1 > 0)$$

The amount of heat Q absorbed by zone b will be determined by the integral

$$Q = \int [-B_0 L_0 \sin \omega t + B_1 L_0 \sin(\omega t - \psi_z)]dt$$

$$= \frac{B_0 L_0}{\omega} \cos \omega t - \frac{B_1 L_0}{\omega} \cos(\omega t - \psi_z)$$

Since the quantity $3.65(\delta T/T)_{\text{nad}}$ is proportional to the absorbed heat Q (for small oscillations) and is in phase with it, then

$$3.65 \left(\frac{\delta T}{T}\right)_{\text{nad}} \sim B_0 \cos \omega t - B_1 \cos(\omega t - \psi_z)$$

i.e., the vector of the quantity $3.65(\delta T/T)_{\text{nad}}$ in the vector diagram will be directed nearly vertically upwards, since $B_1 \ll B_0$. It follows that in taking into account the nonadiabaticity of the oscillations the vector

$$3.65 \frac{\delta T}{T} = 3.65 \left(\frac{\delta T}{T}\right)_{\text{ad}} + 3.65 \left(\frac{\delta T}{T}\right)_{\text{nad}}$$

and, consequently, the vector

$$3.65 \frac{\delta T}{T} + 2 \frac{\delta r}{r} + \frac{\delta(\partial T/\partial r)}{\partial T/\partial r}$$

will occupy the position shown in Figure 2.

For an increase in the nonadiabaticity of the oscillations (which may be achieved, for instance, by increasing the equilibrium value of the luminosity L_0 in our assumed stellar model) the vector $\delta L/L$ in Figure 2 will turn from left to right and the angle of phase shift ψ_z will diminish from π to 0. It is also evident from Figure 2 that, in the case of a finite degree of nonadiabaticity, the following condition must be realized, for the phase shift $\psi_z \approx \pi/2$ to occur in the ionized zone:

$$\left(\frac{\delta L}{L}\right)_{\text{qsad}} > 0 \quad \text{when} \quad \frac{\delta r}{r} > 0 \qquad 20.$$

In our example this means that

$$\left| \left(-1.85 \frac{\delta \rho}{\rho}\right)_{\text{ad}} \right| > \left| \left(3.65 \frac{\delta T}{T} + 2 \frac{\delta r}{r} + \frac{\delta(\partial T/\partial r)}{\partial T/\partial r}\right)_{\text{ad}} \right|$$

Condition 20 will be sufficient provided that the nonadiabaticity of oscillations y_z is not too great [the greater the inequality (Eq. 20), the larger may be the parameter y_z].

The fact that either a delay or an advance in phase of the energy flux by the angle $\psi_z \approx \pm\pi/2$ can lead to negative dissipation even when there is no absorption of the energy flux oscillations by the zone, i.e., the energy flux passes through the zone without changing its amplitude, may be shown most simply by using the example of homologous (homogeneous) oscillations of the volume of the ionized zone (discrete model). If these oscillations of the volume V follow the law $\delta V = A \cos \omega t$ and the oscillations of the energy fluxes entering and issuing from the zone, which are equal in amplitude but shifted in phase by $\psi_z = \pm\pi/2$ (here and further, the upper sign is for the case of delay, the lower one is for that of advance), are respectively $\delta L_a(\omega t) = -B \cos \omega t$ and

$$\delta L_b(\omega t) = -B \cos\left(\omega t \mp \frac{\pi}{2}\right) = \mp B \sin \omega t$$

then, in the case of quasi-adiabatic oscillations, $(\delta T/T)_{\text{ad}} = -C \cos \omega t$, while in the case of fully nonadiabatic oscillations (i.e., when the temperature variation is caused solely by the absorption or emission of heat):

$$\begin{aligned}\left(\frac{\delta T}{T}\right)_{\text{fnad}} &= D\left[\delta L_a\left(\omega t - \frac{\pi}{2}\right) - \delta L_b\left(\omega t - \frac{\pi}{2}\right)\right] \\ &= D\left[-B\cos\left(\omega t - \frac{\pi}{2}\right) + B\cos\left(\omega t \mp \frac{\pi}{2}\right)\right] \\ &= \begin{cases} DB\sqrt{2}\sin\left(\omega t - \frac{3\pi}{4}\right) & \text{if } \psi_z = +\frac{\pi}{2} \\ DB\sqrt{2}\cos\left(\omega t - \frac{3\pi}{4}\right) & \text{if } \psi_z = -\frac{\pi}{2}\end{cases}\end{aligned} \qquad 21.$$

