APPENDIX

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In the foregoing paper Sandage obtains the change of redshift of a given source over a large finite interval of time. He confines his investigation to zero-pressure models of the universe in which the cosmic constant λ is zero. This appendix is concerned with the instantaneous rate of change of redshift in any zero-pressure model of the universe with an arbitrary cosmical constant. A formula is also given for the instantaneous rate of change of the apparent bolometric magnitude of a source of radiation. The formula ([18A] below) supplements the entries under m in Sandage's Tables 1 and 2 in which the changes of apparent magnitude over finite time intervals are listed for two classes of model universes. One result of this calculation is that the apparent brightness of a very distant source may *increase* with time. Corresponding properties are shown to hold for the flux density of a radio source.

The instant t_0 is the moment at which the observer at the origin $r = 0$ makes his observations. The Hubble constant H_0 and the deceleration parameter q_0 are again defined to mean

$$H_0 = \frac{\dot{R}_0}{R_0}, \qquad q_0 = -\frac{(\ddot{R}_0/R_0)}{(\dot{R}_0/R_0)^2}, \tag{1A}$$

where a dot denotes a derivative with respect to t and suffix zero means the value of the function at time t_0. A source of radiation has, by definition, a fixed r co-ordinate and therefore by equation (4) a fixed value of u also.

Consider the radiation from the source that arrives at times t_0 and $t_0 + \Delta t_0$, having left the source at times t and $t + \Delta t$. The redshifts observed at $r = 0$ are, respectively,

$$\begin{aligned} z &= \frac{R_0}{R} - 1, \\ z + \Delta z &= \frac{R(t_0 + \Delta t_0)}{R(t + \Delta t)} - 1. \end{aligned} \tag{2A}$$

We therefore have

$$\begin{aligned} \Delta z &= \frac{R_0 + \dot{R}_0 \Delta t_0}{R + \dot{R}\Delta t} - \frac{R_0}{R} \\ &= \frac{R_0}{R}\left(\frac{\dot{R}_0}{R_0} - \frac{\dot{R}}{R_0}\frac{\Delta t}{\Delta t_0}\right)\Delta t_0. \end{aligned} \tag{3A}$$

But now the constancy of u in equation (3) shows that

$$\frac{\Delta t}{\Delta t_0} = \frac{R}{R_0}.$$

Hence, proceeding to the limit $\Delta t_0 \rightarrow 0$ in (3A), we have, with the aid of (1A) and (2A)

$$\begin{aligned} \left(\frac{dz}{dt}\right)_0 &= -\frac{R_0}{R}\left(\frac{\dot{R}}{R_0} - \frac{\dot{R}_0}{R_0}\right) \\ &= -H_0(1+z)\left(\frac{\dot{R}}{H_0 R_0} - 1\right). \end{aligned} \tag{4A}$$

Now consider zero-pressure models in which the density ρ and pressure p ($= 0$) are given by

$$8\pi G\rho = 3\left(\frac{\dot{R}}{R}\right)^2 + 3\frac{kc^2}{R^2} - \lambda, \tag{5A}$$

$$0 = -\frac{2\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 - \frac{kc^2}{R^2} + \lambda, \tag{6A}$$

where λ is the cosmical constant. Write

$$\sigma_0 = \frac{4\pi G \rho_0}{H_0^2}, \tag{7A}$$

and use (1A) for q_0. Then at time t_0, (5A) and (6A) can be solved for λ and for kc^2/R_0^2 to give

$$\begin{aligned} \lambda &= -(3q_0 - \sigma_0) H_0^2, \\ \frac{kc^2}{R_0^2} &= -(q_0 + 1 - \sigma_0) H_0^2. \end{aligned} \tag{8A}$$

It is also known that (5A) and (6A) can be combined into one equation, viz.,

$$8\pi G \rho_0 \left(\frac{R_0}{R}\right)^3 = 3\left(\frac{\dot{R}}{R}\right)^2 + \frac{3kc^2}{R^2} - \lambda, \tag{9A}$$

where ρ_0 is the present average density of matter Introduce the dimensionless variables

$$Y = \frac{R}{R_0}, \qquad X = H_0 t, \tag{10A}$$

and then (9A) is

$$Y\left(\frac{dY}{dX}\right)^2 = \tfrac{2}{3}\sigma_0 + (q_0 + 1 - \sigma_0) Y - (q_0 - \tfrac{1}{3}\sigma_0) Y^3. \tag{11A}$$

