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The Formation of the Nebulae

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With 3 Figures

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The growth of condensations in the pressure-free cosmological models of general relativity (with $\Lambda = 0$) is studied by using a simplified model of a condensation. It is shown that, although condensations can form from quite small perturbations in the density of the cosmic medium, the perturbations required to account for the formation of the nebulae by the present time are nevertheless much larger than those which would be expected from ordinary statistical theory. The Lemaître point-source model (with $\Lambda \neq 0$) seems no more favourable to the growth of condensations.

It is concluded that unless some source of larger perturbations can be discovered, it will be necessary to abandon the point-source models in favour of others with a longer time-scale, for example, the disturbed Einstein model.

1. Introduction

From the field equations of general relativity

$$-8\pi T_{ik} = G_{ik} - \frac{1}{2} g_{ik} G, \tag{1.1}$$

it is possible to derive three different models of the universe (BONDI, 1952). In two of these (the open models) the universe has infinite extent, and in one (the closed model) it is finite. The three most important observed data of cosmology — HUBBLE's Law, the average density of matter, and the minimum age of the universe — can be accommodated by each the three models by an appropriate choice of constants, though in the closed model agreement with observation is not very good.

As well as fitting these three observations, a satisfactory cosmological theory should have something to say about the formation of the nebulae. Here, however, one meets the difficulty that one does not know the initial state of the universe, and on this the development of condensations certainly depends. The three models satisfying (1.1) all start from a singular state of infinite density at time $t = 0$, but one does not know what physical circumstances this state represents. Indeed, since the equations (1.1) break down at $t = 0$, some assumption about the commencement of the models is necessary and one might prefer to assume that the universe began say, 10^8 years later than the singular state of the models, in a slightly inhomogeneous condition. An assumption of this sort would be quite compatible with the three fundamental data, and would, as we shall see, effectively remove the problem of the formation of the nebulae. Since the singular state at $t = 0$ is, in our present state of knowledge, a miracle, the question really is whether one prefers one's miracle then or later.

To this extent the problem — or absence of problem — of the formation of the nebulae depends on the philosophical assumptions one cares to make. In my own opinion it seems preferable to extrapolate the solutions of (1.1) back as far as possible, that is, to the singular state at $t = 0$. If one does this, one has to conclude that the temperature was initially so high that the very early spatial distribution of matter was completely uniform (on a macroscopic scale), and to account for the formation of the nebulae it becomes necessary to find some cause which has developed inhomogeneity.

The most reasonable cause of non-uniformity seems to be the random fluctuations in density and velocity which would be expected from statistical mechanics. In the case of an ideal gas, the isothermal fluctuation in the density of a group of N molecules is given by

$$\frac{\overline{(\varrho - \bar{\varrho})^2}}{\bar{\varrho}^2} = \frac{1}{N}\,. \tag{1.2}$$

If it could be shown that density fluctuations of this order could produce by the present time condensations like the nebulae, this would be an important step in the solution of the problem.

This question has been studied by LIFSHITZ (1946), who considered the effect of small perturbations of a very general kind in the density and velocity. He concluded that most perturbations would either die out or not grow, but that certain types could eventually become large, though not large enough to produce nebulae or stars in the time available.

LIFSHITZ used equations which are linear in the perturbations, and this enabled him to deal with a very general class of disturbances. However, the field equations themselves are non-linear, and it is at first sight possible that this non-linearity may lead to some process which speeds up (or slows down) the process of condensation in a way not obvious in the linear approximation. It is for this reason that I here study the problem from a rather different point of view. I take a much simplified model of a condensation (explained in section 2) which is supposed to be forming at a time in the history of the universe when the pressure may be neglected. These simplifications enable me to trace the process of condensation without neglecting any non-linear terms in the field equations and to estimate the time required for condensation, starting with a perturbation of given magnitude.

My conclusions are that although there can indeed be a speeding up not predicted in the linear approximation, and although condensations certainly can form eventually in the models, the perturbations given by (1.2) are very much too small to have produced nebulae or stars by the present time. Put in another way, this means that the formation of the nebulae is a tremendously improbable occurrence in these models if ordinary statistical theory is used.

