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THE RATE OF STAR FORMATION

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ABSTRACT

It is assumed that the rate of star formation for population I varies with a power n of the density of interstellar gas and that the initial luminosity function is time-independent. Direct evidence on the value of n is found in the relative distribution, perpendicular to the galactic plane, of gas and young objects. For various values of n, computations were made of the initial luminosity function, the rate of star formation, the exchange of gas between stars and interstellar medium, the number of white dwarfs and their luminosity function, and the abundance of helium. It is concluded, from a comparison of the results with observational data, that n is around 2. The present rate of star formation, then, is five times slower than the average rate. The interstellar gas, of which the surface density on the galactic plane was taken to be 11 $\mathfrak{M}_\odot$ per square parsec, loses 1.4 $\mathfrak{M}_\odot/pc^2$ per 10^9 years by the formation of stars but gains about one-third of this by ejection of gas from evolving stars. The present helium abundance of the interstellar gas may be explained if a star has burned, on the average, 53 per cent of its original hydrogen into helium at the time that ejection takes place. The ejected material was assumed to have a composition equal to the average composition of the star. The effect of star formation on the gas density in the galactic system and other galaxies is briefly discussed.

I. INTRODUCTION

The effects of stellar evolution on the observed luminosity function for main-sequence stars, $\varphi(M_v)$, in the solar neighborhood were first investigated by Salpeter (1955). He reasoned that we observe at present only those stars which were formed less than $T(M_v)$ years ago, where $T(M_v)$ is the lifetime of a star with absolute visual magnitude M_v on the main sequence. For stars with $M_v > 3.5$ the lifetime $T(M_v)$ is larger than the age of the Galaxy, so that we observe all the stars ever formed. For stars brighter than $M_v = 3.5$, however, the ratio of the number we see to the number ever formed will be $T(M_v)/\text{T(Galaxy)}$ on the assumption of a uniform rate of star formation. Thus, if $\psi(M_v)$ is the luminosity function of all stars ever formed in the solar neighborhood, then

$$\varphi(M)_v = \frac{\psi(M_v)\,T(M_v)}{T(\text{Galaxy})}. \tag{1}$$

The initial luminosity function $\psi(M_v)$ thus obtained was compared with the distribution of luminosities in open clusters by Sandage (1957*a*), van den Bergh (1957*a*), and Jaschek

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and Jaschek (1957). They found that the distribution of luminosities is about the same for rich and poor clusters, and they agree that $\psi(M_v)$ fits the distribution in clusters reasonably well.

The decrease in the density of interstellar gas due to star formation was first considered by van den Bergh (1957*b*). He found that, at a constant rate of star formation, the gas in the solar vicinity will be exhausted about 7×10^8 years from now. The author suggested that the gas is being replenished from the nuclear regions of the Galaxy. He also investigated an exponentially decreasing rate of star formation but concluded that the rate of star formation in the past cannot have been very different from the present rate. This conclusion is not substantiated in the present investigation.

In the present paper it is assumed explicitly that the rate of star formation depends on a positive power n of the gas density (Sec. II). Basic data are discussed in Section III. Analytical approximate solutions of the integral equations are found to be sufficiently accurate for our purpose (Sec. IV). Direct evidence on the value of n from the distribution of young objects is given in Section V. For different values of n the initial luminosity function, the rate of star formation, the ejection of gas by evolving stars, the number of white dwarfs, and the helium abundance will be computed (Secs. VI–X). The effects of star formation in regions not near the sun and in other galaxies are briefly discussed (Sec. XI). It is concluded in Section XII that the rate of star formation varies with the square of the gas density.

II. EQUATIONS

Let $N(M_v, t)$ be the number of stars which up to time t had been formed on the main sequence between magnitudes $M_v - \frac{1}{2}$ and $M_v + \frac{1}{2}$. The rate of star formation will be

$$\frac{dN(M_v, t)}{dt}.$$

Assume that the distribution of magnitudes at formation, or the initial luminosity function, is time-independent; then

$$\frac{dN(M_v, t)}{dt} = \psi(M_v)\, f(t)\,, \tag{2}$$

where $f(t)$ may be called the rate function. Let $t = 0$ denote the beginning of the era of star formation, which may tentatively be identified with the time that the Galaxy was formed, and $t = 1$ the present. Our unit of time then is the age of the Galaxy, which in numerical examples will be taken to be 8×10^9 years. Then

$$N(M_v, 1) = \psi(M_v) \int_0^1 f(t)\, dt\,, \tag{3}$$

where $N(M_v, 1)$ is the number of stars formed on the main sequence up to now. According to the definition of $\psi(M_v)$ given in the previous section,

$$\psi(M_v) = N(M_v, 1)\,; \tag{4}$$

so

$$\int_0^1 f(t)\, dt = 1\,. \tag{5}$$

The observed luminosity function of main-sequence stars is

$$\varphi(M_v) = \psi(M_v) \int_{1-T(M\ )}^1 f(t)\, dt\,, \tag{6}$$

where $T(M_v)$ is the lifetime on the main sequence. For stars fainter than $M_v = +3.5$, $T(M_v) > T(\text{Galaxy}) = 1$, and the lower limit of the integral becomes zero; hence $\varphi(M_v) = \psi(M_v)$. For constant rate of star formation, equation (6) is reduced to $\varphi(M_v) = \psi(M_v)T(M_v)$, in accordance with equation (1).

We shall now derive equations for the gas density as a function of time. Present ideas on stellar evolution require that after a star has burned about 10 per cent of its hydrogen, it will move rapidly to the red giant region. It is not clear as yet how a star will evolve afterward, but it is believed that the end product will be a white dwarf. The theory of interior structure of white dwarfs predicts masses less than 1.25 solar masses (Rudkjöbing 1952). All evolving stars have larger masses, so that they supposedly have to eject mass in order to reach the white dwarf stage. Most determinations of the mass of white dwarfs yield values well below the theoretical maximum. Direct determinations are available for three white dwarfs in visual binaries, which give 1.0, 0.4, and 0.4 solar masses. A value of 0.7 solar mass is used in the present investigation. Greenstein (1958) recently found a statistical average of 0.56 solar mass.