(A, B, C, D are constants >0). Therefore, for quasi-adiabatic oscillations in the case of both delay and advance, the dissipation of the oscillation energy will be given by

$$\begin{aligned}W_{\text{qsad}} &= -\oint (\delta L_a - \delta L_b)\left(\frac{\delta T}{T}\right)_{\text{qsad}} dt \\ &= \oint\left[-B\cos\omega t + B\cos\left(\omega t \mp \frac{\pi}{2}\right)\right](-C\cos\omega t)dt \\ &= -\frac{1}{2}BC < 0\end{aligned}$$

(i.e., it will be the same as in the case of total absorption when $\delta L_b(\omega t) \equiv 0$), and for fully nonadiabatic oscillations, according to Equation 21.

$$W_{\text{fnad}} = -\oint (\delta L_a - \delta L_b)\left(\frac{\delta T}{T}\right)_{\text{fnad}} dt = 0$$

both for $\psi_z = +\pi/2$ and for $\psi_z = -\pi/2$.

In the real case of nonadiabatic oscillations, the amount of negative dissipation will evidently be intermediate between the values $-\frac{1}{2}BC$ and 0 obtained in both limiting cases.

In connection with statements in the literature that the phase shift $\psi \approx \pm\pi/2$ is responsible for maximum negative dissipation (33, 34), it is desirable to emphasize the two following circumstances:

(*a*) At fixed amplitudes of the energy fluxes, as one may easily ascertain with the aid of analogous, elementary considerations, the maximum of negative dissipation will arise when $\psi_z = \pi$ but not when $\psi_z = \pm\pi/2$.

(*b*) If the amplitude of the energy flux is not constant when it passes through the ionized zone, i.e., assuming the possibility of absorption of the oscillations, by the zone, then considerable negative dissipation will also occur for the phase shift $\psi_z = 0$.

In general, for total (or nearly total) absorption of the oscillations by the zone, i.e., when $\delta L_b(\omega t) \equiv 0$ the amount of negative dissipation created by the zone will not depend on the amount of the phase shift in the oscillations of the energy flux issuing from the zone. Therefore, there may be stars for which $\psi \approx 0$ (see the more detailed consideration below).

We must emphasize that the condition 20 and the condition of a sufficiently small parameter of nonadiabaticity y_z in the zone are necessary and sufficient conditions for the occurrence of the phase shift $\psi_z \gtrsim \pi/2$ in the radiation flux issuing from the ionized zone. The phase and amplitude of the radiation flux on exit from the star will also depend on the value of the parameter of the nonadiabaticity in those stellar layers which are situated above the critical ionization zone of He^+. Calculation shows that if in the given layer $y \gtrsim 2.5$, then the layer is not able to absorb appreciably the radiation flux and consequently to change the phase and amplitude of the radiation flux passing through it (21, 30). This is also clear (23, 26) from the following considerations.

Let us assume that the parameter of nonadiabaticity of the He^+ critical ionization zone is close to the limiting value $y_z \approx 2.5$ when the zone is still able to absorb the additional radiation flux that comes from the internal quasi-adiabatic region *a* of the star during its contraction, and to create a delay in the flux oscillations by the angle $\psi_z \gtrsim \pi/2$. For example, let $y_z = 1.5$. In such a case, the layer external to the He^+ critical ionization zone, being considerably more rarefied than the zone, will be so strongly nonadiabatic ($y \approx 4$) that it will not be able to change substantially the amplitude and phase of the radiation flux oscillations "formed" by the He^+ ionization zone. The radiation flux comes to the stellar surface with a phase shift $\psi_c \approx +\pi/2$ and with an amplitude of oscillation (diminished by a factor of 3–10 as compared with the amplitude of oscillation of the radiation flux entering the zone) as created by the He^+ zone. In conformity with the phase shift $\psi_c \approx +\pi/2$, we obtain in this case a Cepheid-type variable.

We assume now that the parameter y_z of the ionized zone is relatively small ($-0.35 < \log y_z < -0.2$; i.e., $0.45 < y_z < 0.65$, see Fig. 4), which is characteristic of stars with low surface temperature and with small acceleration of gravity (such stars are the long-period variables of the o Ceti type). Then the region c lying above the He^+ critical ionization zone will turn out to possess a relatively small nonadiabaticity ($y_c \approx 1-2$). Therefore it is able to "reform" the radiation flux oscillations that were formed earlier by the ionized zone.