The plan of the paper is as follows. In section 2 I explain and justify the model chosen to represent a condensation; in sections 3 and 4 respectively I consider the application of the model to the closed and open models satisfying (1.1); in section 5 I discuss briefly condensation in LEMAÎTRE's model; and in the Conclusion, section 6, I summarise the results and discuss some possible ways of overcoming the difficulties suggested by them.

2. *The model of a condensation*

I take a spherically symmetric, pressure-free model of the universe and use comoving coordinates. The most general line-element satisfying these assumptions is, in pseudo-polar coordinates

$$ds^2 = -e^{\lambda}\, dr^2 - e^{\omega}(d\theta^2 + \sin^2\theta\, d\Phi^2) + dt^2, \qquad (2.1)$$

where λ and ω are functions of r and t. The field equations of general relativity, without cosmological term, give the following relevant equations (TOLMAN 1934a):

$$e^{\lambda} = e^{\omega}\, \omega'^2/4\,\alpha(r), \qquad (2.2)$$

$$\frac{1}{2} e^{3\omega/2}\, \dot{\omega}^2 + 2\, e^{\frac{1}{2}\omega}(1-\alpha) = 4\,\beta(r), \qquad (2.3)$$

$$8\pi\varrho = 4\, e^{-3\omega/2}\, \beta'/\omega',$$

where $\cdot$ and $'$ mean $\partial/\partial t$ and $\partial/\partial r$ respectively, and $\alpha(r)$ and $\beta(r)$ are arbitrary functions of r. Equn. (2.3) may be written

$$\int \frac{d e^{\omega/2}}{\sqrt{2\beta\, e^{-\omega/2} - (1-\alpha)}} = t + \gamma(r), \qquad (2.4)$$

where $\gamma(r)$ is another arbitrary function. The integral gives rise to three different cases according as

$$1 - \alpha \gtreqless 0.$$

For the time being, let us suppose that

$$1 - \alpha > 0; \qquad (2.5)$$

then, evaluating the integral (2.4) we find the solution in the following form

$$e^{\frac{1}{2}\omega} = \beta(1-\alpha)^{-1}(1-\cos\psi), \tag{2.6}$$

$$t+\gamma = \beta(1-\alpha)^{-3/2}(\psi-\sin\psi), \tag{2.7}$$

$$8\pi\varrho = \frac{1}{2}\beta'(1-\alpha)^3\beta^{-3}\operatorname{cosec}^6\frac{1}{2}\psi(\omega')^{-1}. \tag{2.8}$$

Let us suppose that the model represents an inhomogeneity inside a region $r=a$ of an ordinary pressure-free expanding universe of general relativity: that is to say, for $r<a$ (2.2), (2.6), (2.7) and (2.8) apply, whereas for $r>a$ we have

$$ds^2 = -[R(t)]^2[(1-kr^2)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\Phi^2)] + dt^2, \tag{2.9}$$

where k has one of the values 0, +1 or —1, and the field equations give

$$8\pi\varrho = \frac{3k}{R^2} + 3\left(\frac{\dot R}{R}\right)^2,$$

$$8\pi p = -\frac{k}{R^2} - \left(\frac{\dot R}{R}\right)^2 - \frac{2\ddot R}{R} = 0.$$

Let us for the present take $k=+1$, so that (2.9) represents a closed, homogeneous model; then the solution for it corresponding to (2.6)—(2.8) is

$$R = K(1-\cos\xi), \tag{2.10}$$

$$t = K(\xi-\sin\xi), \tag{2.11}$$

$$8\pi\varrho = 6K/R^3,$$

where K is a positive constant, and where I have omitted in (2.11) the additive constant which corresponds to γ in (2.7). (This simply amounts to taking the origin of t at $\xi=0$, when the model is in its initial singular state.)

