SOLUTION OF THE PROBLEM OF ARTIFICIAL SATELLITE THEORY WITHOUT DRAG

By DIRK BROUWER
Yale University Observatory

Received October 27, 1959

Abstract. Sections 1–6 give the solution of the main problem for a spheroidal earth with the potential limited to the principal term and the second harmonic which contains the small factor k_2. The solution is developed in powers of k_2 in canonical variables by a method which is basically the same as that used in treating a different problem by von Zeipel (1916). The periodic terms are divided in two classes: the short-period terms contain the mean anomaly in their arguments; the arguments of the long-period terms are multiples of the mean argument of the perigee.

The periodic terms, both of short and long period, are developed to $O(k_2)$; the secular motions are obtained to $O(k_2^2)$. The results are obtained in closed form; no series developments in eccentricity or inclination arise. The solution does not apply to orbits near the critical inclination, 63°.4, but is otherwise valid for any eccentricity < 1 and any inclination.

Section 7 gives the long-period terms and the additions to the secular motions caused by the fourth harmonic in the potential; section 8 gives the contributions by the third and fifth harmonics; section 9 contains formulas for computation.

1. *The equations of motion.* The equations of motion of a small mass attracted by a spheroid are

$$\frac{d^2x}{dt^2} = \frac{\partial U}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial U}{\partial y}, \quad \frac{d^2z}{dt^2} = \frac{\partial U}{\partial z},$$

with

$$U = \frac{\mu}{r} + \frac{\mu k_2}{r^3}(1 - 3\sin^2\beta) + \frac{\mu k_4}{r^5}\left(1 - 10\sin^2\beta + \frac{35}{3}\sin^4\beta\right) + \cdots.$$

The equatorial plane of the spheroid is taken as the xy plane; β is the latitude, and if M is the mass of the spheroid and k the Gaussian constant, $\mu = k^2M$.

The main problem of artificial satellite theory may be considered the problem with $k_4 = 0$. Let I be the instantaneous inclination of the orbital plane with the equatorial plane, g the distance of the pericenter from the ascending node, and f the true anomaly. Then

$$\sin\beta = \sin I \sin(g+f),$$
$$2\sin^2\beta = \sin^2 I\,[1 - \cos(2g+2f)],$$

and the disturbing function can be written

$$R = \frac{\mu k_2}{a^3}\left[(-\tfrac{1}{2} + \tfrac{3}{2}\cos^2 I)\frac{a^3}{r^3} + (\tfrac{3}{2} - \tfrac{3}{2}\cos^2 I)\frac{a^3}{r^3}\cos(2g+2f)\right].$$

Let a, e be the osculating semi-major axis and eccentricity, respectively. The Delaunay variables are then

$$L = (\mu a)^{\frac{1}{2}}, \qquad l = \text{mean anomaly},$$
$$G = L(1-e^2)^{\frac{1}{2}}, \qquad g = \text{argument of the pericenter},$$
$$H = G\cos I, \qquad h = \text{longitude of ascending node}.$$

With these variables the equations become

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l}, & \frac{dl}{dt} &= -\frac{\partial F}{\partial L}, \\ \frac{dG}{dt} &= \frac{\partial F}{\partial g}, & \frac{dg}{dt} &= -\frac{\partial F}{\partial G}, \\ \frac{dH}{dt} &= \frac{\partial F}{\partial h}, & \frac{dh}{dt} &= -\frac{\partial F}{\partial H}, \end{aligned} \tag{1}$$

$$F = \frac{\mu^2}{2L^2} + \frac{\mu^4 k_2}{L^6}\left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3} + \left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G^2}\right)\frac{a^3}{r^3}\cos(2g+2f)\right]. \tag{2}$$

The parts of F not exhibited in terms of the Delaunay variables may be expanded in Fourier series as follows:

$$\frac{a^3}{r^3} = \frac{L^3}{G^3} + \sum_{j=1}^{\infty} 2P_j \cos jl \equiv \frac{L^3}{G^3} + \sigma_1,$$

$$\frac{a^3}{r^3}\cos(2g+2f) = \sum_{j=-\infty}^{+\infty} Q_j \cos(2g+jl) \equiv \sigma_2.$$

The coefficients P_j, Q_j are power series in the eccentricity e; the lowest power of e is e^j in P_j; $e^{|j-2|}$ in Q_j, $j \neq 0$. Since $e^2 = 1 - G^2/L^2$, the derivatives of a function $\psi(e)$ with respect to L or G may be obtained by

$$\frac{\partial\psi}{\partial L} = \frac{1}{e}\frac{\partial\psi}{\partial e}\frac{G^2}{L^3},$$
$$\frac{\partial\psi}{\partial G} = -\frac{1}{e}\frac{\partial\psi}{\partial e}\frac{G}{L^2}. \tag{3}$$

The functions ψ that will arise are functions of a/r and f. Well-known formulas of elliptic motion are

$$\frac{\partial}{\partial e}\left(\frac{a}{r}\right) = \frac{a^2}{r^2}\cos f, \tag{4}$$

$$\frac{\partial f}{\partial e} = \left(\frac{a}{r} + \frac{L^2}{G^2}\right)\sin f. \tag{5}$$

Some important properties of the Hamiltonian F may be noted:

(a) the independent variable t is not explicitly present in F, hence the integral F = constant exists;
(b) the variable h is not present in F;
(c) the coefficient Q_0 in σ_2 is zero.

The value of this coefficient is the constant term of the expansion in terms of the mean anomaly of $a^3r^{-3}\cos 2f$: hence

$$Q_0 = \frac{1}{\pi}\int_0^\pi \frac{a^3}{r^3}\cos 2f dl.$$

By making use of the integral of areas in the form

$$dl = \frac{L}{G}\frac{r^2}{a^2}df$$

this may be written

$$\begin{aligned} Q_0 &= \frac{1}{\pi}\frac{L}{G}\int_0^\pi \frac{a}{r}\cos 2f df \\ &= \frac{1}{\pi}\frac{L^3}{G^3}\int_0^\pi (1 + e\cos f)\cos 2f df \\ &= 0. \end{aligned}$$

This derivation may be generalized to show that the constant term of $(a^p/r^p)\cos qf$ (p and q positive integers), expanded in terms of the mean anomaly, is zero if $q > p - 2 \geqq 0$.

2. *Outline of the method of solution.* Consider a transformation from the variables L, G, H, l, g, h to new variables L', G', H', l', g', h', with the aid of a determining function $S(L', G', H', l, g, h)$. Then, if

$$L = \frac{\partial S}{\partial l}, \quad G = \frac{\partial S}{\partial g}, \quad H = \frac{\partial S}{\partial h},$$
$$l' = \frac{\partial S}{\partial L'}, \quad g' = \frac{\partial S}{\partial G'}, \quad h' = \frac{\partial S}{\partial H'},$$

the equations in the new variables will be

$$\begin{aligned} \frac{dL'}{dt} &= \frac{\partial F^*}{\partial l'}, & \frac{dl'}{dt} &= -\frac{\partial F^*}{\partial L'}, \\ \frac{dG'}{dt} &= \frac{\partial F^*}{\partial g'}, & \frac{dg'}{dt} &= -\frac{\partial F^*}{\partial G'}, \\ \frac{dH'}{dt} &= \frac{\partial F^*}{\partial h'}, & \frac{dh'}{dt} &= -\frac{\partial F^*}{\partial H'}, \end{aligned} \tag{7}$$

with

$$F^*(L', G', H', l', g', —) = F(L, G, H, l, g, —) \tag{8}$$

The dashes in the places for h' and h are used to indicate the absence of these variables.

The problem would be completely solved if a determining function were found such that F^* is a function of the variables L', G', H' only. The differential equations (7) show that then L', G', H' are constants, while l', g', h' are linear functions of the time. Substitution of this solution of the primed quantities into (6) then yields expressions from which the original variables may be obtained in terms of L', G', H', l', g', h' and therefore in terms of t and the constants of integration.

In the present problem it is more convenient to choose the determining function S in such a manner that l' is not present in F^*, while g' is permitted to appear. If this is accomplished, L' and H' will be constants, and the system is essentially reduced to one of one degree of freedom:

$$\frac{dG'}{dt} = \frac{\partial F^*}{\partial g'}, \quad \frac{dg'}{dt} = -\frac{\partial F^*}{\partial G'}$$

in which L' and H' are present as constants. After this system is solved, l', h' are obtained by quadratures from

$$\frac{dl'}{dt} = -\frac{\partial F^*}{\partial L'}, \quad \frac{dh'}{dt} = -\frac{\partial F^*}{\partial H'}.$$

Preferably a canonical transformation by the choice of a suitable determining function leads to an equivalent result.