Since in region c the adiabatic index of the oscillations is close to $\gamma_1 = 5/3$,[13] then, if the oscillations are nonadiabatic, this region, in contrast to the He^+ ionized zone, will, during contraction, lose energy by means of radiation (i.e., $(\delta L_c/L)_{\text{qsad}} < 0$ when $\delta r/r > 0$) but will not retard it. In proceeding to the examination of nonadiabatic oscillations, we must take into account the fact that the He^+ ionization zone below region c "intercepts" the increased radiation flux arising in internal region a at contraction, and prevents it from entering region c (if there were no ionization zone, then the loss of energy by region c through radiation at contraction would be compensated by an approximately equal amount of energy issuing from region b of the star, i.e., the oscillations of region c, regardless of the small mass of the matter contained in it, would prove to be close to adiabatic). Consequently, the nonadiabatic oscillations in region c will be such that, during contraction, region c will lose heat but not gain it. Therefore, the nonadiabatic components of the vectors $\delta T/T$ and $\delta\rho/\rho$ in the vector diagram will now be directed downward, not upward as in the case considered earlier (see Fig. 2), and we will obtain the vector diagram shown in Figure 3. It is evident that the oscillations of the radiation flux issuing from the star will pass ahead of the epoch of maxi-

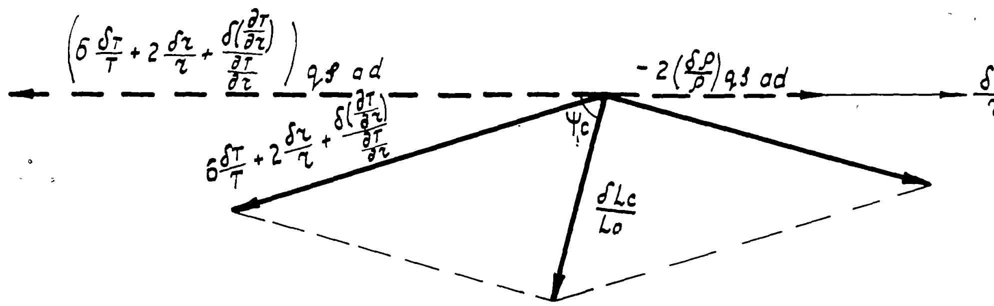

FIG. 3. The vector diagram illustrating the occurrence of the phase advance of the stellar radiation flux in relation to the phase of maximum stellar contraction for the case when the amplitude and phase of the radiation flux are formed in region c (the case of long-period variables of the o Ceti type).

[13] Although the critical ionization zones of He and H are located in region c, this region may be characterized by an effective value of $\gamma_1 = 5/3$. The latter is connected with the fact that the principal absorbing action is manifested by that layer of region c which directly borders the He^+ zone and where $\gamma_1 \approx 5/3$, for this layer is considerably more dense and massive than the critical ionization zones of He and H. These zones are located in such nonadiabatic layers of a giant star that they are incapable of influencing vitally the phase and amplitude of the oscillating radiation flux that passes through them.

mal stellar contraction—ψ_c, the angle of "delay," will be negative: $\psi_c \approx -\pi/2$. We obtain a phase shift between the oscillations in brightness and stellar radius which is characteristic of long-period variables of the o Ceti type.

Thus, in the case of variability of the o Ceti type, the amplitude and phase of the radiation flux are mainly formed in region c, i.e., in the layers lying above the He^+ critical ionization zone, but not in the zone itself, as in the case of Cepheids. In this case, however, the zone takes a very considerable part in the formation of the amplitude and phase of the radiation flux, intercepting the additional radiation flux coming from region a during contraction. In conformity with this, the diagram in Figure 3 pertains to region c above the He^+ zone, whereas the diagram in Figure 2 refers to the He^+ zone itself.

In the range of y_z values intermediate between the intervals satisfying both the case of Cepheids and that of long-period variables of the o Ceti type, we must obtain a continuous transition of phase shifts from $\psi_c \approx -\pi/2$ to $\psi_c \approx +\pi/2$. To the intermediate interval, where $\psi_c \approx 0$, we assign the long-period variables of the RR Herculis type, the short-period variables of the β Can Maj type, and the semiregular variables of the RS Cancri and AF Cygni types (21, 23, 27, 31).

Being guided by similar considerations, one may state straight away that, owing to the effect of nonhomology of oscillations in the star envelope, in the absence of nonadiabatic pulsations by the ionized zone there must arise (as compared with the case of quasi-adiabatic pulsations) an advance in the phase of the brightness curve relative to that of the surface displacement $\delta R/R$ (i.e., $\psi_c < 0$), but not a delay, such as Eddington has obtained (54). Indeed, since the external layers oscillate more strongly than the internal ones, then, during the compression phase, the layer under consideration radiates more energy than it receives from the deeper-lying layers. For this reason, the layer must lose energy during the star's contraction, i.e., we obtain the same diagram (Fig. 3) as in the case of long-period variables of the o Ceti type (with the difference, of course, that the cooling of the layer and correspondingly the phase shift are considerably smaller, namely $\psi_c \approx -1°$), for there is no "interception" of the flux that issues from a region lying deeper than the ionization zone. The corresponding calculation is given in reference (21).