The problem now is to match the two solutions at the comoving boundary $r=a$. According to the boundary conditions of O'Brien and Synge (1952), the following must be continuous at $r=a$:

$$g_{\alpha\beta},\ \partial g_{mn}/\partial x_1,\ T^1_\alpha,\ g_{\alpha m}T^m_\beta - g_{\beta m}T^m_\alpha, \tag{2.12}$$

where α, $\beta = 1, 2, 3, 4$ and $m, n = 2, 3, 4$ and where $g_{\alpha\beta}$ is the metric tensor and T^α_β the energy tensor. In the present case these continuity conditions are found to reduce to

$$e^{\omega(a,t)} = a^2R^2(t) \tag{2.13}$$

$$\omega'(a,t) = 2/a, \tag{2.14}$$

$$\alpha(a) = 1-a^2. \tag{2.15}$$

If we compare (2.6) and (2.7) with (2.10) and (2.11), it is easy to verify that the conditions (2.13)—(2.15) are satisfied if we choose the following:

$$\alpha(a) = 1 - a^2, \qquad \alpha'(a) = -2\,a, \tag{2.16}$$

$$\beta(a) = K\,a^3, \qquad \beta'(a) = 3\,K\,a^2, \tag{2.17}$$

$$\gamma(a) = 0, \qquad \gamma'(a) = 0. \tag{2.18}$$

The purpose of the foregoing investigation is to show that solutions (2.6)—(2.8) exist satisfying the boundary conditions at $r = a$. These conditions do not, of course, uniquely determine the functions α, β and γ. There is still a wide choice of the solution (2.1) open to us. Let us choose for $0 < r < b < a$ (b constant) a solution given by

$$ds^2 = -[R^*(t)]^2\,[(1 - k\,r^2)^{-1}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\Phi^2)] + dt^2, \tag{2.19}$$

where $k = +1$, and where

$$R^* = K^*\,(1 - \cos\eta),$$

$$t + \varepsilon = K^*\,(\eta - \sin\eta),$$

K^* and ε being arbitrary constants. This solution is a special case of (2.6) and (2.7) corresponding to a homogeneous region. It means that in the centre of the condensing region we choose a Friedmann model with constants different from those specifying (2.9) which applies to the universe outside $r = a$. For $b < r < a$ we shall still need the more general inhomogeneous solution (2.1), but we may let this region be as small as we please, and it is clear from the above that we can choose the arbitrary functions α, β and γ so that the boundary conditions (2.12) are satisfied at $r = a$ and $r = b$. Thus we idealize the condensation by a homogeneous part for $r < b$ (of different density from the rest of the universe) and a transition region $b < r < a$ and we study its growth by comparing the behaviour of the homogeneous Friedmann models (2.9) and (2.19).

In (2.19) we took k to be $+1$, so that the model is closed. This is in accordance with the condition (2.5) on α. However, had we chosen $1 - \alpha < 0$ we could have integrated (2.4) and proceeded in an exactly analogous way to the following solution (instead of (2.19)) for $r < b$:

$$ds^2 = -[R^*(t)]^2\,[(1 + r^2)^{-1}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\Phi^2)] + dt^2,$$

$$R^* = K^*(\cosh u - 1),$$

$$t + \varepsilon = K^*(\sinh u - u).$$

If we do this, then to satisfy the boundary conditions at $r = a$ it is necessary that, in the transition zone $b < r < a$, $1 - \alpha$ shall change sign. But this evidently will not affect the solution inside $r = b$, provided only that the solution in the transition zone satisfies at $r = b$ a set of conditions similar, *mutatis mutandis*, to (2.16)—(2.18).

Indeed, the constant k in (2.19) may be chosen quite irrespective of that in (2.9): we may choose the former to be $+1$, -1 or 0 whether the universe outside $r = a$ is closed ($k = +1$) or open ($k = 0$ or -1).