3. *First-order solution.* The determining function is obtained by a method used by H. v. Zeipel (1916) in a qualitative study of the motions of minor planets. In this procedure a development in powers of the small parameter k_2 is introduced. If F_1 is written for R, the original Hamiltonian is

$$F = F_0 + F_1,$$

in which the subscript denotes the power of k_2 present as a factor. F_0 is a function of L only. Let also

$$S = S_0 + S_1 + S_2 \cdots,$$
$$F^* = F_0^* + F_1^* + F_2^* \cdots.$$

It will be convenient to choose

$$S_0 = L'l + G'g + H'h.$$

If then the expressions for L, G, H, l', g', h' given by (6) are substituted into

$$F(L, G, H, l, g, —) = F^*(L', G', H', —, g', —),$$

there results

$$F_0\left(\frac{\partial S}{\partial l}\right) + F_1\left(\frac{\partial S}{\partial l}, \frac{\partial S}{\partial g}, \frac{\partial S}{\partial h}, l, g, —\right)$$
$$= F_0^* + F_1^*\left(L', G', H', —, \frac{\partial S}{\partial G'}, —\right)$$
$$+ F_2^*\left(L', G', H', —, \frac{\partial S}{\partial G'}, —\right).$$

Expanding everywhere by Taylor's theorem to the second power of k_2, the result is

$$F_0(L') + \frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l}$$
$$+ \frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2$$
$$+ F_1(L', G', H', l, g, —)$$
$$+ \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g}$$
$$= F_0^* + F_1^*(L', G', H', —, g, —)$$
$$+ \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'} + F_2^*(L', G', H', —, g, —).$$

Parts of corresponding order in k_2 on both sides yield:

order 0,

$$F_0(L') = F_0^*(L'); \tag{9}$$

order 1,

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + F_1 = F_1^*; \tag{10}$$

order 2,

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_2}{\partial l} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 + \frac{\partial F_1}{\partial L'}\frac{\partial S_1}{\partial l}$$
$$+ \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g} = F_2^* + \frac{\partial F_1^*}{\partial g}\frac{\partial S_1}{\partial G'}. \tag{11}$$

The expansion could be carried on indefinitely, but for current practical requirements consideration of parts beyond the second order in k_2 appears to be unnecessary. Put

$$F_1 = F_{1s} + F_{1p},$$

in which F_{1s} is the part independent of l, F_{1p} the part dependent on l. In the terminology of planetary theory F_{1s} would be the secular part of the disturbing function, F_{1p} the periodic part. Hence, if for the sake of brevity the notation

$$A = -\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}, \quad B = +\frac{3}{2} - \frac{3}{2}\frac{H^2}{G^2} \tag{12}$$

is used,

$$F_{1s} = \frac{\mu^4 k_2}{L^3 G^3} A,$$

$$F_{1p} = \frac{\mu^4 k_2}{L^6} (A\sigma_1 + B\sigma_2).$$

Now equation (10) is split up into two equations

$$\begin{aligned} \frac{\partial S_1}{\partial l} &= \frac{\mu^2 k_2}{L'^3} (A\sigma_1 + B\sigma_2), \\ F_1^* &= \frac{\mu^4 k_2}{L'^3 G'^3} A, \end{aligned} \tag{13}$$

in which use is made of

$$\frac{\partial F_0}{\partial L'} = -\frac{\mu^2}{L'^3}.$$

By integration S_1 may be obtained in the form of an infinite series as

$$S_1 = \frac{\mu^2 k_2}{L'^3}\left[A \sum_1^\infty \frac{2}{j} P_j \sin jl\right.$$
$$\left. + B \sum_{-\infty}^{+\infty} \frac{1}{j} Q_j \sin(2g + jl)\right]. \tag{14}$$

No constants of integration are required, since only partial derivatives of S_1 with respect to L', G', H', l, g, h will be needed.

For the following developments it will be advantageous to obtain a closed expression for S_1. The method used at the end of Section 2 for obtaining the integral of functions of the type $a^p r^{-p} \cos qf$ can be used to obtain

$$\int \sigma_1 dl = \int \left(\frac{a^3}{r^3} - \frac{L^3}{G^3}\right) dl = \frac{L^3}{G^3}[f - l + e \sin f],$$

$$\int \sigma_2 dl = \int \frac{a^3}{r^3} \cos(2g + 2f) dl = \frac{L^3}{G^3}\left[\frac{1}{2}\sin(2g + 2f) + \frac{e}{2}\sin(2g + f) + \frac{e}{6}\sin(2g + 3f)\right].$$

Hence

$$S_1 = \frac{\mu^2 k_2}{G'^3}\left\{A(f - l + e \sin f) + B\left[\frac{1}{2}\sin(2g + 2f) + \frac{e}{2}\sin(2g + f) + \frac{e}{6}\sin(2g + 3f)\right]\right\}, \quad (15)$$

$$\frac{\partial S_1}{\partial g} = \frac{\mu^2 k_2}{G'^3} B\left[\cos(2g + 2f) + e\cos(2g + f) + \frac{e}{3}\cos(2g + 3f)\right]. \quad (16)$$

Introduce

$$\gamma_2 = \frac{\mu^2 k_2}{L'^4};$$

then

$$L = L' + \frac{\partial S_1}{\partial l} = L'\left\{1 + \gamma_2\left[\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^3}{r^3} - \frac{L'^3}{G'^3}\right) + \left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G'^2}\right)\frac{a^3}{r^3}\cos(2g + 2f)\right]\right\}, \quad (17)$$

$$G = G' + \frac{\partial S_1}{\partial g}, = G'\left\{1 + \gamma_2 \frac{L'^4}{G'^4}\left(\frac{3}{2} - \frac{3}{2}\frac{H^2}{G'^2}\right)\left[\cos(2g + 2f) + e\cos(2g + f) + \frac{e}{3}\cos(2g + 3f)\right]\right\} \quad (18)$$

$$H = H'. \quad (19)$$

S_1 was obtained as a function of G', H, e, f, g. Hence L' is present through e and f only. To obtain $\partial S_1/\partial L'$ it is convenient to obtain first

$$\frac{\partial S_1}{\partial e} = \frac{\mu^2 k_2}{G'^3}\left\{A\left[(1 + e\cos f)\frac{\partial f}{\partial e} + \sin f\right] + B\left[\cos(2g + 2f)(1 + e\cos f)\frac{\partial f}{\partial e} + \frac{1}{2}\sin(2g + f) + \frac{1}{6}\sin(2g + 3f)\right]\right\},$$

in which use was made of

$$\cos(2g + 2f) + \frac{e}{2}\cos(2g + f) + \frac{e}{2}\cos(2g + 3f) = \cos(2g + 2f)(1 + e\cos f).$$

With

$$(1 + e\cos f)\frac{\partial f}{\partial e} = \left(\frac{a^2}{r^2}\frac{G^2}{L^2} + \frac{a}{r}\right)\sin f$$

there results

$$\frac{\partial S_1}{\partial e} = \frac{\mu^2 k_2}{G'^3}\left\{A\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2} + \frac{a}{r} + 1\right)\sin f + \tfrac{1}{2}B\left[\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2} - \frac{a}{r} + 1\right)\sin(2g + f) + \left(\frac{a^2}{r^2}\frac{G'^2}{L'^2} + \frac{a}{r} + \frac{1}{3}\right)\sin(2g + 3f)\right]\right\}.$$

With the aid of the relations (3) there follows at once

$$\begin{aligned} l &= l' - \frac{\partial S_1}{\partial L'} \\ &= l' - \frac{\gamma_2}{e}\frac{L'}{G'}\left\{\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f + \left(\frac{3}{4}-\frac{3}{4}\frac{H^2}{G'^2}\right)\right. \\ &\qquad \left.\times\left[\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)+\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)\right]\right\}, \end{aligned} \tag{20}$$

$$\begin{aligned} g &= g' - \frac{\partial S_1}{\partial G'} \\ &= g' + \frac{\gamma_2}{e}\frac{L'^2}{G'^2}\left\{\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f + \left(\frac{3}{4}-\frac{3}{4}\frac{H^2}{G'^2}\right)\right. \\ &\qquad \left.\times\left[\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)+\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)\right]\right\} \\ &\quad + \gamma_2\frac{L'^4}{G'^4}\left\{\left(-\frac{3}{2}+\frac{15}{2}\frac{H^2}{G'^2}\right)(f-l+e\sin f)+\left(\frac{9}{2}-\frac{15}{2}\frac{H^2}{G'^2}\right)\right. \\ &\qquad \left.\times\left[\frac{1}{2}\sin(2g+2f)+\frac{e}{2}\sin(2g+f)+\frac{e}{6}\sin(2g+3f)\right]\right\}, \end{aligned} \tag{21}$$

$$\begin{aligned} h &= h' - \frac{\partial S_1}{\partial H} \\ &= h' - 3\gamma_2\frac{L'^4}{G'^4}\left[f-l+e\sin f-\frac{1}{2}\sin(2g+2f)-\frac{e}{2}\sin(2g+f)-\frac{e}{6}\sin(2g+3f)\right]\frac{H}{G'}. \end{aligned} \tag{22}$$

In the calculation of the coordinates, l and g are not needed separately but only the sum:

$$\begin{aligned} l+g &= l'+g'+\frac{\gamma_2}{e}\left(\frac{L'^2}{G'^2}-\frac{L'}{G'}\right)\left\{\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+1\right)\sin f+\left(\frac{3}{4}-\frac{3}{4}\frac{H^2}{G'^2}\right)\right. \\ &\qquad \left.\times\left[\left(-\frac{a^2}{r^2}\frac{G'^2}{L'^2}-\frac{a}{r}+1\right)\sin(2g+f)+\left(\frac{a^2}{r^2}\frac{G'^2}{L'^2}+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)\right]\right\} \\ &\quad + \gamma_2\frac{L'^4}{G'^4}\left\{\left(-\frac{3}{2}+\frac{15}{2}\frac{H^2}{G'^2}\right)(f-l+e\sin f)+\left(\frac{9}{2}-\frac{15}{2}\frac{H^2}{G'^2}\right)\right. \\ &\qquad \left.\times\left[\frac{1}{2}\sin(2g+2f)+\frac{e}{2}\sin(2g+f)+\frac{e}{6}\sin(2g+3f)\right]\right\}. \end{aligned} \tag{23}$$

Since $(1-e^2)^{-1}-(1-e^2)^{-\frac{1}{2}}$ is divisible by e^2, the first part of the difference $(l+g)-(l'+g')$ has e as a factor and not as a divisor.