The lifetime in the giant stage is only 20 per cent or less of the lifetime on the main sequence. We shall neglect the giant stage altogether and accordingly suppose that a star after leaving the main sequence loses all its mass in excess of 0.7 solar mass and becomes a white dwarf.

The total mass of all stars formed up to time t is

$$\mathfrak{M}_T(t) = \sum_{M_v} \psi(M_v)\, \mathfrak{M}(M_v) \int_0^t f(t)\, dt\,. \tag{7}$$

The mass of gas ejected by evolving stars up to time t is

$$\mathfrak{M}_E(t) = \sum_{M_v} \psi(M_v)\, [\mathfrak{M}(M_v) - 0.7] \int_0^{t-T(M_v)} f(t)\, dt\,. \tag{8}$$

We assume that no gas or stars enter or leave the region under consideration and that at $t = 0$ all mass was in the form of gas, so that $\mathfrak{M}_G(0)$ is the total mass density. The mass of gas left at time t is

$$\mathfrak{M}_G(t) = \mathfrak{M}_G(0) - \mathfrak{M}_T(t) + \mathfrak{M}_E(t)\,. \tag{9}$$

The above equations, except for some changes, were developed by van den Bergh (1957*b*).

It would seem most probable that the rate of star formation depends on the gas density, and we shall assume that the number formed per unit interval of time varies with a power of the gas density,

$$f(t) \sum_{M_v} \psi(M_v) = C\,[\mathfrak{M}_G(t)]^n\,. \tag{10}$$

Substitution of equations (7), (8), and (9) in equation (10) gives

$$f(t)\,\Sigma\psi = C\Big[\mathfrak{M}_G(0) - \sum \psi\mathfrak{M} \int_0^t f(t)\, dt + \sum \psi\,(\mathfrak{M} - 0.7) \int_0^{t-T} f(t)\, dt\Big]^n\,, \tag{11}$$

where

$$\psi \equiv \psi(M_v)\,,\quad \mathfrak{M} = \mathfrak{M}(M_v)\,,\quad T \equiv T(M_v)\,,\qquad \text{and}\qquad \sum \equiv \sum_{M_v}\,.$$

It may easily be shown that equations (5), (6), and (11) define a unique solution for $f(t)$, $\psi(M_v)$, and the constant C, once $\varphi(M_v)$ and the present gas density $\mathfrak{M}_G(1)$ are given. The solution will be discussed in Section IV.

III. BASIC DATA

a) The Main-Sequence Luminosity Function

The data adopted for the general luminosity function were taken from van Rhijn (1936), which McCuskey (1956) found to give a good representation within 500 pc in the galactic plane. For M_v fainter than $+4$, data given by Kuiper (1942) and Luyten (1939) were also considered, and a smooth curve was drawn. The fraction f of stars which are main-sequence stars is rather uncertain. Both Salpeter (1955) and Sandage (1957*a*) determined f from a summary of data given by Trumpler and Weaver (1953) but arrived at rather different results for bright stars. The values given in Table 1 were derived by separating giants and dwarfs according to $B - V$ and are close to the values found by Sandage. For $M_v = 0$ and $+1$, however, the ratio was derived from McCuskey's data, with the luminosity function of giants as found by Halliday (1955). The resulting values are lower than those found for neighboring magnitudes, which is to be expected from the convergence of evolutionary tracks indicated by Sandage (1957*b*). The hump shown by van Rhijn's function at $M_v = +1$ may well be entirely due to this effect. The f-values were slightly adjusted within the limits set by their accuracy, so as to yield a fairly smooth luminosity function for main-sequence stars.

TABLE 1

DATA FOR BRIGHT STARS

M_v	φ_{total} (10^{-4} pc^{-3})	f	φ_{ms} (10^{-4} pc^{-3})	Equiv. Width (pc)	Φ (pc^{-2})	$\mathfrak{M}$ (⊙)	B C. (mag.)	K	T
−5.... ..	0.00059	0 41	0 00024	220	0 000005	25.1	−3 08	4 0	0.002
−4........	0.0038	.41	0 00155	220	.000034	15.8	−2 67	3 7	.004
−3.... .	0.0132	46	0.00603	220	.000133	10 0	−2 43	3.3	.007
−2........	0.0513	.48	0.0245	225	.00055	7 6	−2.21	3 0	.015
−1........	0.209	.52	0.110	230	.00253	5.57	−1.86	2.5	.032
0........	0.955	.46	0.437	240	.0105	3.96	−1.38	1.8	.065
1........	3.89	.33	1.29	250	.0322	2.89	−0.99	1.3	.124
2........	5.13	.69	3.55	300	.106	2.03	−0.47	1.1	.296
3........	9.55	0.87	8.32	425	0.354	1.50	−0.16	1.0	0.661

The luminosity function thus arrived at refers to a volume of 1 pc^3 near the galactic plane. However, the luminosity function at different values of z, the distance to the galactic plane, is different, and it would seem improbable that the luminosity function in the plane has any special significance. The primary question here is: Why do faint stars have a wider distribution in z than bright stars? Faint stars are, on the average, much older than bright stars, and the difference may be due to a gradual increase in velocity dispersion with time. Spitzer and Schwarzschild (1953) have investigated the effect of gravitational encounters with interstellar cloud complexes. They find that masses of 10^6 solar masses about 350 pc apart in an infinite and homogeneous interstellar medium would increase the velocity dispersion of a group of stars from 10 to 20 km/sec in 1.5×10^9 years. This result may have to be modified for two reasons. First, cloud complexes of this size must be confined to spiral arms. Actually, absorption spectra at 21-cm wave length in the direction of bright radio sources indicate almost complete absence of gas between spiral arms (Muller 1957). Most stars will be outside a spiral arm in part of their orbits. Second, the average age of faint stars is probably about 6×10^9 years (cf. Sec. VII). The two modifications work in opposite directions, and the original result is probably of the right order. This theory may then explain the increase in velocity dispersion from spectral type B to late F. However, the still larger velocity dispersion of M dwarfs cannot be explained and may be due to a part of these stars being population II objects (Spitzer and Schwarzschild 1951). Oort (1958*a*) has shown that the width of the distribution of open clusters in z increases with age, which is supposedly due to the Spitzer-Schwarzschild mechanism.