IV. Universal Interpretation of the Different Types of Pulsating Variable Stars

The physical picture of the formation of phase shifts in the envelopes of variable stars given in Section II is confirmed by calculations (21, 30, 32); it is set forth in particular detail in reference (32).

Figure 2 of (30), which is reproduced in Figure 4, represents the results of calculations of nonadiabatic oscillations of three particular four-layer spherical models of an envelope with a chemical composition consisting of

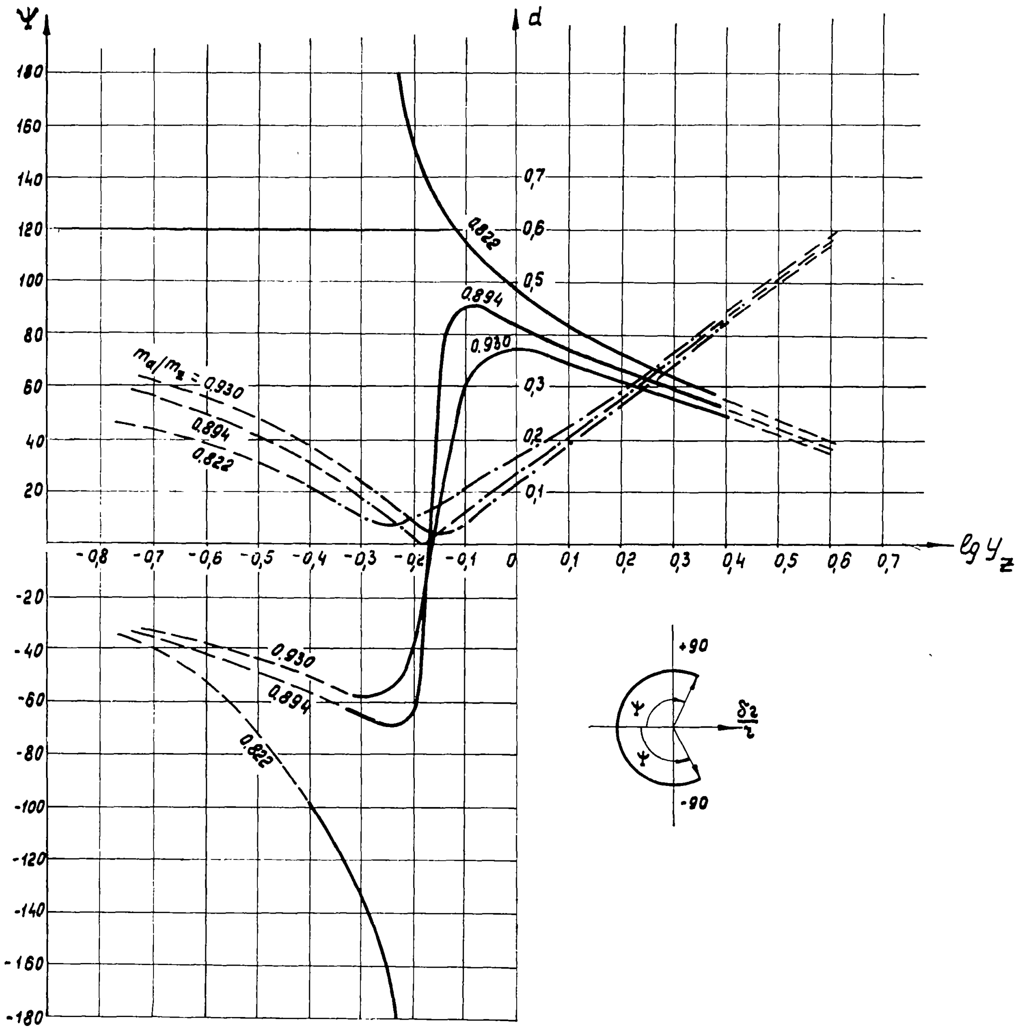

FIG. 4. The phase shift ψ (curves———) between the oscillations of the radiation flux issuing from the star, and the stellar radius $(-\delta r/r)$ and the value d (the curves—·—) of the ratio of the amplitude of the radiation flux issuing from the star to that of the radiation flux entering the zone, as a function of the logarithm of parameter of nonadiabaticity of the oscillations of the He^+ critical ionization zone. The dashes (– – –) denote those parts of the curves for which dissipation of the total energy of stellar oscillations is positive and which therefore cannot be realized. The curves are plotted for different values of the ratio m_a/m_z. Here m_a is the mass of region c (the atmosphere) and m_z is the mass of the He^+ critical ionization zone.