4. *The second-order terms.* For use in the evaluation of $F_2{}^*$ it will be convenient to put

$$\rho_2 = \frac{L'^3}{G'^3}\left[\cos(2g+2f)+e\cos(2g+f)+\frac{e}{3}\cos(2g+3f)\right], \tag{24}$$

so that

$$\frac{\partial S_1}{\partial g} = \frac{\mu^2 k_2}{L'^3}B\rho_2.$$

Also, define τ_1, τ_2 by

$$\tau_1 = \frac{1}{e}\frac{\partial\sigma_1}{\partial e} = \frac{3}{e}\frac{a^4}{r^4}\cos f - 3\frac{L^5}{G^5},$$

$$\tau_2 = \frac{1}{e}\frac{\partial\sigma_2}{\partial e} = \frac{1}{e}\left[\frac{1}{2}\frac{a^4}{r^4} - \frac{a^3}{r^3}\frac{L^2}{G^2}\right]\cos(2g+f) + \frac{1}{e}\left[\frac{5}{2}\frac{a^4}{r^4} + \frac{a^3}{r^3}\frac{L^2}{G^2}\right]\cos(2g+3f), \tag{25}$$

which are easily obtained with the aid of the expressions (4) and (5). Then

$$\frac{\partial F_{1p}}{\partial L'} = \frac{\mu^4 k_2}{L'^7}\left[-6(A\sigma_1 + B\sigma_2) + \frac{G'^2}{L'^2}(A\tau_1 + B\tau_2)\right], \tag{26}$$

$$\frac{\partial F_{1p}}{\partial G'} = -\frac{\mu^4 k_2}{L'^7}\left[\frac{G'}{L'}(A\tau_1 + B\tau_2) + \frac{3L'}{G'}\frac{H^2}{G'^2}(\sigma_1 - \sigma_2)\right]. \tag{27}$$

If the terms dependent on l are not needed to the second order in k_2, the only part of interest in (11) will be

$$\left[\frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 + \frac{\partial F_{1p}}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_1}{\partial G'}\frac{\partial S_1}{\partial g}\right]_s = F_2{}^*,$$

where the subscript s designates the parts independent of l. The contribution to the right-hand member that has $\partial F_1{}^*/\partial g$ as a factor vanishes because $F_1{}^*$ is a function of L', G', H' only.

The following products are easily obtained:

$$\frac{1}{2}\frac{\partial^2 F_0}{\partial L'^2}\left(\frac{\partial S_1}{\partial l}\right)^2 = \frac{3}{2}\frac{\mu^6 k_2^2}{L'^{10}}(A\sigma_1 + B\sigma_2)^2$$

$$\frac{\partial F_{1p}}{\partial L'}\frac{\partial S_1}{\partial l} = \frac{\mu^6 k_2^2}{L'^{10}}\left[-6(A\sigma_1 + B\sigma_2)^2 + \frac{G'^2}{L'^2}(A\sigma_1 + B\sigma_2)(A\tau_1 + B\tau_2)\right]$$

$$\frac{\partial F_{1p}}{\partial G'}\frac{\partial S_1}{\partial g} = \frac{\mu^6 k_2^2}{L'^{10}}\left[3\frac{L'}{G'}\frac{H^2}{G'^2}B(-\sigma_1\rho_2 + \sigma_2\rho_2) - \frac{G'}{L'}(AB\tau_1\rho_2 + B^2\tau_2\rho_2)\right]$$

$$\frac{\partial F_{1s}}{\partial G'}\frac{\partial S_1}{\partial g} = \frac{\mu^6 k_2^2}{L'^{10}}\left[-3\frac{L'^4}{G'^4}\left(AB + B\frac{H^2}{G'^2}\right)\rho_2\right]$$

and hence,

$$\begin{aligned} F_2{}^* = \frac{\mu^6 k_2^2}{L'^{10}}\Bigg\{ & A^2\left(-\frac{9}{2}\sigma_1^2 + \frac{G'^2}{L'^2}\sigma_1\tau_1\right) \\ & + AB\left[-9\sigma_1\sigma_2 + \frac{G'^2}{L'^2}(\sigma_1\tau_2 + \sigma_2\tau_1) - \frac{G'}{L'}\tau_1\rho_2 - 3\frac{L'^4}{G'^4}\rho_2\right] \\ & + B^2\left(-\frac{9}{2}\sigma_2^2 + \frac{G'^2}{L'^2}\sigma_2\tau_2 - \frac{G'}{L'}\tau_2\rho_2\right) \\ & + B\frac{H^2}{G'^2}\left[\frac{L'}{G'}(-3\sigma_1\rho_2 + 3\sigma_2\rho_2) - 3\frac{L'^4}{G'^4}\rho_2\right]\Bigg\}_s. \end{aligned} \tag{28}$$

The parts independent of l may be found by evaluating integrals of the same type as those which occurred in the preceding section. The only exceptions are the integrals

$$\frac{e}{\pi}\int_0^\pi \cos f dl = -e^2$$

$$\frac{1}{\pi}\int_0^\pi \cos 2f dl = \frac{1}{e^2}\left[2\frac{G^3}{L^3} - 3\frac{G^2}{L^2} + 1\right]$$

$$\frac{e}{\pi}\int_0^\pi \cos 3f dl = -\frac{4}{e^2}\left[2\frac{G^3}{L^3} - 3\frac{G^2}{L^2} + 1\right] + 3e^2.$$

The separate parts of F_2^* are as follows:

constant part:

$$
\begin{aligned}
-\frac{9}{2}\sigma_1^2 &= -\frac{27}{16}\frac{L^5}{G^5} + \frac{9}{2}\frac{L^6}{G^6} + \frac{135}{8}\frac{L^7}{G^7} - \frac{315}{16}\frac{L^9}{G^9} \\
+\frac{G^2}{L^2}\sigma_1\tau_1 &= +\frac{15}{16} \quad -3 \quad -\frac{105}{8} \quad +\frac{315}{16} \\
\text{Sum} &= -\frac{3}{4}\frac{L^5}{G^5} + \frac{3}{2}\frac{L^6}{G^6} + \frac{15}{4}\frac{L^7}{G^7} \\
-\frac{9}{2}\sigma_2^2 &= -\frac{27}{32}\frac{L^5}{G^5} + \frac{135}{16}\frac{L^7}{G^7} - \frac{315}{32}\frac{L^9}{G^9} \\
+\frac{G^2}{L^2}\sigma_2\tau_2 &= +\frac{15}{32} \quad -\frac{105}{16} \quad +\frac{315}{32} \\
-\frac{G}{L}\tau_2\rho_2 &= +\frac{2}{3} \quad -\frac{5}{2} \\
\text{Sum} &= +\frac{7}{24}\frac{L^5}{G^5} - \frac{5}{8}\frac{L^7}{G^7} \\
+3\frac{L}{G}\sigma_2\rho_2 &= -\frac{L^5}{G^5} + \frac{5}{2}\frac{L^7}{G^7}
\end{aligned}
$$

coefficient of $\cos 2g$:

$$
\begin{aligned}
-9\sigma_1\sigma_2 &= -\frac{9}{4}\frac{L^5}{G^5} + 18\frac{L^7}{G^7} - \frac{63}{4}\frac{L^9}{G^9} \\
+\frac{G^2}{L^2}(\sigma_1\tau_2+\sigma_2\tau_1) &= +\frac{5}{4} \quad -14 \quad +\frac{63}{4} \\
-\frac{G}{L}\tau_1\rho_2 &= +\frac{5}{4} \quad -\frac{21}{4} \qquad +2\frac{L^5}{G^5}\frac{L}{L+G} \\
-3\frac{L^4}{G^4}\rho_2 &= \quad +1 \qquad -2 \\
\text{Sum} &= +\frac{1}{4}\frac{L^5}{G^5} - \frac{1}{4}\frac{L^7}{G^7} \\
-3\frac{L}{G}\sigma_1\rho_2 &= +\frac{3}{2}\frac{L^5}{G^5} - \frac{5}{2}\frac{L^7}{G^7} \qquad +2\frac{L^5}{G^5}\frac{L}{L+G} \\
-3\frac{L^4}{G^4}\rho_2 &= \quad +1 \qquad -2 \\
\text{Sum} &= +\frac{3}{2}\frac{L^5}{G^5} - \frac{3}{2}\frac{L^7}{G^7}
\end{aligned}
$$

coefficient of $\cos 4g$:

$$
\begin{aligned}
-\frac{9}{2}\sigma_2^2 &= -\frac{9}{64}\frac{L^5}{G^5} + \frac{9}{32}\frac{L^7}{G^7} - \frac{9}{64}\frac{L^9}{G^9} \\
+\frac{G^2}{L^2}\sigma_2\tau_2 &= +\frac{5}{64} \quad -\frac{7}{32} \quad +\frac{9}{64} \\
-\frac{G}{L}\tau_2\rho_2 &= +\frac{1}{16} \quad -\frac{1}{16} \\
\text{Sum} &= 0 \\
+3\frac{L}{G}\sigma_2\rho_2 &= 0
\end{aligned}
$$

The terms with cos 4g disappear, and the expression for F_2^* becomes, after substitution of the appropriate expressions for A^2, AB, B^2 and BH^2/G^2,

$$F_2^* = \frac{\mu^6 k_2^2}{L'^{10}}\left[+\frac{15}{32}\frac{L'^5}{G'^5}\left(1-\frac{18}{5}\frac{H^2}{G'^2}+\frac{H^4}{G'^4}\right)\right.$$
$$\left.+\frac{3}{8}\frac{L'^6}{G'^6}\left(1-6\frac{H^2}{G'^2}+9\frac{H^4}{G'^4}\right)-\frac{15}{32}\frac{L'^7}{G'^7}\left(1-2\frac{H^2}{G'^2}-7\frac{H^4}{G'^4}\right)\right]$$
$$+\frac{\mu^6 k_2^2}{L'^{10}}\left[-\frac{3}{16}\left(\frac{L'^5}{G'^5}-\frac{L'^7}{G'^7}\right)\left(1-16\frac{H^2}{G'^2}+15\frac{H^4}{G'^4}\right)\right]\cos 2g', \quad (29)$$

in which g has been changed into g', which is permissible in a part that has k_2^2 as a factor.

5. *Secular and long-period terms.* The problem is now reduced to the solution of the system of canonical equations with the Hamiltonian

$$F^* = \frac{\mu^2}{2L'^2}+\frac{\mu^4 k_2}{L'^3 G'^3}\left(-\frac{1}{2}+\frac{3}{2}\frac{H^2}{G'^2}\right)+F_2^*,$$

in which

$$F_2^* = F_{2s}^* + F_{2p}^*,$$

the former being a function of L', G', H only, the latter depending also on g'.