On this basis, then, we should add the faint stars at larger z to the luminosity function in the plane or, what is equivalent, compute a luminosity function in a cylinder perpendicular to the galactic plane.

Another possibility is that the layer of interstellar gas has collapsed very gradually in time and that stars of different luminosities reflect the distribution of gas at the average time they were formed. Blaauw (1958), who discussed this case as well as the previous one, indicates that in this case those faint stars now observed near the plane, which originated at large distances z, would have to be eliminated from the luminosity function. However, in this case the increase in velocity dispersion of galactic clusters discussed above would be inexplicable.

We shall assume, then, that all stars were formed in a layer of gas which always had the same distribution in z. Let the equivalent width in z be the quantity with which the

TABLE 2

DATA FOR FAINT STARS

M_v	φ (10^{-4} pc^{-3})	Equiv. Width (pc)	Φ (pc^{-2})	$\mathfrak{M}$ ($\odot$)	M_v	φ (10^{-4} pc^{-3})	Equiv. Width (pc)	Φ (pc^{-2})	$\mathfrak{M}$ ($\odot$)
4	20	580	1 13	1.16	12	117	800	9.40	0.23
5	28	740	2 09	0.96	13	132	800	10.54	.16
6	34	800	2 71	0 82	14	129	800	10.30	.12
7	40	800	3.18	0 72	15	107	800	8.58	08
8	45	800	3 58	0.63	16	79	800	6.35	.06
9	54	800	4 30	0.55	17	48	800	3.83	.04
10	69	800	5 54	0.46	18	23	800	1.87	.03
11	93	800	7.46	0.33	19	8	800	0.63	0.02

spatial densities in the plane have to be multiplied to give surface densities on the plane. It was assumed that the spatial densities in the plane as given by van Rhijn's function are actually average densities in a region between $z = \pm 50$ pc. Equivalent widths in z for stars of different luminosity were computed from data given by Bok and MacRae (1941), Parenago (1950), and Oort (1936, 1938). The adopted equivalent widths and the luminosity function $\Phi(M_v)$ in a cylinder perpendicular to the plane are given in Tables 1 and 2.

b) Masses, Bolometric Corrections, and Lifetimes on the Main Sequence

The masses of stars of different luminosity were taken mostly from Strand (1957). Data given by van de Kamp (1954) were consulted to obtain a direct relation between mass and absolute visual magnitude. For $M_v = -5$ and -4, data by Pearce (1957) were used. For stars with $M_v = 10$ and fainter, the relation $\log \mathfrak{M} = 0.15\ (7.77 - M_v)$ was used.

The bolometric corrections as a function of effective temperature, T_e, were taken from Kuiper (1938). The relation of T_e with spectral type was adopted from Morgan and Keenan (1951). For spectral type O8, T_e was taken to be 29500°, a little lower than the value given by Petrie (1957), whose value for B0 is somewhat higher than Morgan and Keenan's. The relation between spectral type and M_v was taken for $M_v = -5$, -4, and -3 from Johnson and Hiltner (1956) and for fainter stars from the (Sp, $B - V$)

relation given by Johnson and Morgan (1953) and the $(B - V, M_v)$ relation for the unevolved main sequence by Sandage (1957*b*).

The lifetime on the main sequence would be proportional to the ratio of mass to luminosity if stars were homologous in all stages of their initial evolution. This is not the case, since massive stars have convective cores while faint stars have none. Hoyle (1958) recently gave most valuable data on evolutionary tracks of stars of different mass. He defined a constant K, such that the lifetime on the main sequence is

$$T \backsim K \frac{\mathfrak{M}}{L}, \tag{12}$$

where $K = 1$ for $\mathfrak{M} = 1.52$. He finds that for 30 $\mathfrak{M}_\odot$ the factor K will be about 4. Values of K have been interpolated from data given. Lifetimes on the main sequence were then calculated from equation (12). The lifetimes were divided by the lifetime for a star with $M_v = +3.5$, which is, by definition, the age of the Galaxy, and are given in Table 1.

The evolutionary track of a star near the main sequence was first computed by Schönberg and Chandrasekhar (1942) and is given by Sandage (1957*b*) in convenient form. Before a star moves off into the giant region, it will brighten by about 1 mag. The main-sequence luminosity function observed at present is affected by this. The bright stars observed have been formed quite recently, and their age distribution up to their lifetime will be uniform. The average brightening will then be about $\frac{1}{3}$ mag. For stars now around $M_v = +3$, the distribution of ages will depend on the value of n. If $n = 2$, most of these stars have been formed in the earlier epochs, and the average brightening will have been more than $\frac{1}{3}$ mag. Trial corrections for these effects have shown that the bright end of the observed luminosity function of main-sequence stars should be lowered by a factor around 2. This is in agreement with a determination by Mathis (1958). For the case $n = 2$, however, it was found impossible to allow for the effects of initial evolution without getting into mathematical instabilities. Hoyle (1958) finds that most stars brighten less than 1 mag. before leaving the main sequence, and it may be possible to obtain a satisfactory solution in the future.