In this equation $Y = (1+z)^{-1}$ so long as $Y \leq 1$, which means that $t \leq t_0$ In Sandage's models where $\lambda = 0$, we have $\sigma_0 = 3q_0$, and so (11A) becomes

$$\begin{aligned} \left(\frac{dY}{dX}\right)^2 &= \frac{2q_0}{Y} - (2q_0 - 1) \\ &= 1 + 2q_0 z, \qquad (z \geq 0 \text{ only}). \end{aligned} \tag{12A}$$

Also

$$\frac{dY}{dX} = \frac{\dot{R}}{H_0 R_0}, \tag{13A}$$

and therefore (4A) becomes

$$\left(\frac{dz}{dt}\right)_0 = -H_0(1+z)\left(\frac{dY}{dX} - 1\right).$$

If we take $H_0 = 75$ km/sec/mpc, so that $H_0^{-1} = 13 \times 10^9$ years, and if we multiply the last equation by $c = 3 \times 10^5$ km/sec and remember that $Y = (1+z)^{-1}$, the last equation becomes, with the aid of (11A),

$$\begin{aligned} c\left(\frac{dz}{dt}\right)_0 &= -\frac{300 \times 10^{-6}}{13} \\ &\times \left\{ \left[\tfrac{2}{3}\sigma_0(1+z) + q_0 + 1 - \sigma_0 - \frac{q_0 - \sigma_0/3}{(1+z)^2} \right]^{1/2} - 1 \right\} \text{km/sec/yr}. \end{aligned} \tag{14A}$$

This equation has been used to calculate the values of $c(dz/dt)_0$ in Table 1A, which refer to an object with redshift equal to 0.4. The values of λ corresponding to the selected values of q_0 and σ_0 are also shown. The first three models are Sandage's. The fourth model is Milne's universe, in which $R = ct$, so that the rate of change of z for all values of the redshift is zero. The last two models have densities that lie in the range 3×10^{-31} to 3×10^{-30} gm/cm^3, which accords with Oort's estimate of the average density of matter contained in the galaxies. There may, of course, be matter in the universe which is not connected with any galaxy. The observational evidence for the amount of this intergalactic matter, if it exists, is not yet available.

The foregoing results refer to general relativity models. In the model of the steady-state theory, as well as in the de Sitter universe of general relativity, we have $R = Be^{Ht}$, where B is a constant and the Hubble constant H is also now independent of time. Hence

$$Y = (1 + z)^{-1} = e^{H(t-t_0)} ,$$

and therefore (4A) with $H_0 = H$, $R_0 = Be^{Ht_0}$ gives

$$c\left(\frac{dz}{dt}\right)_0 = -cH(1+z)[e^{H(t-t_0)} - 1]$$

$$= cHz ,$$

which is Sandage's equation (35).

The rate of change of the apparent bolometric magnitude of a source at the present time t_0 can also be obtained for all models. Let l_0, m_0 be the present apparent bolometric luminosity and

TABLE 1A

RATE OF CHANGE OF z FOR AN OBJECT WITH $z = 0.4$

q_0 .	1	0 5	0 3516	0	0 5	1 15
σ_0 ..	3	1 5	1 0548	0	0 06	0 45
λ/H_0^2..	0	0	0	0	−1 44	−3 0
$c(dz/dt)_0 \times 10^6$.	−11	−5 9	−4 2	0	−3 9	−8 7

magnitude, respectively, of the source and let l_s, m_s be the corresponding quantities of a standard comparison source. Then we have, by definition,

$$\left(\frac{l_s}{l}\right)_0 = 10^{2(m-m_s)_0/5},$$

and logarithmic differentiation gives

$$\left(\frac{1}{l}\frac{dl}{dt}\right)_0 = -\tfrac{2}{5}E\left(\frac{dm}{dt}\right)_0, \tag{15A}$$

if we assume that l_s, m_s are constants. Here $E = \ln 10 = 2.303$. Now equation (22) is equivalent to

$$l_0 = \frac{L}{4\pi D_0^2},$$

where D_0 is the luminosity distance of the source. We assume that L is constant, so that logarithmic differentiation again yields

$$\left(\frac{1}{l}\frac{dl}{dt}\right)_0 = -2\left(\frac{1}{D}\frac{dD}{dt}\right)_0. \tag{16A}$$