85.7 per cent hydrogen and 14.3 per cent helium. As our calculations have shown, the phase-amplitude relations of the oscillations of brightness and stellar radius and also the degree of pulsational instability are, for a given chemical composition, chiefly determined by the value of the parameter y_z and the value of the ratio m_a/m_z, where m_a is the mass of the atmosphere (i.e., the mass of region c) and m_z is the mass of the zone (i.e., the mass of region b), and are rather slightly dependent on the remaining parameters of the envelope (30). Consequently, the diagrams in Figure 4 are universal, i.e., they are also applicable to models of other envelopes of the same chemical composition.

In Figure 4, the continuous lines represent the phase shift $\psi \equiv \psi_c$ between the epoch of maximum brightness and that of minimum stellar radius[14] as a function of the common logarithm of the parameter y_z. The point-dash line represents the dependence on log y_z of the ratio d of the amplitude of the radiation flux emerging from the star to that of the radiation flux entering the He^+ critical ionization zone (we recall that the He^+ zone is taken to be the layer in which $\gamma_1 - 1 = (d\ln T/d\ln\rho)_{ad} < 0.26$). The quantity d characterizes the damping action of the He^+ critical ionization zone upon the amplitude of the radiation flux passing through the stellar envelope. The dash-dash line represents those parts of the curves for which the total dissipation of the energy of stellar oscillations is positive (see Fig. 5) and which therefore have no physical meaning. The curves are plotted for three values of $m_a/m_z = 0.822$, 0.894, 0.930.

It will not be amiss to explain that in reality the two points on the phase shift curve for $m_a/m_z = 0.822$ (see Fig. 4), for which $\psi = +180°$ and $\psi = -180°$, are in fact only one, i.e., the two branches of this curve must be regarded as joined at the point log $y_z = -0.23$. In other words, to obtain a single-valued

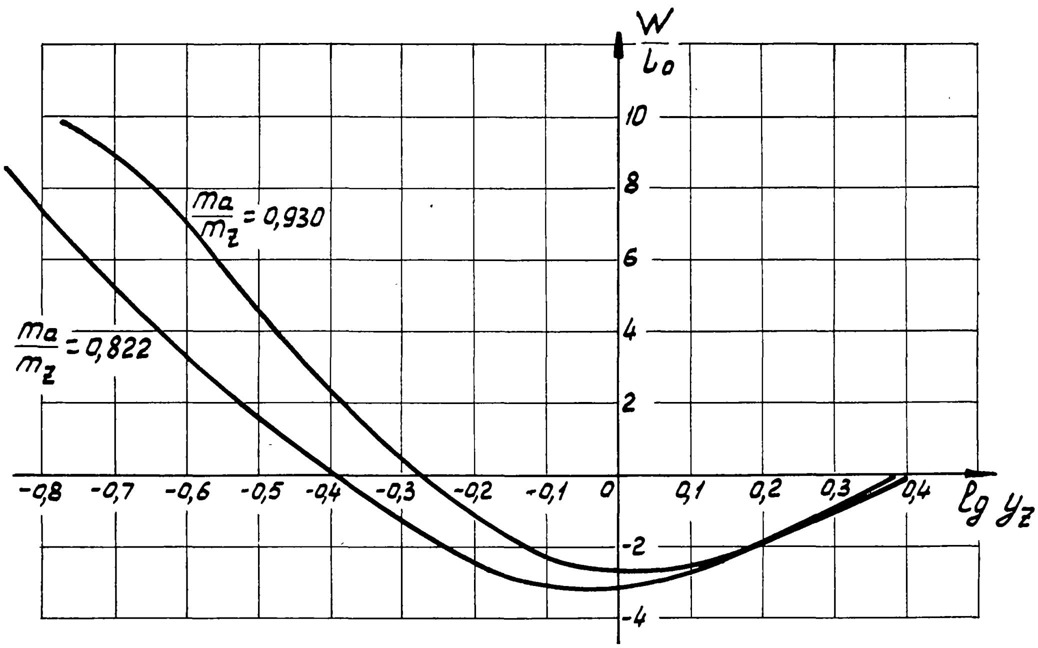

FIG. 5. The reduced dissipation W/L_0 of the total energy of stellar oscillations as a function of the parameter of nonadiabaticity of the oscillations of the He^+ critical ionization zone, when $m_a/m_z = 0.822$; 0.930.