On the other hand, as a consequence of neglecting effects of initial evolution, we are forced to assume that stars formed at $M_v = 3.5$ evolve off the main sequence after an interval of time equal to the age of the Galaxy. In reality, this will be the case for stars which formed at $M_v = +4.5$, which have now brightened to $+3.5$ and are just evolving off the main sequence. The ratio of the lifetimes at $M_v = +4.5$ and $+3.5$ is about 2. We thus see that both the lifetimes and the bright end of the luminosity function used are about a factor of 2 too large. The effects will roughly cancel each other in the determination of $\psi(M_v)$. The situation is somewhat unsatisfactory, however, and needs future investigation.

c) Total Mass Density near the Sun

The total mass $\mathfrak{M}_G(0)$ in a cylinder of 1 pc^2 cross-section, perpendicular to the galactic plane, near the sun may be computed from a model of the distribution of mass (Schmidt 1956). This model was based on a spatial density near the sun of about 0.09 $\mathfrak{M}_\odot/\text{pc}^3$. Modern data lead to a value of about 0.15 $\mathfrak{M}_\odot/\text{pc}^3$ (Oort 1958*b*). The effect on the surface density, which was computed to be 55 $\mathfrak{M}_\odot/\text{pc}^2$, may not be large but needs future investigation.

The projected density of gas may be derived from observations of neutral hydrogen at 21-cm wave length. From data given by Westerhout (1957) the mean density of neutral hydrogen at the sun's distance from the center is found to be about 0.55 atom/cc. One of the assumptions made was that the distribution of hydrogen is fairly homogeneous, with smooth variations in density. Absorption measurements against bright radio sources (Muller 1957) have shown that individual clouds have optical depths up to 4.

As a consequence, the hydrogen densities given probably have to be multiplied by a factor of 1.5 or 2. This would make the density of neutral hydrogen 0.020–0.027 $\mathfrak{M}\odot/\text{pc}^3$. The helium content of the interstellar gas is taken to be 34 per cent, the value found by Mathis (1957) for the Orion Nebula. The gas density, then, is 0.030–0.040 $\mathfrak{M}\odot/\text{pc}^3$. The equivalent width of the gas layer is 273 pc, as found from the vertical distribution of hydrogen (Schmidt 1957). The surface density of gas, then, is 8–11 $\mathfrak{M}\odot/\text{pc}^2$. Not yet included are the contributions of hydrogen in ionized or molecular form. Ionized hydrogen may contribute about 10 per cent, but the amount of molecular hydrogen is unknown. All calculations in the present paper are based on a gas density of 11 $\mathfrak{M}\odot/\text{pc}^2$, or 20 per cent of the mass density.

It is of interest to inquire which part of the total density can be accounted for in terms of observed objects. Main-sequence stars contribute about 25, gas 11, white dwarfs probably 4, and giants and supergiants about 1 $\mathfrak{M}\odot/\text{pc}^2$. Thus we can account for around 41 $\mathfrak{M}\odot/\text{pc}^2$, while the total mass density is 55 $\mathfrak{M}\odot/\text{pc}^2$. A similar situation exists in volume densities near the galactic plane, where the above contributions are about 0.033, 0.041, 0.005, and 0.002 $\mathfrak{M}\odot/\text{pc}^3$, respectively. This is 0.08 $\mathfrak{M}\odot/\text{pc}^3$ in total, while the total density is probably around 0.15 $\mathfrak{M}\odot/\text{pc}^3$ (Oort 1958*b*).

The contribution of stars fainter than $M_v = 10$, which is taken to be 9 $\mathfrak{M}\odot/\text{pc}^2$, is quite uncertain. It may well be that the faint end of the luminosity function or the extrapolation of the mass-luminosity relation to faint stars, or both, are too low. Actually, the mass obtained by Miss Lippincott (1955) for Ross 614 B seems considerably higher than predicted by the mass-luminosity relation used.

We shall assume that the mass which cannot be accounted for is in the form of faint stars and define the mass of stars fainter than $M_v = +3.5$ as

$$\sum_{M_v>3.5} \Phi\mathfrak{M} = \mathfrak{M}_G(0) - \mathfrak{M}_G(1) - \mathfrak{M}_{wd} - \sum_{M_v<3.5} \Phi\mathfrak{M}$$
$$= 43.10 - \mathfrak{M}_{wd} . \tag{13}$$

The number of white dwarfs is the number of evolved stars, $\Sigma(\psi - \Phi)$, and their mass, accordingly, is

$$\mathfrak{M}_{wd} = 0.7 \sum (\psi - \Phi) . \tag{14}$$

It may be noted that we have neglected population II objects, i.e., those objects which were formed before the gas had contracted to a layer. Their contribution to the surface density near the sun, which is unknown, is probably small.

IV. ANALYTICAL AND EXACT SOLUTIONS

We shall first derive analytical solutions of equation (11), which are of an approximate nature, and then investigate their accuracy by comparison with an exact solution. An analytical solution for the rate function $f(t)$ from integral equation (11) may be found by neglecting the term representing ejection from evolving stars.

Let the ratio of present gas density to total mass density, or original gas density, be P; hence

$$\mathfrak{M}_G(1) = P\,\mathfrak{M}_G(0) . \tag{15}$$

Then, from equation (11),

$$f(1) = P^n f(0) . \tag{16}$$

If mass loss is neglected, $f(t)$ may be found in analytical form from equations (5), (7), (9), (11), and (16). We get, for $n = 1$,

$$f(t) = f(0)\, e^{-t/\tau} , \tag{17}$$

where τ and $f(0)$ are determined by

$$e^{-1/\tau} = P \ , \tag{18}$$

$$(1-P)\ \tau f(0) = 1\ ; \tag{19}$$

and for $n > 1$,

$$f(t) = f(0)\left[1 + (n-1)\frac{t}{\tau}\right]^{-n/(n-1)}, \tag{20}$$

with

$$\tau = \frac{n-1}{P^{1-n} - 1}, \tag{21}$$

and again

$$(1-P)\ \tau f(0) = 1\ . \tag{22}$$

Detailed computations have been made for the cases $n = 0$, 1, and 2.