Let S^* be a new determining function,

$$S^* = L''l' + G''g' + H''h' + S_1^*(L'', G'', H'', g');$$

the equation $F^* = F^{**}$ may then be written

$$F_0^* + F_1^*\left(L'', G''+\frac{\partial S_1^*}{\partial g'}, H''\right) + F_{2s}^* + F_{2p}^* = F_0^{**} + F_1^{**} + F_2^{**}.$$

This equation reduces to

$$F_0^* = F_0^{**},$$
$$F_1^* = F_1^{**},$$
$$\frac{\partial F_1^*}{\partial G''}\frac{\partial S_1^*}{\partial g'} + F_{2p}^* = 0, \quad (33)$$
$$F_{2s}^* = F_2^{**}. \quad (34)$$

Since

$$\frac{\partial F_1^*}{\partial G''} = \frac{3}{2}\frac{\mu^4 k_2}{L'^3 G''^4}\left(1-5\frac{H^2}{G''^2}\right),$$

in which L' has been written for L'', H for H'', equation (33) gives

$$\frac{\partial S_1^*}{\partial g'} = G''\gamma_2\left[\frac{1}{8}\left(\frac{L'^2}{G''^2}-\frac{L'^4}{G''^4}\right)\left(1-16\frac{H^2}{G''^2}+15\frac{H^4}{G''^4}\right)\right]\left(1-5\frac{H^2}{G''^2}\right)^{-1}\cos 2g'.$$

Substituted into

$$G' = G'' + \frac{\partial S_1^*}{\partial g'},$$

there results

$$G' = G''\left\{1+\gamma_2\left(\frac{L'^2}{G''^2}-\frac{L'^4}{G''^4}\right)\left[\frac{1}{8}\left(1-11\frac{H^2}{G''^2}\right)-5\frac{H^4}{G''^4}\left(1-5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 2g''\right\}. \quad (35)$$

Also

$$S_1^* = G''\gamma_2\left(\frac{L'^2}{G''^2}-\frac{L'^4}{G''^4}\right)\left[\frac{1}{16}\left(1-11\frac{H^2}{G''^2}\right)-\frac{5}{2}\frac{H^4}{G''^4}\left(1-5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 2g''$$

and

$$l' = l'' - \frac{\partial S_1^*}{\partial L'}$$
$$= l'' + \gamma_2\frac{L'}{G''}\left[\frac{1}{8}\left(1-11\frac{H^2}{G''^2}\right)-5\frac{H^4}{G''^4}\left(1-5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 2g'', \quad (36)$$

$$
\begin{aligned}
g' &= g'' - \frac{\partial S_1^*}{\partial G''} \\
&= g'' + \gamma_2 \left[\frac{1}{16} \frac{L'^2}{G''^2} \left(1 - 33 \frac{H^2}{G''^2}\right) - \frac{3}{16} \frac{L'^4}{G''^4} \left(1 - \frac{55}{3} \frac{H^2}{G''^2}\right) \right. \\
&\qquad + \left(- \frac{25}{2} \frac{L'^2}{G''^2} + \frac{35}{2} \frac{L'^4}{G''^4} \right) \frac{H^4}{G''^4} \left(1 - 5 \frac{H^2}{G''^2}\right)^{-1} \\
&\qquad \left. - 25 \left(\frac{L'^2}{G''^2} - \frac{L'^4}{G''^4} \right) \frac{H^6}{G''^6} \left(1 - 5 \frac{H^2}{G''^2}\right)^{-2} \right] \sin 2g'', \qquad (37)
\end{aligned}
$$

$$
\begin{aligned}
h' &= h'' - \frac{\partial S_1^*}{\partial H} \\
&= h'' + \gamma_2 \left(\frac{L'^2}{G''^2} - \frac{L'^4}{G''^4} \right) \\
&\qquad \times \left[\frac{11}{8} \frac{H}{G''} + 10 \frac{H^3}{G''^3} \left(1 - 5 \frac{H^2}{G''^2}\right)^{-1} + 25 \frac{H^5}{G''^5} \left(1 - 5 \frac{H^2}{G''^2}\right)^{-2} \right] \sin 2g'', \qquad (38)
\end{aligned}
$$

in which g' has been changed into g''.

The Hamiltonian F^{**} is a function of L', G'', H only. It is given by equations (32) and (34),

$$
F^{**} = \frac{\mu^2}{2L'^2} + \frac{\mu^4 k_2}{L'^3 G''^3} \left(- \frac{1}{2} + \frac{3}{2} \frac{H^2}{G''^2} \right) + F_2^{**},
$$

where the last term is the first part of the right hand member of equation (29). Let n_0 be defined by

$$
n_0 = \frac{\mu^2}{L'^3}
$$

Then:

$$
\begin{aligned}
\frac{dl''}{dt} &= - \frac{\partial F^{**}}{\partial L'} \\
&= n_0 \left\{ 1 + 3\gamma_2 \frac{L'^3}{G''^3} \left(- \frac{1}{2} + \frac{3}{2} \frac{H^2}{G''^2} \right) \right. \\
&\qquad + \gamma_2^2 \left[\frac{75}{32} \frac{L'^5}{G''^5} + \frac{3}{2} \frac{L'^6}{G''^6} - \frac{45}{32} \frac{L'^7}{G''^7} + \left(- \frac{135}{16} \frac{L'^5}{G''^5} - 9 \frac{L'^6}{G''^6} + \frac{45}{16} \frac{L'^7}{G''^7} \right) \frac{H^2}{G''^2} \right. \\
&\qquad \left. \left. + \left(\frac{75}{32} \frac{L'^5}{G''^5} + \frac{27}{2} \frac{L'^6}{G''^6} + \frac{315}{32} \frac{L'^7}{G''^7} \right) \frac{H^4}{G''^4} \right] \right\}, \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dg''}{dt} &= - \frac{\partial F^{**}}{\partial G''} \\
&= n_0 \left\{ 3\gamma_2 \frac{L'^4}{G''^4} \left(- \frac{1}{2} + \frac{5}{2} \frac{H^2}{G''^2} \right) \right. \\
&\qquad + \gamma_2^2 \left[\frac{75}{32} \frac{L'^6}{G''^6} + \frac{9}{4} \frac{L'^7}{G''^7} - \frac{105}{32} \frac{L'^8}{G''^8} + \left(- \frac{189}{16} \frac{L'^6}{G''^6} - 18 \frac{L'^7}{G''^7} + \frac{135}{16} \frac{L'^8}{G''^8} \right) \frac{H^2}{G''^2} \right. \\
&\qquad \left. \left. + \left(\frac{135}{32} \frac{L'^6}{G''^6} + \frac{135}{4} \frac{L'^7}{G''^7} + \frac{1155}{32} \frac{L'^8}{G''^8} \right) \frac{H^4}{G''^4} \right] \right\}, \qquad (40)
\end{aligned}
$$

$$\frac{dh''}{dt} = -\frac{\partial F^{**}}{\partial H}$$

$$= n_0 \left\{ -3\gamma_2 \frac{L'^4}{G''^4}\frac{H}{G''} + \gamma_2^2 \left[\left(\frac{27}{8}\frac{L'^6}{G''^6} + \frac{9}{2}\frac{L'^7}{G''^7} - \frac{15}{8}\frac{L'^8}{G''^8} \right)\frac{H}{G''} + \left(-\frac{15}{8}\frac{L'^6}{G''^6} - \frac{27}{2}\frac{L'^7}{G''^7} - \frac{105}{8}\frac{L'^8}{G''^8} \right)\frac{H^3}{G''^3} \right] \right\}. \quad (41)$$

TABLE I. FACTORS ARISING IN THE ARGUMENT OF THE PERIGEE, EQUATION (36)

I	$\frac{25\cos^6 I}{(1-5\cos^2 I)^2}$	$\frac{5\cos^4 I}{1-5\cos^2 I}$
0°	1.562	−1.250
10	1.539	−1.222
20	1.476	−1.142
30	1.395	−1.023
40	1.350	−0.890
50	1.552	−0.801
55	2.140	−0.839
60	6.250	−1.202
61	10.58	−1.576
62	25.72	−2.381
63	234.76	−6.956
64	115.72	+4.716
65	12.45	+1.120
66	3.790	+0.792
70	0.232	+0.165
80	0.001	+0.005
90	0.000	0.000

This completes the solution of the problem for orbits with inclinations sufficiently far from the critical inclination. In the vicinity of the critical inclincation the results may become illusory.

The data given in Table I will serve to show that for an orbit with moderately small eccentricity, no significant loss of accuracy occurs if the inclination is even as close as 1°.5 from the critical inclination, 63°26′. The second column in this table gives the value of the factor that contains $(1-5\cos^2 I)^{-2}$ as it appears in g', multiplied by $\gamma_2 e^2(1-e^2)^{-2}$. In h' the coefficient is $-\sqrt{5}$ times that in g'. No term of this type occurs in l'. The third column in the table indicates the magnitude of the parts of the coefficients that contain $(1-5\cos^2 I)^{-1}$ as a factor. In l' and g' these numbers are multiplied by $\mp\gamma_2$ times a factor that reduces to unity for $e=0$; in G' and h' the coefficients are diminished by a function of e that contains e^2 as a factor.