[14] For Cepheids $\psi \approx 90°$; for long-period variables of the o Ceti type $\psi = -90°$. The scheme of reading angles with respect to the oscillations of the stellar radius $\delta r/r$ is represented in the lower right corner of Fig. 4.

relation the curves of Figure 2 should be represented not on a plane, but on a cylinder, the magnitude of log y_z being plotted along the generatrix of the cylinder and the magnitude of the cyclic coordinate ψ along the circumference.

In Figure 5, the dissipation of the total energy of the stellar oscillations W/L_0, in luminosity units, is plotted against values of the logarithm of y_z obtained with the aid of calculations on the four-layer discrete model (30). The diagrams of Figure 5 are universal in the same sense as those of Figure 4.

As we now see, it follows from the diagrams in Figure 4 that, by using different values of the two parameters m_a/m_z and y_z of the same model, we can obtain all of the main types of pulsating variable stars, differing from one another by the phase relationships between the oscillations of brightness and of radius. As the parameters of the envelope—the acceleration of gravity g and the effective surface temperature T_{eff}—are varied (for fixed chemical composition), the parameter m_a/m_z remains almost constant. Therefore the parameter y_z is revealed as the principal one determining the appearance of this or that type of pulsational variability (30).

As we proceed along the spectral sequence from A-type stars to M-type stars, we find that the gravitational acceleration in the atmospheres of giants and supergiants diminishes, and hence, according to the theory of stellar atmospheres and also from Equation 17 (in which for Cepheids $n \approx 2$, and for long-period variables $n \approx 1.5$), the He^+ critical ionization zone must lie in the deeper, denser layers of the star. Hence, the parameter y_z must diminish and travel from right to left in Figure 4 (in the direction from early to late spectra). As a result, we obtain the following sequence of different types of stellar behavior (27, 30).

(*a*) log $y_z \gtrsim 0.4$. The critical ionization zone of He^+ is extremely close to the stellar surface; on account of its great nonadiabaticity, it cannot create the negative dissipation necessary for excitation of stellar auto-oscillations. The star does not pulsate.

(*b*) $-0.1 \lesssim \log y_z \lesssim 0.4$. The critical ionization zone of He^+ is located in denser layers of the star, while the oscillations of region c (the atmosphere) above it are strongly nonadiabatic. Oscillations occur with a phase shift $\psi \approx +90°$, which is typical of Cepheids and variables of the RV Tauri type.

In this case the phase shift predicted by the theory for Cepheids is not exactly equal to $+90°$, but may occur, depending on the parameters of the stellar envelope (see Fig. 4), between the limits $50° < \psi < 120°$. Just such limits in the values of ψ are obtained from observations of Cepheids.

(*c*) $-0.25 \lesssim \log y_z \lesssim 0.1$. Two cases are possible here:

(*i*) The amplitude and the phase of the radiation flux issuing from the star are relatively stable. This case is realized if $m_a/m_z > (m_a/m_z)_{\text{cr}}$[15] and

[15] Here $(m_a/m_z)_{\text{cr}}$ is the critical value when the transition from positive to negative values of ψ is most rapid. The critical value increases with increasing helium content. For a four-layer model with 14.3% helium $(m_a/m_z)_{\text{cr}} = 0.894$ (Fig. 4).

fluctuations in the variation of the parameter log y_z (the origin of which is not considered here) are essentially less than the width of the interval Δ log y_z, where the transition from $\psi \approx +90°$ to $\psi \approx -90°$ takes place. As the width of this interval, for the given chemical composition of 14.3 per cent He and 85.7 per cent H, is minimal when $m_a/m_z \leq (m_a/m_z)_{cr} = 0.894$ (it is then Δ log $y_z = 0.1$, see Fig. 4), case (*i*) is likely to arise in those stellar envelopes in which m_a/m_z is appreciably greater than 0.894 (for example, when $m_a/m_z = 0.930$, then Δ log $y_z \approx 0.2$, see Fig. 4). According to Figure 4, the phase shift will be comparatively stable and close to zero. We obtain the phase relations between the oscillations of the brightness and radius that are observed in long-period variables of the RR Herculis (64) and β Can Maj types. Maximum brightness occurs at the time of maximum contraction of the star.