$n = 0$.—In this case the rate of star formation is constant, and $f(t) \equiv 1$.

$n = 1$.—The rate of star formation is proportional to the gas density. From equations (17)–(19) with $P = 0.20$, we get

$$f(t) = 2.012\, e^{-1.6094t}\ . \tag{23}$$

$n = 2$.—The rate of star formation varies with the square of the gas density and is given by

$$f(t) = \frac{5}{(1+4t)^2}. \tag{24}$$

The accuracy of the above solutions may be judged by comparison with exact solutions. These are found by starting numerical integration of equations (7), (8), (9), and (11) with assumed values of $f(0)$ and $\psi(M_v)$. At $t = 1$, equations (5), (6), and (16) should be fulfilled. This will not be the case, and the process is repeated with different values for $f(0)$ and $\psi(M_v)$ until a fit is obtained. Table 3 gives some values of $f(t)$ thus obtained

TABLE 3

COMPARISON OF EXACT AND APPROXIMATE SOLUTIONS FOR $f(t)$

t	Exact	Approx.	t	Exact	Approx.
0	5.55	5.00	0.209.....	1.42	1.48
0.013......	4.77	4.52	0.589......	0.46	0 44
0.074......	2.88	2.98	1....	0 22	0.20
0.123......	2.14	2.25			

for $n = 2$, $P = 0.20$. The maximum error of the approximate, analytical solution is 11 per cent. However, the exact solution for $f(t)$ in this case was based on the assumption $K \equiv 1$ (cf. eq. [12]). In that case the mass ejected by evolving stars was about double the amount we find in the present paper. It may be expected, then, that the maximum error in $f(t)$ will be 6 per cent for $n = 2$ and 2 per cent for $n = 1$. The errors are much smaller than the uncertainties involved in the basic data and the assumptions, and it would seem that the analytical solutions for $f(t)$ are adequate for our purpose. In the above approximate solutions for $f(t)$ the ejection term in equation (11) was neglected in a formal mathematical sense only. The actual ejection from evolving stars may be calculated from equation (8) with an accuracy of the same order as the accuracy of $f(t)$.

V. SOME DIRECT EVIDENCE OF THE VALUE OF n

The way in which the rate of star formation varies with gas density is best studied by comparing the distribution of gas and of young objects. It is advantageous to compare distributions over z, since in this co-ordinate the gas has a well-defined distribution. The distribution of neutral hydrogen in z was investigated by Schmidt (1957), who found it to be identical over a large range of distance from the galactic center. From his data the dispersion in z may be computed to be 144 pc. If the rate of star formation varies linearly with the gas density, the dispersion of young objects should be 144 pc. If, on the other hand, the rate of star formation varies with the second power of the gas density ($n = 2$), then the dispersion of young objects should equal the dispersion of the gas density squared, or 78 pc. Similarly for $n = 3$ we should expect a dispersion of 57 pc.

A representative sample of young objects may be a group of 134 classical cepheids closer than 3 kpc, observed by Walraven, Muller, and Oosterhoff (1958). They adopted a correction to Shapley's period-luminosity relation of -1.7 mag. and obtained a dispersion in z of 65 pc. Weaver (1958) has indicated that the authors should have adopted, for the period-color relation used, a correction to Shapley's relation of about -2.1 mag. so as to be in agreement with proper-motion data. This would increase the distances obtained by about 20 per cent. The dispersion of cepheids in z would then be about 80 pc. Open clusters with earliest types O–B4 have an average z-component of 44 pc (Oort 1958*a*), or a dispersion of 58 pc.

TABLE 4

INITIAL LUMINOSITY FUNCTIONS

M_v	ψ(pc^{-2})			ψ_{cl}	
	$n=0$	$n=1$	$n=2$		
-5	0.00	0.01	0.01	....	0.01
-4	0.01	0.02	0.04	0.02	0.05
-3	0.02	0.05	0 09	0.04	0.06
-2	0 04	0.09	0.18	0.08	0.11
-1	0.08	0 19	0.38	0.13	0.17
0	0.16	0.38	0.77	0.27	0.34
1	0.26	0.58	1.17	0.58	0.66
2	0.36	0.69	1.37	1.07	0.96
3	0.54	0.75	1.26	1.37	1.11
4	1.13	1.13	1.13	1.57	1.56
5	2.09	2.09	2 09	1.65	1.66

Comparison of these observed dispersions with the values given above indicate $n = 2$–3.

Indirect evidence on the value of n will be obtained in the following sections. A summary of the results will be given in Section XII.

VI. THE INITIAL LUMINOSITY FUNCTION

The initial luminosity functions $\psi(M_v)$ for different values of n are given in Table 4 and shown in Figure 1. Also given are two determinations of the luminosity function in open clusters. The first one is by van den Bergh (1957*a*), who investigated the mass spectrum of nine open clusters. He found similar spectra for different clusters and computed a luminosity function from the mean distribution of masses. The values given in Table 4 were normalized to the computed initial luminosity functions in the interval 3.5–5.5. The cluster luminosity function in the last column of Table 4 was constructed from data collected by Sandage (1957*a*) for five clusters. Four of these clusters were included in van den Bergh's material. No counts less than 1 mag. below the break-off point were considered, so as to avoid the effects of initial evolution near the main sequence. The difference between the two columns gives some indication of the uncertainty of the cluster luminosity function.

For bright stars, the lifetime on the main sequence, $T(M_v)$, is short, and equation (6) may be written as

$$\Phi(M_v) \cong f(1)\,T(M_v)\,\psi(M_v)\,. \tag{25}$$

For different values of n, only $f(1)$ changes its value (cf. Table 5). Therefore, the form of the initial luminosity function for M_v brighter than $+1$ for different values of n is practically identical.