6. *Comparison with results obtained by Hill's method.* For $e=0$, the motions obtained in equations (39), (40), (41) may be compared with my results obtained by Hill's method (Brouwer 1958). With $L'/G''=1$ these expressions reduce to

$$\begin{aligned}
\frac{dl''}{dt} &= n_0\left[1+\gamma_2\left(-\frac{3}{2}+\frac{9}{2}\cos^2 I''\right)+\gamma_2^2\left(+\frac{39}{16}-\frac{117}{8}\cos^2 I''+\frac{411}{16}\cos^4 I''\right)\right],\\
\frac{dg''}{dt} &= n_0\left[\gamma_2\left(-\frac{3}{2}+\frac{15}{2}\cos^2 I''\right)+\gamma_2^2\left(+\frac{21}{16}-\frac{171}{8}\cos^2 I''+\frac{1185}{16}\cos^4 I''\right)\right],\\
\cos I\frac{dh''}{dt} &= n_0\left[\gamma_2\left(-3\cos^2 I''\right)+\gamma_2^2\left(+6\cos^2 I''-\frac{57}{2}\cos^4 I''\right)\right],\\
\frac{d(l''+g'')}{dt} &= n_0\left[1+\gamma_2\left(-3+12\cos^2 I''\right)+\gamma_2^2\left(+\frac{15}{4}-36\cos^2 I''+\frac{399}{4}\cos^4 I''\right)\right]. \quad (42)
\end{aligned}$$

The sidereal mean motion is

$$n_s = \frac{d(l''+g'')}{dt}+\cos I''\frac{dh''}{dt},$$

$$n_s = n_0\left[1+\gamma_2(-3+9\cos^2 I'')+\gamma_2^2\left(+\frac{15}{4}-30\cos^2 I''+\frac{285}{4}\cos^4 I''\right)\right],$$

or

$$n_0 = n_s\left[1+\gamma_2(+3-9\cos^2 I'')+\gamma_2^2\left(+\frac{21}{4}-24\cos^2 I''+\frac{39}{4}\cos^4 I''\right)\right].$$

Thus

$$\frac{d(l''+g'')}{dt} = n_s\left[1 + \gamma_2(+3\cos^2 I'') + \gamma_2^2\left(+3\cos^2 I'' + \frac{3}{2}\cos^4 I''\right)\right],$$
$$\cos I''\frac{dh''}{dt} = n_s\left[\gamma_2(-3\cos^2 I'') + \gamma_2^2\left(-3\cos^2 I'' - \frac{3}{2}\cos^4 I''\right)\right]. \quad (43)$$

In the Hill method applied to the artificial satellite problem I found (Brouwer 1958, p. 435)

$$\frac{d(l''+g'')}{dt} = n_{sH}\left[1 + 3\gamma_2'\cos^2 I_H + \gamma_2'\left(\frac{21}{2}\cos^2 I_H - 30\cos^4 I_H\right)\right], \quad (44)$$

the subscript H being used with n_s, I.

The expression in square brackets is equivalent to the value of $1/(1+p)$ for $\gamma_4 = 0$ with γ_2 replaced by γ_2'. In order to compare with the result obtained above it should be noted that γ_2', $\cos I_H$ and n_{sH} differ from γ_2, H/G'' and n_s.

In the present theory, if $L' = (\mu a_0)^{\frac{1}{2}}$,

$$\gamma_2 = \frac{\mu^2 k_2}{L'^4} = \frac{k_2}{a_0^2}, \quad n_0 = \frac{\mu^2}{L'^3} = \frac{\mu^{\frac{1}{2}}}{a_0^{\frac{3}{2}}}.$$

In the Hill theory

$$\gamma_2' = \frac{k_2}{a^2}, \quad n_{sH} = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}.$$

Hence

$$\frac{\gamma_2}{\gamma_2'} = \frac{a^2}{a_0^2} = \left(\frac{n_0}{n_{sH}}\right)^{\frac{4}{3}}$$
$$= 1 + 4\gamma_2(1 - 3\cos^2 I_H),$$

from equations (42) and (44). Hence

$$\gamma_2' = \gamma_2 - 4\gamma_2^2(1 - 3\cos^2 I_H). \quad (45)$$

The inclination I_H in the Hill method was defined as the maximum latitude. In the present theory this does not apply to the inclination obtained from $\cos I'' = H/G''$ in view of the presence of the term of the first order in γ_2 and zero order in e in G/G'. For $e = 0$ the relation is

$$G = G'[1 + \tfrac{3}{2}\gamma_2\sin^2 I\cos(2g+2l)].$$

Then

$$\Delta\cos^2 I = -\frac{2H^2}{G'^2}\frac{\Delta G}{G'}$$
$$= -3\gamma_2\sin^2 I\cos^2 I\cos(2g+2l)$$

or

$$\sin^2 I = \sin^2 I''[1 + 3\gamma_2\cos^2 I''\cos(2g+2l)].$$

Now

$$\sin^2\beta = \sin^2 I\sin^2(g+l)$$
$$= \sin^2 I''[1 + 3\gamma_2\cos^2 I''\cos(2g+2l)] \times \sin^2(g+l).$$

This is a maximum for $2g + 2l = 180°$, when

$$\sin^2\beta_{max} = \sin^2 I_H = \sin^2 I''[1 - 3\gamma_2\cos^2 I''],$$

from which

$$\cos^2 I_H = \cos^2 I''[1 + 3\gamma_2\sin^2 I'']. \quad (46)$$

The introduction of (45) and (46) into (44) yields

$$\frac{d(l''+g'')}{dt} = n_{sH}\left[1 + 3\gamma_2\cos^2 I'' + \gamma_2^2\left(\frac{15}{2}\cos^2 I'' - 3\cos^4 I''\right)\right]. \quad (47)$$

Finally

$$n_s = \frac{d(l''+g'')}{dt} + \cos I''\frac{dh''}{dt},$$
$$n_{sH} = \frac{d(l''+g'')}{dt} + \cos I_H\frac{dh''}{dt},$$

or

$$n_{sH} - n_s = (\cos I_H - \cos I'')\frac{dh''}{dt}$$
$$= \frac{3}{2}\gamma_2\sin^2 I''\cos I''\frac{dh''}{dt}$$
$$= -\frac{9}{2}n_s\gamma_2^2\sin^2 I''\cos^2 I''.$$

Introduction into (47) yields

$$\frac{d(l''+g'')}{dt} = n_s[1 + 3\gamma_2\cos^2 I'' + \gamma_2^2(3\cos^2 I'' + \tfrac{3}{2}\cos^4 I'')],$$

in agreement with the expression (43) obtained previously.

In addition to establishing the agreement between the results for $e = 0$ obtained by two different methods, the discussion serves as an illustration of the caution necessary in the comparison of results to the second power of γ_2 obtained by different treatments of the same problem.

7. *The fourth harmonic.* The coefficients of these terms are so small that it is unnecessary to consider the terms of short period. Hence the contributions to the Hamiltonian may be limited to the parts independent of l, and only the first powers are needed.

The expression for U_4 is

$$U_4 = \frac{\mu k_4}{r^5}\left(1 - 10\sin^2\beta + \frac{35}{3}\sin^4\beta\right).$$

Since

$$\sin^2\beta = \frac{1}{2}\sin^2 I\,[1 - \cos(2g + 2f)],$$

$$\sin^4\beta = \frac{1}{8}\sin^4 I\,[3 - 4\cos(2g + 2f) + \cos(4g + 4f)],$$

this becomes

$$U_4 = \frac{\mu^6 k_4}{L^{10}}\left[\left(\frac{3}{8} - \frac{15}{4}\cos^2 I + \frac{35}{8}\cos^4 I\right)\frac{a^5}{r^5} + \left(-\frac{5}{6} + \frac{20}{3}\cos^2 I - \frac{35}{6}\cos^4 I\right)\frac{a^5}{r^5}\cos(2g + 2f)\right.$$
$$\left. + \left(\frac{35}{24} - \frac{35}{12}\cos^2 I + \frac{35}{24}\cos^4 I\right)\frac{a^5}{r^5}\cos(4g + 4f)\right].$$

The secular part is

$$\frac{\mu^6 k_4}{L^3 G^7}\left[\left(\frac{3}{8} - \frac{15}{4}\frac{H^2}{G^2} + \frac{35}{8}\frac{H^4}{G^4}\right)\left(\frac{5}{2} - \frac{3}{2}\frac{G^2}{L^2}\right) + \left(-\frac{5}{6} + \frac{20}{3}\frac{H^2}{G^2} - \frac{35}{6}\frac{H^4}{G^4}\right)\left(\frac{3}{4} - \frac{3}{4}\frac{G^2}{L^2}\right)\cos 2g\right].$$

The part independent of g is added to F_{2s}^*,

$$\Delta_4 F_{2s}^* = \frac{\mu^6 k_4}{L'^{10}}\left(\frac{15}{16}\frac{L'^7}{G'^7} - \frac{9}{16}\frac{L'^5}{G'^5}\right)\left(1 - 10\frac{H^2}{G'^2} + \frac{35}{3}\frac{H^4}{G'^4}\right);$$

the part having $\cos 2g$ as a factor is added to F_{2p}^*,

$$\Delta_4 F_{2p}^* = -\frac{5}{8}\frac{\mu^6 k_4}{L'^{10}}\left(\frac{L'^7}{G'^7} - \frac{L'^5}{G'^5}\right)\left(1 - 8\frac{H^2}{G'^2} + 7\frac{H^4}{G'^4}\right)\cos 2g'.$$

Now

$$\frac{\partial}{\partial g}\Delta_4 S_1^* = -\frac{2}{3}\frac{L'^3 G''^4}{\mu^4 k_2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\Delta_4 F_{2p}^*$$
$$= \frac{5}{12}\frac{\mu^2}{L'^4}\frac{k_4}{k_2}G''\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\left(1 - 8\frac{H^2}{G''^2} + 7\frac{H^4}{G''^4}\right)\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\cos 2g'$$
$$= \frac{5}{12}\frac{\mu^2}{L'^4}\frac{k_4}{k_2}G''\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\left[1 - 3\frac{H^2}{G''^2} - 8\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 2g'.$$

By integration,

$$\Delta_4 S_1^* = \frac{5}{24}\frac{\mu^2}{L'^4}\frac{k_4}{k_2}G''\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\left[1 - 3\frac{H^2}{G''^2} - 8\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 2g'.$$

If now

$$\gamma_4 = \frac{\mu^4 k_4}{L'^8},$$

so that

$$\frac{\mu^2 k_4}{L'^4 k_2} = \frac{\gamma_4}{\gamma_2},$$

it follows that the additions to the right-hand members of equations (35) to (38) are

$$\Delta_4 G' = \frac{5}{12}\frac{\gamma_4}{\gamma_2} G''\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\left[1 - 3\frac{H^2}{G''^2} - 8\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 2g'',$$

$$\Delta_4 l' = -\frac{5}{12}\frac{\gamma_4}{\gamma_2}\frac{L'}{G''}\left[1 - 3\frac{H^2}{G''^2} - 8\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 2g'',$$

$$\Delta_4 g' = \frac{\gamma_4}{\gamma_2}\left[\frac{5}{8}\frac{L'^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right) - \frac{5}{24}\frac{L'^2}{G''^2}\left(1 - 9\frac{H^2}{G''^2}\right) + \left(-\frac{35}{3}\frac{L'^4}{G''^4} + \frac{25}{3}\frac{L'^2}{G''^2}\right)\right.$$
$$\left.\times\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + \frac{50}{3}\left(-\frac{L'^4}{G''^4} + \frac{L'^2}{G''^2}\right)\frac{H^6}{G''^6}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\sin 2g'',$$

$$\Delta_4 h' = +\frac{5}{4}\frac{\gamma_4}{\gamma_2}\left(\frac{L'^4}{G''^4} - \frac{L'^2}{G''^2}\right)\frac{H}{G''}\left[1 + \frac{16}{3}\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + \frac{40}{3}\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\sin 2g''.$$