But it can be proved (McVittie 1956) that

$$\left(\frac{1}{D}\frac{dD}{dt}\right)_0 = \frac{2\dot{R}_0}{R_0} - \frac{\dot{R}}{R_0}$$

$$= 2H_0\left(1 - \frac{\dot{R}}{2H_0R_0}\right).$$

Thus, by (13A),

$$\left(\frac{1}{D}\frac{dD}{dt}\right)_0 = 2H_0\left(1-\frac{1}{2}\frac{dY}{dX}\right), \tag{17A}$$

where dY/dX is given in general by (11A) and, for Sandage's models in particular, by (12A). Thus combining (15A), (16A), and (17A), we obtain

$$\left(\frac{dm}{dt}\right)_0 = \frac{10H_0}{E}\left(1-\frac{1}{2}\frac{dY}{dX}\right). \tag{18A}$$

For sources with small redshifts, Y is only slightly smaller than unity, say $Y = 1 - \epsilon$. It then follows from (11A) that $dY/dX = 1 + \epsilon q_0$. Hence $(dm/dt)_0 > 0$, and therefore the apparent bolometric magnitudes increase with time, i.e., the sources appear to get fainter as time proceeds. But, clearly, in models where the expansion begins from $Y = 0$, this is not true for sources of very large redshift. As an illustration, consider Sandage's models. In these we have, by (12A) and (18A),

$$\left(\frac{dm}{dt}\right)_0 = \frac{10H_0}{E}\left[1-\frac{(1+2q_0z)^{1/2}}{2}\right].$$

Hence for any source which now shows a redshift exceeding $z = 1.5/q_0$, the value of $(dm/dt)_0$ is *negative*. These sources will therefore increase in brightness as time proceeds. This explains the nature of the run of m against z in Sandage's Tables 1 and 2 for large z values.

Since in the steady-state theory and in the de Sitter universe of general relativity,

$$\frac{dY}{dX} = Y = (1+z)^{-1},$$

equation (18A) becomes

$$\left(\frac{dm}{dt}\right)_0 = \frac{5H}{E}\left(\frac{2z+1}{z+1}\right),$$

and so is always positive. Moreover, $(dm/dt)_0$ tends to the finite limit $10\, H/E$ as soon as z becomes large compared with unity.

Corresponding results can be obtained for the flux density of a radio source, an observable that plays the part of an apparent magnitude of optical astronomy. If S denotes the flux density of such a source, we have (McVittie 1961)

$$\frac{S}{S_s} = \left(\frac{1+z}{1+z_s}\right)^{1+x}\frac{D_s^2}{D^2},$$

where suffix s again refers to the standard comparison source and x is the spectral index which is a negative number probably lying in the range -0.8 to -1.2. If the quantities referring to the standard source are regarded as constants, we have, on differentiating the last formula with respect to the time and using (4A), (13A), and (17A),

$$\left(\frac{1}{S}\frac{dS}{dt}\right)_0 = -H_0(1-x)\left(\frac{3-x}{1-x}-\frac{dY}{dX}\right). \tag{19A}$$

The right-hand side is negative for sources with small z, as for bolometric apparent magnitudes. But, in general relativity models of the kind considered above, $(dS/dt)_0$ changes sign for a sufficiently large value of z. In Sandage's models this happens when

$$z = \frac{4-2x}{(1-x)^2}\frac{1}{q_0}.$$

As we have seen, the coefficient of $1/q_0$ in the corresponding result for apparent bolometric magnitude is 1.5. It now varies from 1.32 when $x = -1.2$ to 1.73 when $x = -0.8$. This example shows that the details of the energy-distribution function of the spectrum of the source and the type of apparent magnitude that is considered will affect the precise value of the redshift at the turning point in formulae such as (18A) and (19A).

In the steady-state theory, or the de Sitter universe, (19A) becomes

$$\left(\frac{1}{S}\frac{dS}{dt}\right)_0 = -H_0(1-x)\left(\frac{3-x}{1-x}-\frac{1}{1+z}\right),$$

and therefore the right-hand side is always negative for $z \geq 0$. Moreover,

$$\left(\frac{1}{S}\frac{dS}{dt}\right)_0 \to -H_0(3-x),$$

when z is large compared with unity.