It should be kept in mind that we have neglected effects of initial evolution near the main sequence. The consequences of this procedure were discussed in Section III*b*. In

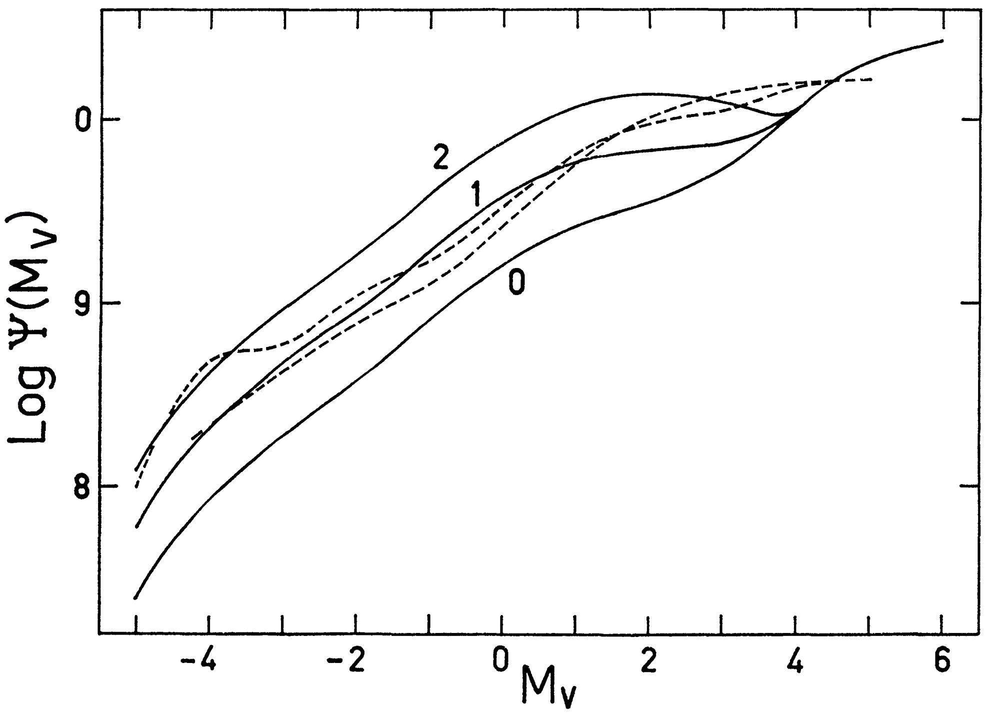

FIG. 1.—The initial luminosity functions are shown as solid lines, with the corresponding value of n indicated. The cluster luminosity functions are shown as dotted lines.

TABLE 5

THE RATE FUNCTION $f(t)$

t	n = 0	n = 1	n = 2	t	n = 0	n = 1	n = 2
0	1	2.01	5.00	0.6	1	0.77	0.43
0.2	1	1.46	1.54	0.8	1	0.55	.28
0.4	1	1.06	0.74	1	1	0.40	0.20

view of the uncertainties involved, the agreement between the luminosity function in clusters and those computed for $n = 1$ and $n = 2$ is satisfactory. We conclude that the value of n would be between 1 and 2, if the cluster luminosity function may be identified with the initial luminosity function.

VII. THE RATE OF STAR FORMATION

Equations for the rate function $f(t)$ for different values of n have been given in Section IV. Some values are given in Table 5. The present value of the rate function $f(1)$ is five times smaller for $n = 2$ than it is for $n = 0$. This does not mean that the rate of star formation for all absolute magnitudes is five times slower than in the case of uniform formation. The present rate of star formation is $\psi(M_v)f(1)$. For bright stars this is $\Phi(M_v)/T(M_v)$, according to equation (25), and thus independent of the value of n. For stars fainter than $M_v = +3.5$ the rate of formation is $\Phi(M_v)f(1)$, and this does depend on the value of n.

We thus see that, while the number of bright stars formed per year at present is fixed by the observations, the number of faint stars which are now being formed per year is proportional to $f(1)$.

TABLE 6

CONSUMPTION AND EJECTION OF GAS AND THE GAS DENSITY ($\mathfrak{M}_\odot/\mathrm{pc}^2$)

t	$\mathfrak{M}_T(t)$			$\mathfrak{M}_E(t)$			$\mathfrak{M}_G(t)$		
	$n=0$	$n=1$	$n=2$	$n=0$	$n=1$	$n=2$	$n=0$	$n=1$	$n=2$
0	0	0	0	0	0	0	55	55	55
0.2	9.3	17 2	31 3	0 3	1.3	4.4	46.0	39.0	28.2
0.4	18.6	29.6	43 3	0.8	2.8	7.9	37.2	28.2	19.6
0.6	27.9	38.7	49 6	1.2	4 1	9.8	28.3	20.4	15.1
0 8	37.2	45.2	53 6	1.8	5.2	11.3	19.6	14.9	12.7
1	46.5	50 0	56 3	2.5	6.0	12.3	11	11	11
Present rate	46.5	20.1	11.3	3.0	3.5	3.7	−43.5	−16.6	− 7.6

The average age of unevolved stars, i.e., stars fainter than $M_v = +3.5$, is given in Table 7. It increases with n because, for larger values of n, star formation is shifted to the beginning. For $n = 2$ the average age is 75 per cent of the age of the Galaxy.

VIII. EXCHANGE OF GAS BETWEEN STARS AND INTERSTELLAR MEDIUM

The mass of gas used in the formation of stars, $\mathfrak{M}_T(t)$, the mass of gas ejected by evolving stars, $\mathfrak{M}_E(t)$, and the gas density of the interstellar medium, $\mathfrak{M}_G(t)$, are computed from equations (7)–(9). Table 6 gives the results.