The additions to the secular motions are

$$\Delta_4\frac{dl''}{dt} = -\Delta_4\frac{\partial F_2^*}{\partial L'} = n_0\gamma_4\left[-\frac{45}{16}\left(\frac{L'^5}{G''^5} - \frac{L'^7}{G''^7}\right)\left(1 - 10\frac{H^2}{G''^2} + \frac{35}{3}\frac{H^4}{G''^4}\right)\right],$$

$$\Delta_4\frac{dg''}{dt} = -\Delta_4\frac{\partial F_2^*}{\partial G''} = n_0\gamma_4\left[-\frac{15}{16}\left(3\frac{L'^6}{G''^6} - 7\frac{L'^8}{G''^8}\right) + \frac{45}{8}\left(7\frac{L'^6}{G''^6} - 15\frac{L'^8}{G''^8}\right)\frac{H^2}{G''^2}\right.$$
$$\left. - \frac{35}{16}\left(27\frac{L'^6}{G''^6} - 55\frac{L'^8}{G''^8}\right)\frac{H^4}{G''^4}\right],$$

$$\Delta_4\frac{dh''}{dt} = -\Delta_4\frac{\partial F_2^*}{\partial H} = n_0\gamma_4\left[-\frac{5}{4}\left(3\frac{L'^6}{G''^6} - 5\frac{L'^8}{G''^8}\right)\left(3\frac{H}{G''} - 7\frac{H^3}{G''^3}\right)\right].$$

8. *The third and fifth harmonics.* Odd harmonics are to be considered only if symmetry of the earth with respect to the equator does not exist. A study of the motion of 1958 β_2 led to the introduction of these odd harmonics by O'Keefe, Eckels and Squires (1959).

The expression for U_3 is

$$U_3 = \frac{\mu A_{3.0}}{r^4}\left(\frac{5}{2}\sin^3\beta - \frac{3}{2}\sin\beta\right).$$

With

$$\sin^3\beta = \sin^3 I\left[\frac{3}{4}\sin(g + f) - \frac{1}{4}\sin(3g + 3f)\right]$$

this becomes

$$U_3 = \frac{\mu A_{3.0}}{r^4}\left[\left(-\frac{3}{2}\sin I + \frac{15}{8}\sin^3 I\right)\sin(g + f) - \frac{5}{8}\sin^3 I\sin(3g + 3f)\right].$$

For the sake of brevity, let $e'\sin I'$ stand for $(1 - G'^2/L'^2)^{\frac{1}{2}}(1 - H^2/G'^2)^{\frac{1}{2}}$, $e''\sin I''$ for the same expression with G' replaced by G''. The secular part may then be written

$$\Delta_3 F_{2p}^* = \frac{\mu^5 A_{3.0}}{L'^3 G'^5}e'\sin I'\left(\frac{3}{8} - \frac{15}{8}\frac{H^2}{G'^2}\right)\sin g'.$$

There is no part independent of g'.

It follows that

$$\frac{\partial}{\partial g'}\Delta_3 S_1^* = -\frac{1}{4}\frac{\mu}{L'^2}\frac{A_{3.0}}{k_2}\frac{L'^2}{G''^2}G''e''\sin I''\sin g',$$

$$\Delta_3 S_1^* = +\frac{1}{4}\frac{\mu}{L'^2}\frac{A_{3.0}}{k_2}\frac{L'^2}{G''^2}G''e''\sin I''\cos g'.$$

If

$$\gamma_3 = \frac{\mu^3 A_{3.0}}{L'^6},$$

$$\Delta_3 G' = -\frac{1}{4}\frac{\gamma_3}{\gamma_2}\frac{L'^2}{G''^2} G'' e'' \sin I'' \sin g''.$$

With

$$\frac{\partial}{\partial L}(e \sin I) = \frac{\sin I}{e}\frac{G^2}{L^3},$$

$$\frac{\partial}{\partial G}(e \sin I) = -\frac{\sin I}{e}\frac{G}{L^2} + \frac{e}{\sin I}\frac{H^2}{G^3},$$

$$\frac{\partial}{\partial H}(e \sin I) = -\frac{e}{\sin I}\frac{H}{G^2},$$

the following results are easily obtained:

$$\Delta_3 l' = -\frac{1}{4}\frac{\gamma_3}{\gamma_2}\frac{\sin I''}{e''}\frac{G''}{L'}\cos g'',$$

$$\Delta_3 g' = +\frac{1}{4}\frac{\gamma_3}{\gamma_2}\frac{L'^2}{G''^2}\left(\frac{\sin I''}{e''} - \frac{e''}{\sin I''}\frac{H^2}{G''^2}\right)\cos g'',$$

$$\Delta_3 h' = +\frac{1}{4}\frac{\gamma_3}{\gamma_2}\frac{L'^2}{G''^2}\frac{H}{G''}\frac{e''}{\sin I''}\cos g''.$$

The fifth harmonic is

$$U_5 = \frac{\mu A_{5.0}}{r^6}\left[\frac{15}{8}\sin\beta - \frac{35}{4}\sin^3\beta + \frac{63}{8}\sin^5\beta\right].$$

With

$$\sin^5\beta = \sin^5 I\left[\frac{5}{8}\sin(g+f) - \frac{5}{16}\sin(3g+3f) + \frac{1}{16}\sin(5g+5f)\right]$$

this becomes

$$U_5 = \frac{\mu A_{5.0}}{a^6}\left[\left(\frac{15}{8}\sin I - \frac{105}{16}\sin^3 I + \frac{315}{64}\sin^5 I\right)\frac{a^6}{r^6}\sin(g+f)\right.$$
$$\left. + \left(\frac{35}{16}\sin^3 I - \frac{315}{128}\sin^5 I\right)\frac{a^6}{r^6}\sin(3g+3f) + \frac{63}{128}\sin^5 I\frac{a^6}{r^6}\sin(5g+5f)\right\}.$$

$$U_5 = \frac{\mu^7 A_{5.0}}{L^{12}}\sin I\left[\frac{15}{64}\left(1 - 14\frac{H^2}{G^2} + 21\frac{H^4}{G^4}\right)\frac{a^6}{r^6}\sin(g+f)\right.$$
$$\left. - \frac{35}{128}\left(1 - 10\frac{H^2}{G^2} + 9\frac{H^4}{G^4}\right)\frac{a^6}{r^6}\sin(3g+3f) + \frac{63}{128}\left(1 - 2\frac{H^2}{G^2} + \frac{H^4}{G^4}\right)\frac{a^6}{r^6}\sin(5g+5f)\right].$$

The secular part is

$$\Delta_5 F_{2p}{}^* = \frac{\mu^7 A_{5.0}}{L'^3 G'^9} e' \sin I'$$
$$\times\left[\frac{15}{128}\left(1 - 14\frac{H^2}{G'^2} + 21\frac{H^4}{G'^4}\right)\left(7 - 3\frac{G'^2}{L'^2}\right)\sin g'\right.$$
$$\left. - \frac{35}{256}\left(1 - 10\frac{H^2}{G'^2} + 9\frac{H^4}{G'^4}\right)\left(1 - \frac{G'^2}{L'^2}\right)\sin 3g'\right].$$

Hence

$$\frac{\partial}{\partial g'}\Delta_5 S_1{}^* = \frac{\mu^3 A_{5.0}}{G''^5 k_2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} e'' \sin I''$$
$$\times\left[-\frac{5}{64}\left(1 - 14\frac{H^2}{G''^2} + 21\frac{H^4}{G''^4}\right)\left(7 - 3\frac{G''^2}{L'^2}\right)\sin g' + \frac{35}{384}\left(1 - 10\frac{H^2}{G''^2} + 9\frac{H^4}{G''^4}\right)\left(1 - \frac{G''^2}{L'^2}\right)\sin 3g'\right],$$

$$\Delta_5 S_1{}^* = \frac{\mu^3 A_{5.0}}{L'^6 k_2}\frac{L'^6}{G''^6} G'' e'' \sin I''$$
$$\times\left\{\frac{5}{64}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g' - \frac{35}{1152}\left(1 - \frac{G''^2}{L'^2}\right)\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g'.\right.$$