(*ii*) The amplitude and phase of the radiation flux are unstable. This case occurs only when fluctuations in the variation of the parameter log y_z are of the same order or greater than the width of the interval Δ log y_z in which the phase shift changes from $\psi \approx +90°$ to $\psi \approx -90°$. Fulfillment of this case is likely in those stellar envelopes where Δ log y_z is minimal (this will occur for $m_a/m_z = (m_a/m_z)_{cr} = 0.894$, when Δ log $y_z = 0.1$, see Fig. 4). Then, stars for which log y_z falls in the interval $-0.25 \lesssim$ log $y_z \lesssim -0.1$ should exhibit variations in the phase shift of the radiation flux which are close to random, approximately from $-90°$ to $+90°$, and variations in the amplitude of the radiation flux by a factor of 3–4 (see Fig. 4), i.e., such stars will be subjected to irregular variations in brightness. In this case the oscillations must take on a "semiregular" character which means that although the time between two brightness maxima (the "period" of oscillation) may change greatly from one cycle to another, yet it always fluctuates about a certain mean value, which coincides with the period of oscillation of the inner region, and which is surely constant to a high degree of accuracy. To this interpretation of "semiregular" stellar variability, we assign the "semiregular" variables of the AF Cygni and RS Cancri types (30).

(*d*) $-0.35 \lesssim$ log $y_z \lesssim -0.25$. The critical ionization zone is located deep under the stellar surface. Oscillations occur with a phase shift $\psi \approx -90°$ (see Fig. 4), which is typical for variables of the *o* Ceti type.

(*e*) log $y_z \lesssim -0.35$. The He^+ critical ionization zone is situated so deep beneath the stellar surface that, because the oscillations of the atmosphere are to a high degree adiabatic, the envelope cannot create sufficient negative dissipation for the excitation of stellar auto-oscillations.

V. Comparison of the Theory with the Observations

It follows from Figure 5 that, although the degree of pulsational instability decreases as the ratio m_a/m_z increases, it depends on this parameter rather slightly.

On the other hand, the value of the phase shift is essentially determined by the parameter m_a/m_z. Thus, in the case of Cepheids, when the helium content is 14.3 per cent and $m_a/m_z > (m_a/m_z)_{cr} = 0.894$, the phase shift ψ is always less than $+90°$, whereas Cepheids do exist with phase shifts ψ greater than 90° and as high as 120°.

Hence it follows from Figure 4 that for such Cepheids either m_a/m_z is smaller than 0.894, or the helium content is much higher than 14.3 per cent.

It is obvious that the entire universal interpretation of different types of pulsational variability set forth in Section III (provided that all variable stars have the same helium content) is based on the fact that the ratio m_a/m_z may be smaller or greater than $(m_a/m_z)_{cr}$.

In this connection it should be noted that until recently the pulsation theory of stellar variability encountered the difficulty that our calculations of the envelopes of classical Cepheids always led to the value $(m_a/m_z) \approx$ I, I-I, 2 (unpublished), which is considerably greater than the value $(m_a/m_z)_{cr} = 0.894$ obtained for a helium content of 14.3 per cent from calculation of the nonadiabatic oscillations of the envelope with the help of a four-layer discrete model (30). This difficulty has been removed by more exact calculations of the nonadiabatic oscillations of the envelope, which have been carried out by V. I. Aleshin on the basis of a ten-layer discrete model of the envelope, and a helium content of 14.3 per cent. The resulting new value of $(m_a/m_z)_{cr}$ is about 1.1.

It would be highly desirable to carry out envelope calculations for a number of specific variable stars in order to ascertain their values of m_a/m_z. It goes without saying that, in making these calculations, one must not use a static theory of the envelope but should take into account, along with the turbulent and radiation pressure, the diminution of the effective acceleration of gravitation caused by the transfer of mechanical momentum by shock waves (65, 66), and the increase in the size of the zone of critical ionization of He^+ because of its erosion by turbulent convective currents. It should be noted that the ordinary static theory of atmospheres gives $m_a/m_z = 1.1$–1.2.

It is evident from Figure 4 that the theory gives the correct relationships between the amplitudes of the brightness and radial-velocity oscillations for all of the above-mentioned types of variable stars (30). To make certain of this, we take as an example the case of δ Cephei.