It may be seen that for $n = 2$ the total mass of gas used up to now in formation of stars exceeds the total mass, which is 55 $\mathfrak{M}_\odot/\mathrm{pc}^2$. The last row of Table 6 gives the present rates of change per unit time (= the age of the Galaxy). Table 7 gives the rates of consumption and ejection per 10^9 years. The present rate of ejection is quite insensitive to the value of n, because a major part of the presently ejected material comes from stars brighter than $M_v = +1$, of which the present rate of formation does not depend on n, as discussed in the previous section.

IX. WHITE DWARFS

The number of white dwarfs is simply $\Sigma(\psi - \Phi)$; it is given in Table 7. These numbers cannot be compared with the observations at once, because the brightness of a white dwarf depends on its age. On the hypothesis of cooling at constant radius, Schwarzschild (1958) gives the time t_L in years required for a white dwarf to cool down from some high initial luminosity to luminosity L as

$$t_L = 1.2 \times 10^7 \frac{\mathfrak{M}}{L^{5/7}} . \tag{26}$$

Greenstein (1958) derived bolometric magnitudes for 27 white dwarfs, which fit well a linear relation with M_v, for $M_v < 14.5$,

$$M_v = 4.0 + 0.72 M_{\rm bol} . \tag{27}$$

TABLE 7

MISCELLANEOUS RESULTS

	$n=0$	$n=1$	$n=2$
Average age of faint stars (10^9 years)	4.0	5 0	6.0
Total consumption of gas ($\mathfrak{M}\odot$/pc^2)	46.5	50.0	56.3
Consumption of gas used in forming stars $M_v<3.5$ ($\mathfrak{M}\odot$/pc^2)	4.0	8.4	16.5
Total ejection of gas ($\mathfrak{M}\odot$/pc^2)	2.5	6.0	12.3
Present rate of consumption ($\mathfrak{M}\odot$/pc^2 per 10^9 years)	5 8	2.5	1.4
Present rate of ejection ($\mathfrak{M}\odot$/pc^2 per 10^9 years)	0.38	0.44	0.47
Total number of white dwarfs (pc^{-2})	0.96	2.26	4.77
Number of white dwarfs with $M_v<14$ (pc^{-2})	0.55	0.99	1.60
Number of white dwarfs with $M_v<12$ (pc^{-2})	0.09	0.13	0.15

TABLE 8

LUMINOSITY FUNCTION OF WHITE DWARFS (pc^{-2})

M_v	14.5	14.0	13.0	12.0	11.0	10.0	9.0	
$n=0$	0 19	0.21	0.32	0.14	0.056	0.022	0.009	0.005
$n=1$	0.71	0.56	0.63	0.22	0 078	0.030	0.012	0.007
$n=2$	2.16	1.01	1.15	0.30	0 095	0.035	0.013	0.009

Let us assume that a white dwarf with $M_{\rm bol} = 15.3$ has an age equal to the age of the Galaxy (according to eq. [26]; this would be 8.4×10^9 years if the mass is 0.7 $\mathfrak{M}\odot$). Then, if all white dwarfs have the same mass,

$$\log t_L = -6.0 + 0.4 M_v , \tag{28}$$

again for $M_v < 14.5$, with t_L now expressed in the age of the Galaxy; that is, for $M_v = 14.5$, $t_L = 0.63$; hence all white dwarfs which originated as such before $t = 0.37$ are now fainter than 14.5. The number of white dwarfs formed up to time t is

$$\sum \psi \int_0^{t-T} f(t)\, dt .$$

The computed distribution of absolute magnitudes of white dwarfs is given in Table 8. The number of faint white dwarfs is quite sensitive to n. Unfortunately, the observational material, because of the low luminosity of these objects, is meager. We cannot do much better than expect that all white dwarfs brighter than $M_v = 14$ are known within 5 pc and all those brighter than $M_v = 12$ within 10 pc. The observed numbers are 4 and 3, respectively. The distribution of these objects in z is completely unknown, but we would expect the equivalent width to be not much less than that of faint stars. Let us assume an equivalent width of 400–800 pc, in which cases the "observed" surface densities would be 3–6 for $M_v < 14$ and 0.3–0.6 for $M_v < 12$. This should be compared with the computed numbers given in Table 7. The case $n = 2$ comes closest to the observed numbers, but the uncertainty is large. The contribution of population II objects in the observed numbers should be small, most of them being fainter than $M_v = 14$.

X. THE ABUNDANCE OF HELIUM

In an extensive paper on the synthesis of elements in stars, Burbidge, Burbidge, Fowler, and Hoyle (1957) have shown that the element-building process occurring in stellar interiors and subsequent ejection may well be responsible for the presently observed abundances. It was stated that in the case of helium the necessary efficiency is near unity and that the estimates are not accurate enough to establish whether all helium may have been formed by nucleosynthesis in stellar interiors.

TABLE 9

THE ABUNDANCE OF HELIUM $Y(t)$

t	$n=0$ $E=1$	$n=1$ $E=1$	$n=2$ $E=0.53$
0.	0	0	0
0.2.	0.01	0.03	0.07
0.4.	.02	.08	.16
0.6.	.04	.13	.23
0.8.	.07	.19	.29
1.	0.12	0.25	0.34