With the aid of

$$\gamma_5 = \frac{\mu^5 A_{5.0}}{L'^{10}}$$

the resulting long-period terms become:

$$\Delta_5 G = \frac{\gamma_5}{\gamma_2}\frac{L'^6}{G''^6} G'' e'' \sin I''$$
$$\times\left\{-\frac{5}{64}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin g'' + \frac{35}{384}\left(1 - \frac{G''^2}{L'^2}\right)\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\sin 3g''\right\},$$

$$\Delta_5 l' = \frac{\gamma_5}{\gamma_2}\frac{L'^3}{G''^3} e'' \sin I''$$
$$\times\left\{-\frac{5}{64}\left(13 - 9\frac{G''^2}{L'^2}\right)\left(1 - \frac{G''^2}{L'^2}\right)^{-1}\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g'' + \frac{35}{384}\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g''\right\},$$

$$\Delta_5 g' = \frac{\gamma_5}{\gamma_2}\frac{L'^6}{G''^6}\left[\frac{G''^2}{L'^2}\frac{\sin I''}{e''} - \frac{H^2}{G''^2}\frac{e''}{\sin I''}\right]$$
$$\times\left\{\frac{5}{64}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g''\right.$$
$$\left. - \frac{35}{1152}\left(1 - \frac{G''^2}{L'^2}\right)\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g''\right\}$$
$$+ \frac{\gamma^5}{\gamma_2}\frac{L'^6}{G''^6} e'' \sin I''\left\{\frac{5}{64}\left(35 - 9\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g''\right.$$
$$- \frac{35}{1152}\left(5 - 3\frac{G''^2}{L'^2}\right)\left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g''$$
$$- \frac{15}{32}\left(7 - 3\frac{G''^2}{L'^2}\right)\frac{H^2}{G''^2}\left[3 + 16\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + 40\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\cos g''$$
$$\left. + \frac{35}{576}\left(1 - \frac{G''^2}{L'^2}\right)\frac{H^2}{G''^2}\left[5 + 32\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + 80\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\cos 3g''\right\},$$

$$\Delta_5 h' = \frac{\gamma_5}{\gamma_2}\frac{L'^6}{G''^6}\frac{H}{G''}\frac{e''}{\sin I''}$$
$$\times \left\{ \frac{5}{64}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[1 - 9\frac{H^2}{G''^2} - 24\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos g'' - \frac{35}{1152}\left(1 - \frac{G''^2}{L'^2}\right)\right.$$
$$\left.\times \left[1 - 5\frac{H^2}{G''^2} - 16\frac{H^2}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1}\right]\cos 3g''\right\} + \frac{\gamma_5}{\gamma_2}\frac{L'^6}{G''^6}\frac{H}{G''}e''\sin I''$$
$$\times \left\{ \frac{15}{32}\left(7 - 3\frac{G''^2}{L'^2}\right)\left[3 + 16\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + 40\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\cos g''\right.$$
$$\left. - \frac{35}{576}\left(1 - \frac{G''^2}{L'^2}\right)\left[5 + 32\frac{H^2}{G''^2}\left(1 - 5\frac{H^2}{G''^2}\right)^{-1} + 80\frac{H^4}{G''^4}\left(1 - 5\frac{H^2}{G''^2}\right)^{-2}\right]\cos 3g''\right\}.$$

9. *Formulas for computation.* For convenience of computation the perturbations in the Keplerian elements a, e, I are given instead of those in L, G, H.

The adopted force function is

$$U = \frac{\mu}{r} + \frac{\mu k_2}{r^3}(1 - 3\sin^2\beta)$$
$$+ \frac{\mu k_4}{r^5}\left(1 - 10\sin^2\beta + \frac{35}{3}\sin^4\beta\right)$$
$$+ \frac{\mu A_{3.0}}{r^4}\left(-\frac{3}{2}\sin\beta + \frac{5}{2}\sin^3\beta\right)$$
$$+ \frac{\mu A_{5.0}}{r^6}\left(\frac{15}{8}\sin\beta - \frac{35}{4}\sin^3\beta + \frac{63}{8}\sin^5\beta\right),$$

in which k_2 is a small quantity, and k_4, $A_{3.0}$, $A_{5.0}$ are assumed to be of order k_2^2.

The secular motions have been computed to $O(k_2^2)$, the coefficients of periodic terms to $O(k_2)$.

Basic constants:

a'' = semi-major axis constant

e'' = eccentricity constant

I'' = inclination constant

$n_0 = \mu^{\frac{1}{2}} a''^{-\frac{3}{2}} = 17.04337\ (a_0/R)^{-\frac{3}{2}}$ rev./day

R = equatorial radius

Abbreviations:

$$\eta = (1 - e''^2)^{\frac{1}{2}} \qquad \theta = \cos I''$$

$$\gamma_2 = \frac{k_2}{a''^2} \qquad \gamma_4 = \frac{k_4}{a''^4} \qquad \gamma_3 = \frac{A_{3.0}}{a''^3} \qquad \gamma_5 = \frac{A_{5.0}}{a''^5}$$

$$\gamma_2' = \gamma_2\eta^{-4} \qquad \gamma_4' = \gamma_4\eta^{-8} \qquad \gamma_3' = \gamma_3\eta^{-6} \qquad \gamma_5' = \gamma_5\eta^{-10}$$

It is customary to use for the second harmonic the coefficient J; Jeffreys (1954) used for the fourth harmonic the coefficient D. The relations between J, D and γ_2, γ_4 are

$$\gamma_2 = \frac{1}{3}J\left(\frac{R}{a''}\right)^2, \quad \gamma_4 = \frac{3}{35}D\left(\frac{R}{a''}\right)^4.$$

Strictly speaking, $e'' + \delta_1 e$, $\theta' = \cos(I'' + \delta_1 I)$, $\eta' = [1 - (e'' + \delta_1 e)^2]^{\frac{1}{2}}$ should be used in the computation of the periodic terms, but since the short-period terms are obtained to $O(k_2)$, it is of no consequence if contributions of $O(k_2)$ are omitted in expressions that have γ_2 as a factor. Similarly, l'', g'' might be used in computing f', r'; but since l', g' are available, their use does not complicate the calculation.

The formulas are applicable for any eccentricity $e < 1$ and any inclination with the exception of inclinations near the critical inclination, for which $1 - 5\cos^2 I$ appears as a small divisor.

The appearance of e'' as a divisor in the short-period terms in e is apparent only. The expressions that are multiplied by e''^{-1} contain e'' as a factor, either implicitly or explicitly.

In the short-period terms in l and g a divisor e'' occurs also, but for the calculation of the position only $g + l +$ equation of the center is needed. In $g + l$ the divisor e'' is not present.

Singularities in some of the elements also occur for very small inclinations; again, no singularity is present in the coordinates. In such cases it may be found convenient to modify the formulas and obtain expressions for the perturbations in coordinates.

Secular terms:

l'' = "mean" mean anomaly

$$= n_0 t\left\{1 + \frac{3}{2}\gamma_2'\eta(-1 + 3\theta^2) + \frac{3}{32}\gamma_2'^2\eta[-15 + 16\eta + 25\eta^2 + (30 - 96\eta - 90\eta^2)\theta^2\right.$$
$$\left. + (105 + 144\eta + 25\eta^2)\theta^4] + \frac{15}{16}\gamma_4'\eta e''^2[3 - 30\theta^2 + 35\theta^4]\right\} + l_0''$$

g'' = mean argument of perigee

$$= n_0 t \left\{ \frac{3}{2}\gamma_2'(-1+5\theta^2) + \frac{3}{32}\gamma_2'^2[-35+24\eta+25\eta^2+(90-192\eta-126\eta^2)\theta^2 + (385+360\eta+45\eta^2)\theta^4] + \frac{5}{16}\gamma_4'[21-9\eta^2+(-270+126\eta^2)\theta^2+(385-189\eta^2)\theta^4] \right\} + g_0''$$

h'' = mean longitude of ascending node

$$= n_0 t \left\{ -3\gamma_2'\theta + \frac{3}{8}\gamma_2'^2[(-5+12\eta+9\eta^2)\theta+(-35-36\eta-5\eta^2)\theta^3] + \frac{5}{4}\gamma_4'(5-3\eta^2)\theta(3-7\theta^2) \right\} + h_0''$$

Long-period terms:

$$\delta_1 e = \left\{ \frac{1}{8}\gamma_2' e''\eta^2[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}] - \frac{5}{12}\frac{\gamma_4'}{\gamma_2'} e''\eta^2[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}] \right\} \cos 2g''$$
$$+ \left\{ \frac{1}{4}\frac{\gamma_3'}{\gamma_2'}\eta^2 \sin I'' + \frac{5}{64}\frac{\gamma_5'}{\gamma_2'\eta^2} \sin I''\,(4+3e''^2)[1-9\theta^2-24\theta^4(1-5\theta^2)^{-1}] \right\} \sin g''$$
$$- \frac{35}{384}\frac{\gamma_5'}{\gamma_2'} e''^2\eta^2 \sin I''\,[1-5\theta^2-16\theta^4(1-5\theta^2)^{-1}] \sin 3g''$$

$$\delta_1 I = -\frac{e''\delta_1 e}{\eta^2 \tan I''}$$

$$l' = l'' + \left\{ \frac{1}{8}\gamma_2'\eta^3[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}] - \frac{5}{12}\frac{\gamma_4'}{\gamma_2'}\eta^3[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}] \right\} \sin 2g''$$
$$+ \left\{ -\frac{1}{4}\frac{\gamma_3'}{\gamma_2'}\frac{\eta^3}{e''} \sin I'' - \frac{5}{64}\frac{\gamma_5'}{\gamma_2'}\frac{\eta^3}{e''} \sin I''\,(4+9e''^2)[1-9\theta^2-24\theta^4(1-5\theta^2)^{-1}] \right\} \cos g''$$
$$+ \frac{35}{384}\frac{\gamma_5'}{\gamma_2'}\eta^3 e'' \sin I''\,[1-5\theta^2-16\theta^4(1-5\theta^2)^{-1}] \cos 3g''$$

$$g' = g'' + \left\{ -\frac{1}{16}\gamma_2'[+(2+e''^2) - 11(2+3e''^2)\theta^2 - 40(2+5e''^2)\theta^4(1-5\theta^2)^{-1} - 400e''^2\theta^6(1-5\theta^2)^{-2}] + \frac{5}{24}\frac{\gamma_4'}{\gamma_2'}[2+e''^2-3(2+3e''^2)\theta^2-8(2+5e''^2)\theta^4(1-5\theta^2)^{-1} - 80e''^2\theta^6(1-5\theta^2)^{-2}] \right\} \sin 2g'' + \left\{ \frac{1}{4}\frac{\gamma_3'}{\gamma_2'}\left(\frac{\sin I''}{e''} - \frac{e''\theta^2}{\sin I''}\right) + \frac{5}{64}\frac{\gamma_5'}{\gamma_2'} \right.$$
$$\times \left[\left(\frac{\eta^2 \sin I''}{e''} - \frac{e''\theta^2}{\sin I''}\right)(4+3e''^2) + e'' \sin I''\,(26+9e''^2)\right][1-9\theta^2-24\theta^4(1-5\theta^2)^{-1}]$$
$$\left. - \frac{15}{32}\frac{\gamma_5'}{\gamma_2'} e''\theta^2 \sin I''\,(4+3e''^2)[3+16\theta^2(1-5\theta^2)^{-1}+40\theta^4(1-5\theta^2)^{-2}] \right\} \cos g''$$
$$+ \left\{ -\frac{35}{1152}\frac{\gamma_5'}{\gamma_2'}\left[e'' \sin I''\,(3+2e''^2) - \frac{e''^3\theta^2}{\sin I''}\right][1-5\theta^2-16\theta^4(1-5\theta^2)^{-1}] \right.$$
$$\left. + \frac{35}{576}\frac{\gamma_5'}{\gamma_2'} e''^3\theta^2 \sin I''\,[5+32\theta^2(1-5\theta^2)^{-1}+80\theta^4(1-5\theta^2)^{-2}] \right\} \cos 3g''$$