APPENDIX REFERENCES

McVittie, G. C. 1956, *General Relativity and Cosmology* (London: Chapman & Hall), eq. (8 524).
———. 1961, *Fact and Theory in Cosmology* (London: Eyre & Spottiswoode), p. 116.

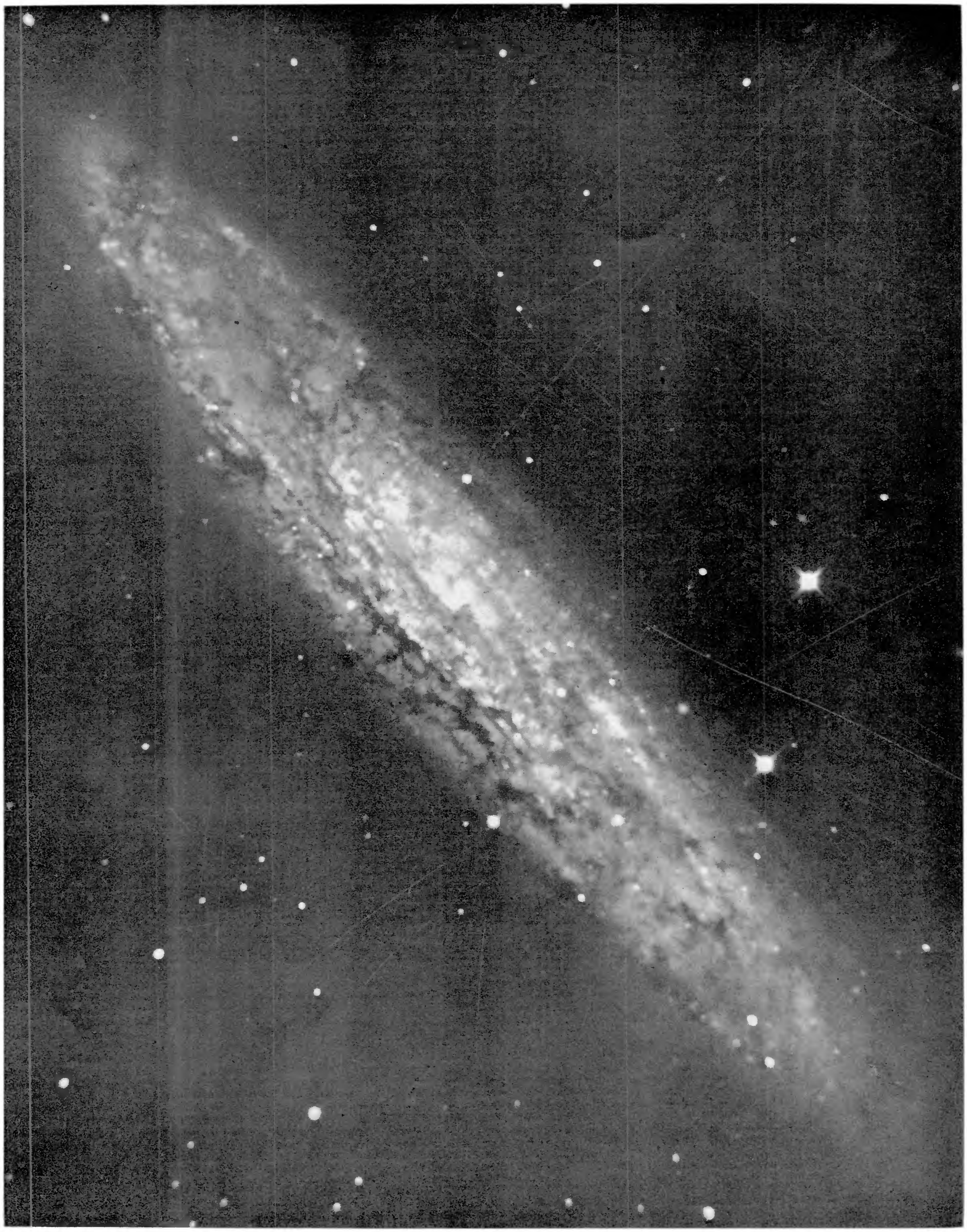

FIG. 1.—NGC 253, photographed at prime focus of 82-inch telescope on 103a-O Eastman Kodak plate. East is at top. north at left. Scale: 1 mm = 6″9.