We shall investigate this problem more in detail for the solar neighborhood. However, the mechanism for ejection of matter from evolving stars is still unknown, and so is the composition of the ejected material. Thus a theory on the enrichment of helium in the interstellar gas will have to be of a very schematic nature. We shall assume that at $t = 0$ the helium abundance of the interstellar gas, $Y(t)$, was zero. A star born at time $t - T$ will use an amount of helium $\mathfrak{M}Y(t - T)$ out of the interstellar gas. Its hydrogen mass is $\mathfrak{M} - \mathfrak{M}Y(t - T)$, if we neglect heavier elements. Now assume that out of this hydrogen mass a fraction E has been burned into helium at time t when ejection takes place and that the ejected material has the average composition of the whole star. Then the amount of helium ejected will be

$$(\mathfrak{M} - 0.7)\{Y(t-T) + E[1 - Y(t-T)]\}.$$

It was found that without substantial loss of accuracy we may replace the argument $t - T$ by t, in computing the rate of change of the mass of interstellar helium,

$$\frac{d\mathfrak{M}_{\mathrm{He}}(t)}{dt} = -\sum \psi \mathfrak{M} f(t)\, Y(t) + \sum \psi (\mathfrak{M} - 0.7)\, f(t-T)\, [E + (1-E)\, Y(t)]\,. \quad (29)$$

The value of E was chosen such that, after integration of equation (29), the present value of the helium abundance $Y(1)$ is 34 per cent, the value found by Mathis (1957) in the Orion Nebula. This was found to be impossible for the cases $n = 0$ and $n = 1$. For the case $n = 2$ a value $E = 0.53$ was found. Table 9 gives the change in helium abundance with time. It is interesting to see that at $t = 0.44$, i.e., 4.5×10^9 years ago, the helium abundance of the interstellar gas was 17 per cent for $n = 2$, in close agreement with about 20 per cent found by Schwarzschild, Howard, and Härm (1957) for the abundance of the initial composition of the sun.

XI. REGIONS NOT NEAR THE SUN, AND GALAXIES

Up to this point we have considered star formation in the neighborhood of the sun only. It is rather tempting to try to estimate the effects of star formation in regions elsewhere in the galactic system and in galaxies as a whole. Let us assume that both the initial luminosity function $\psi(M_v)$ and the way star formation varies with gas density are universal. The situation is the reverse of what we had before. Now the luminosity function $\Phi(M_v)$ and the present gas density $\mathfrak{M}_G(1)$ are unknown, while $\psi(M_v)$ and the constant C of equation (10) are known. In a region elsewhere in the galactic plane the total density will be a factor D larger than near the sun. This means that $\mathfrak{M}_G(0)$ in equation (11) should be replaced by $D\mathfrak{M}_G(0)$.

For the case $n = 1$ it may easily be seen that $f(t)$ is the same as in the solar neighborhood, the only change being that $\psi(M_v)$ is to be multiplied by D. Accordingly, the present percentage of gas would be the same as near the sun, or 20 per cent.

For the case $n = 2$, however, the rate function $f(t)$ will depend on D. An approximate solution gives

$$f(t) = \frac{5D}{[1 + (5D - 1)t]^2}. \qquad (30)$$

Accordingly, the gas density would be

$$\mathfrak{M}_G(t) = \frac{55D}{1 + (5D - 1)t}, \qquad (31)$$

or exactly 11 at $t = 1$. This would mean that for $n = 2$ the present gas density would be independent of D, which cannot be strictly true, of course. The approximation may be judged from one exact solution in which $\mathfrak{M}_G(0) = 550$, or $D = 10$. In this solution it was assumed that the factor K (cf. eq. [12]) was identical with 1. This will have rather exaggerated the ejection of gas from evolving stars, so that the result, $\mathfrak{M}_G(1) = 24$ or 4.4 per cent, must have been too large.

We thus see that while for $n = 1$ the percentage of gas should be constant over the galactic plane, for $n = 2$ (or larger) the gas density itself should vary only slightly. Observations of neutral hydrogen at 21-cm wave length (Schmidt 1957; Westerhout 1957) show that, except for spiral structure, the gas density between the sun and 3 kpc from the center is constant within a factor of 2. The total density increases by a factor of around 10 over this range (Schmidt 1956). This clearly indicates that n should be 2 or larger.

The number of bright stars observed in the luminosity function depends on the gas density only and thus will be roughly constant over the galactic plane. The faint end of the luminosity function is not at all constant but varies roughly with the total density. The ratio of faint to bright stars will increase toward the galactic center. Also the ratio of M67 type giants to, say, main-sequence A stars will increase, and, as a consequence, the central parts should be redder than the solar neighborhood. Also the helium abundance should increase toward the center.

Let us now consider galaxies as a whole. For the case $n = 2$ or higher, we would expect that in systems with high mean density, like the giant ellipticals, the present percentage of gas would be low, the abundance of helium large, and the color red. A system with low mean density like the Small Magellanic Cloud should have a larger percentage of gas, a small abundance of helium, and a bluish color. Those predictions which can be checked by observations seem to be confirmed. It is hoped to study the evolution of galaxies more in detail in the future.

XII. DISCUSSION

We have obtained the following evidence on the value of n:

a) The distribution of young objects perpendicular to the galactic plane suggests $n = 2$–3.

b) The assumption that the distribution of luminosities in open clusters is identical with the initial luminosity function leads to $n = 1$–2.

c) The number of white dwarfs observed leads to n at least 2, but the uncertainty is large.

d) The present abundance of helium in interstellar gas and the original helium abundance of the sun may be explained for $n = 2$ or larger, if the original matter out of which the Galaxy was formed contained no helium.

e) The roughly uniform distribution of hydrogen observed over the galactic plane requires a value of n around 2 or larger.

The combined evidence strongly suggests a value of n around 2. The only objection to a value larger than 2 stems from the luminosity function in open clusters.

We shall adopt $n = 2$, so that the rate of star formation varies with the square of the gas density. Some further results, in addition to the ones listed above, are given below.

f) The present rate of formation of stars is about five times slower than the average rate.

g) The present rate of gas consumption due to the formation of stars is 1.4 $\mathfrak{M}_\odot/\mathrm{pc}^2$ per 10^9 years. The present rate of ejection of gas by evolving stars is a little less than 0.5 $\mathfrak{M}_\odot/\mathrm{pc}^2$ per 10^9 years.

h) The mean density of a galaxy may determine, as a first approximation, its present gas content and thus its evolution stage.

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