$$
\begin{aligned}
h' = h'' &+ \left\{ -\frac{1}{8}\gamma_2' e''^2\theta[11 + 80\theta^2(1-5\theta^2)^{-1} + 200\theta^4(1-5\theta^2)^{-2}] \right. \\
&\left. + \frac{5}{12}\frac{\gamma_4'}{\gamma_2'} e''^2\theta[3 + 16\theta^2(1-5\theta^2)^{-1} + 40\theta^4(1-5\theta^2)^{-2}] \right\} \sin 2g'' \\
&+ \left\{ \frac{1}{4}\frac{\gamma_3'}{\gamma_2'}\frac{e''\theta}{\sin I''} + \frac{5}{64}\frac{\gamma_5'}{\gamma_2'}\frac{e''\theta}{\sin I''}(4+3e''^2)[1 - 9\theta^2 - 24\theta^4(1-5\theta^2)^{-1}] \right. \\
&\left. + \frac{15}{32}\frac{\gamma_5'}{\gamma_2'} e''\theta \sin I''\,(4+3e''^2)[3 + 16\theta^2(1-5\theta^2)^{-1} + 40\theta^4(1-5\theta^2)^{-2}] \right\} \cos g'' \\
&+ \left\{ -\frac{35}{1152}\frac{\gamma_5'}{\gamma_2'}\frac{e''^3\theta}{\sin I''}[1 - 5\theta^2 - 16\theta^4(1-5\theta^2)^{-1}] \right. \\
&\left. - \frac{35}{576}\frac{\gamma_5'}{\gamma_2'} e''^3\theta \sin I''\,[5 + 32\theta^2(1-5\theta^2)^{-1} + 80\theta^4(1-5\theta^2)^{-2}] \right\} \cos 3g''
\end{aligned}
$$

Short-period terms included:

$$
a = a''\left\{1 + \gamma_2\left[(-1+3\theta^2)\left(\frac{a''^3}{r'^3} - \eta^{-3}\right) + 3(1-\theta^2)\frac{a''^3}{r'^3}\cos(2g'+2f')\right]\right\}
$$

$$
\begin{aligned}
e = e'' + \delta_1 e + \frac{\eta^2}{2e''}&\left\{\gamma_2\left[(-1+3\theta^2)\left(\frac{a''^3}{r'^3} - \eta^{-3}\right) + 3(1-\theta^2)\left(\frac{a''^3}{r'^3} - \eta^{-4}\right)\cos(2g'+2f')\right]\right. \\
&\left. - \gamma_2'(1-\theta^2)[3e''\cos(2g'+f') + e''\cos(2g'+3f')]\right\}
\end{aligned}
$$

$$
I = I'' + \delta_1 I + \tfrac{1}{2}\gamma_2'\theta(1-\theta^2)^{\frac{1}{2}}[3\cos(2g'+2f') + 3e''\cos(2g'+f') + e''\cos(2g'+3f')]
$$

$$
\begin{aligned}
l = l' - \frac{\eta^3}{4e''}\gamma_2' &\left\{2(-1+3\theta^2)\left(\frac{a''^2}{r'^2}\eta^2 + \frac{a''}{r'} + 1\right)\sin f' \right. \\
&\left. + 3(1-\theta^2)\left[\left(-\frac{a''^2}{r'^2}\eta^2 - \frac{a''}{r'} + 1\right)\sin(2g'+f') + \left(\frac{a''^2}{r'^2}\eta^2 + \frac{a''}{r'} + \frac{1}{3}\right)\sin(2g'+3f')\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
g = g' + \frac{\eta^2}{4e''}\gamma_2' &\left\{2(-1+3\theta^2)\left(\frac{a''^2}{r'^2}\eta^2 + \frac{a''}{r'} + 1\right)\sin f' \right. \\
&\left. + 3(1-\theta^2)\left[\left(-\frac{a''^2}{r'^2}\eta^2 - \frac{a''}{r'} + 1\right)\sin(2g'+f') + \left(\frac{a''^2}{r'^2}\eta^2 + \frac{a''}{r'} + \frac{1}{3}\right)\sin(2g'+3f')\right]\right\} \\
&+ \tfrac{1}{4}\gamma_2'\{6(-1+5\theta^2)(f' - l' + e''\sin f') \\
&\qquad + (3-5\theta^2)[3\sin(2g'+2f') + 3e''\sin(2g'+f') + e''\sin(2g'+3f')]\}
\end{aligned}
$$

$$
\begin{aligned}
h = h' - \tfrac{1}{2}\gamma_2'\theta[&6(f' - l' + e''\sin f') - 3\sin(2g'+2f') \\
&- 3e''\sin(2g'+f') - e''\sin(2g'+3f')].
\end{aligned}
$$

f', r' are to be computed from

$$
E' - e''\sin E' = l'
$$

$$
\tan\tfrac{1}{2}f' = \left(\frac{1+e''}{1-e''}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}E' \qquad\qquad \frac{r'}{a''}\sin f' = (1-e''^2)^{\frac{1}{2}}\sin E'
$$

$$
\frac{a''}{r'} = \frac{1+e''\cos f'}{1-e''^2} \qquad \text{or} \qquad \frac{r'}{a''}\cos f' = \cos E' - e''
$$

$$
\frac{r'}{a''} = 1 - e''\cos E'
$$

For the calculation of the coordinates at any time the complete values of e and l should be used for the solution of Kepler's equation,

$$E - e \sin E = l$$

and subsequently r and f, which may then be used in the formulas:

$$x = r[\cos(g+f)\cos h - \sin(g+f)\sin h \cos I]$$
$$y = r[\cos(g+f)\sin h + \sin(g+f)\cos h \cos I]$$
$$z = r \sin(g+f) \sin I$$

A convenient alternative form is:

$$x = A_x(\cos E - e) + B_x \sin E$$
$$y = A_y(\cos E - e) + B_y \sin E$$
$$z = A_z(\cos E - e) + B_z \sin E$$

$$A_x = a[\cos g \cos h - \sin g \sin h \cos I]$$
$$B_x = -a(1-e^2)^{\frac{1}{2}}[\sin g \cos h + \cos g \sin h \cos I]$$
$$A_y = a[\sin g \cos h \cos I + \cos g \sin h]$$
$$B_y = a(1-e^2)^{\frac{1}{2}}[\cos g \cos h \cos I - \sin g \sin h]$$
$$A_z = a \sin g \sin I$$
$$B_z = a(1-e^2)^{\frac{1}{2}} \cos g \sin I$$

Noted added in proof. The lack of uniformity in notation of the coefficients of the second and fourth harmonics of the earth's potential in papers dealing with the motion of artificial satellites calls for a comment on this subject.

The table below contains a listing of some of the designations used and their relations to the coefficients B_p in the expression of the force function of a body with rotational symmetry,

$$F = \frac{\mu}{r}\left[1 + \sum_{p=2}^{\infty} \frac{B_p P_p(\sin\beta)}{r^p}\right],$$

in which P_p are Legendre polynomials and $\mu = GM$. The expression is an adaptation of the Laplacian expression given by Tisserand.

In addition to the equivalents of B_2 and B_4 the table gives those of the ratio B_4/B_2^2, which is unity for the special case treated by Vinti (1959), in which the terms with small divisors near the critical inclination vanish. No effort has been made to make the tabulation complete.

Laplace	B_2	B_4	B_4/B_2^2	Tisserand, *Méc. Cél.* II, 320, 1890
H. Struve	$-\frac{2}{3}k$	$\frac{2}{3}l$	$\frac{3}{2}l/k^2$	*Suppl. I, Obs. Pulkovo* 1888
W. de Sitter	$-\frac{2}{3}JR^2$	$\frac{4}{15}KR^4$	$\frac{3}{5}K/J^2$	*B. A. N.* **2**, 97, 1924
D. Brouwer	$-2k_2$	$\frac{8}{3}k_4$	$\frac{2}{3}k_4/k_2^2$	*A. J.* **51**, 223, 1946
H. Jeffreys	$-\frac{2}{3}JR^2$	$\frac{8}{35}DR^4$	$\frac{18}{35}D/J^2$	*M. N.* **14**, 433, 1954
Y. Kozai	$-\frac{2}{3}A_2$	$\frac{8}{35}A_4$	$\frac{18}{35}A_4/A_2^2$	
P. Herget and P. Musen	$-2k_2$	$8k_4$	$2k_4/k_2^2$	*A. J.* **63**, 430, 1958
J. O'Keefe et al.	$+A_{2,0}/\mu$	$+A_{4,0}/\mu$	$\mu A_{4,0}/A_{2,0}^2$	*A. J.* **64**, 235, 1959
B. Garfinkel	$-2k$	k'	$\frac{1}{4}k'/k^2$	This issue
J. Vinti	$-J_2R^2$	$-J_4R^4$	J_4/J_2^2	*J. of Res. Nat. Bureau of Standards* **62B**, 105, 1959

In the table R represents the earth's equatorial radius. Ignoring the presence of R^2 and differences in sign, essentially three different coefficients for the second harmonic have been used in recent papers. For the coefficients of the fourth harmonic six different choices are listed. I now regret that I introduced k_2, k_4 in my paper in 1946. The principal reason was that they give a particularly simple form for the expression of the potential in the equatorial plane. If I could have foreseen the increase in interest in the subject and the confusion to which I was contributing, I would have chosen the coefficients B_p or the alternative form

$$F = \frac{\mu}{r}\left[1 - \sum_{p=2}^{\infty} J_p \left(\frac{R}{r}\right)^p P_p(\sin\beta)\right],$$

which was used by Vinti (1959). I intend to revert to this form and recommend this to other authors.