Let the mass of δ Cephei be $M = 6.75\odot$. The bolometric amplitude is $\Delta m_b = 0.58$ (67), which corresponds to a relative amplitude of the luminosity oscillations $\delta L/L_0 = 0.25$ (because $0^m .58/2.5 = \log [(1+0.25)/(1-0.25)]$. From integration of the radial-velocity curve, we have $\delta R = 1.84 \cdot 10^6$km (68), which, with $R_{\delta \mathrm{Cep}} = 3.36 \cdot 10^{12}$, gives $\delta R/R = 0.055$. When the luminosity is $L_0 = 8.6 \cdot 10^{36}$ erg/sec (30) we have $\log y_z = -0.022$ according to the relation $y_z = 1.104 \cdot 10^{-37} L_0$, which is valid for $M = 6.75\odot$ (30). From the curve of Fig. 4 with $m_a/m_z = 0.894$, we find that for $\log y_z = -0.022$ (as it must be in δ Cephei), the phase shift $\psi = +90°$ and $d \approx 0.16$. Since on the surface when $\delta R/R = 1$, $(\delta L/L_0)_R = 8.4 \times 0.16 = 1.34$ [8.4 is the value of the relative amplitude of the flux oscillations $\delta L/L$ upon entry into the He^+ critical ionization zone, see (29)], then we have, for $\delta R/R=0.055$, $(\delta L/L_0)_R=1.34\times 0.055 =0.07$. This value, as it should be, is smaller than the observed one $\delta L/L_0 =0.25$.

The fact is that, in the linear approximation which we use, the damping action of the zone upon the amplitude of the radiation flux will always be greater than it is in the presence of auto-oscillations. The auto-oscillations are established just because, as the amplitude of oscillation increases, the damping action of the zone upon the amplitude of the radiation flux (and also on its phase-shifting action) diminishes and, as a consequence, the ratio of the quantity of negative dissipation created by the zone to that of positive dissipation produced by region a will also diminish. Therefore, in the regime of auto-oscillations, the quantity $(\delta L/L_0)_R$ turns out to be greater than the value $(\delta L/L_0)_R=0.07$ obtained in the linear approximation, and will probably be close to the observed one.

If we had neglected the damping action of the zone upon the amplitude of the radiation flux, the latter would be three to six times larger than the observed value (21).

The fact that, for a mass of 6.75 $\odot$, a radius of $3.36\cdot 10^{12}$ cm, and a luminosity of $8.6\cdot 10^{36}$ erg/sec, which according to the period-luminosity relation with Baade's zero-point corresponds to the period of δ Cephei, $P=5^d.37$, we obtain for our model of δ Cephei pulsational instability and a phase shift $\psi\approx 90°$, typical of Cepheids, means that for classical Cepheids (to which δ Cephei belongs) the theory leads to the correct value of the zero-point of the period-density relation $P\sqrt{\bar{\rho}/\bar{\rho}_\odot}=0.041$ (63) (i.e., masses which are $\sim$1.6 times smaller than those from the mass-luminosity relation for stars of the main sequence).

The inclination of the curve of the period-luminosity relation may be obtained from the condition $y_z\sim PL_0/\Delta M=\text{const}$. Eliminating ΔM by means of Equation 16 and making use of luminosity relation $L_0\sim M^m$ and the period-density relation $P\sim\sqrt{R^3}/M$, we find

$$P \sim L_0^{1/m+3/(2n-1)} \qquad 22.$$

The theoretical period-luminosity relation, Equation 22, satisfies the observations (69) if, with $m=3$, we take the polytropic index of the Cepheid envelope to be $n=3$, and that for the long-period variables to be $n=2.5$.[16]

From the circumstance that the interval $\Delta \log y_z\approx 0.5$ (see Fig. 4) satisfies the variables of the Cepheid type, it follows that, for a given value of the period, the internal dispersion of Cepheid luminosity is about 1^m, the bluer variables corresponding to higher luminosities; the Cepheids of class C (according to the Eggen classification) have a higher luminosity than those of the same period in classes A and B (28).

[16] The values of n obtained are close to the real ones: $n\approx 2-2.5$ (Cepheids) and $n\approx 1.5$ (long-period variables). Taking into account the particularly tentative nature of Equation 22, we may conclude, without more detailed calculations, that the theoretical period-luminosity relation at least does not contradict the observations.

The above agrees with the results obtained by Arp and Sandage from observation (70, 71).

The theory also describes rather well the short-period variables of the RR Lyrae type. From calculations on the envelope of RR Lyrae, using a radius $R=4.57\cdot10^{11}$ cm and an absolute bolometric magnitude $M_b=-0.04$ ($L_0=2.82\cdot10^{35}$ erg/sec), Yang Hai Shou showed that when the mass is within the limits $0.5M\odot<M<1M\odot$ (72), the star becomes pulsationally unstable with a phase shift $\psi\approx+90°$.

Shortness of space does not permit us to dwell on a number of corroborations of the theory of maintenance of stellar auto-oscillations by means of the He^+ critical ionization zone, which were expounded in reference (30).

In conclusion we should like to state that although much work remains to be done to clear up the applicability of the theory to particular types of variable stars and to explain a number of effects (such as, for example, Blazhko's effect for short-period variables), we believe that the theory of stellar pulsational variability will scarcely undergo substantial changes in the future.

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