3. Condensations in a closed universe

In accordance with the results of the previous section, we shall take as a model for a condensation a comoving sphere $r = b$ in which the line-element is

$$ds^2 = -[R^*(t)]^2 [(1 - r^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\Phi^2)] + dt^2, \; (B) \tag{3.1}$$

where

$$R^* = K^*(1 - \cos\eta), \tag{3.2}$$

$$t + \varepsilon = K^*(\eta - \sin\eta), \tag{3.3}$$

$$8\pi\varrho^* = 6\,K^*/R^{*3}. \tag{3.4}$$

Outside $r = a (> b)$ we suppose that there is an expanding homogeneous world-model with line-element

$$ds^2 = -[R(t)]^2 [(1 - r^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\Phi^2)] + dt^2, \; (A) \tag{3.5}$$

where

$$R = K(1 - \cos\xi), \tag{3.6}$$

$$t = K(\xi - \sin\xi), \tag{3.7}$$

$$8\pi\varrho = 6\,K/R^3. \tag{3.8}$$

For $b < r < a$ there is a transition zone whose purpose is simply to ensure satisfaction of the boundary conditions at $r = a$ and $r = b$. Since we are using comoving coordinates, matter initially inside $r = b$ and outside $r = a$ will stay there, so no matter enters or leaves the transition zone. The important criterion for the development of condensation is the change in the ratio ϱ^*/ϱ.

Model (3.5) is supposed to represent the actual universe as a whole, so we may calculate K by using contemporary observed data. I shall take

$$\varrho = 2 \times 10^{-28}\ \text{gm./cm}^3., \tag{3.9}$$

$$\dot{R}/R = 2.8 \times 10^{-10}\ (\text{yrs.})^{-1}. \tag{3.10}$$

The average density ϱ is not known accurately, but (3.9) is within the range allowed by observation; $\dot{R}/R$ is HUBBLE's constant. Substituting (3.9) and (3.10) into (3.6)—(3.8) we find

$$K = 9.3 \times 10^9\ \text{yrs.} \tag{3.11}$$

$$T = 2.2 \times 10^9\ \text{yrs.}$$

$$\xi_T = 1.15\ \text{rad.}$$

T, ξ_T being the present values of t, ξ. The figure of 2.2×10^9 years for "the age of the universe" is rather low; in fact, even with the recent correction to HUBBLE's constant, the closed, pressure-free model (with $\Lambda = 0$) has an uncomfortably short time-scale. This is seen from the fact (TOLMAN, 1934b, p. 415) that the age of such a model (irrespective of density) is certainly not greater than $2R/3\dot{R}$, that is, about $2.4 \times \times 10^9$ years. However, in view of the present uncertainties in the observations, it seems unwise to rule out the closed model.

We shall now compare the two models A (3.5) and B (3.1). Let us for the moment put $\varepsilon = 0$ in (3.3): then both models start at a singular state of zero volume and infinite density at $t = 0$ ($\xi = \eta = 0$). They also finish at singular states when $\xi = \eta = 2\pi$, and their life-times are $2\pi K$ and $2\pi K^*$ respectively. The model B has the shorter life-time if $K^* < K$. This case is illustrated in Fig. 1 which shows graphs of the radii of the models (R and R^*) plotted against time. As a rough approximation we may take the singular state at t_B as representing the formation of a condensation of the matter inside $r = b$.[1] Thus to account for the fact that the nebulae have already formed we require that t_B shall be before the present time, i.e.

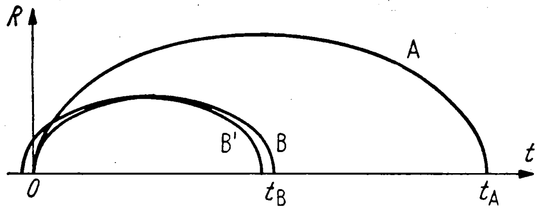

Fig. 1. Condensation in closed models

$$2\pi K^* = t_B < T = 2.2 \times 10^9 \text{ yrs.} \tag{3.12}$$

Let us suppose that the universe was initially homogeneous so that in its early history the density in the region $r < b$ followed curve A. Then if at a certain time, say $t = t_0$, a disturbance took place which caused the matter in $r < b$ to follow instead curve B, a condensation would have formed by time t_B. The problem which I wish to study is that of finding the magnitude of the perturbation — in terms of the change in ϱ_0 and $\dot{\varrho}_0$ at time t_0 — which would have been necessary to initiate the condensation.

The effect of putting $\varepsilon = 0$ in (3.3) is to make the initial singular state of model B occur at time $t = 0$. This is necessary if the inhomogeneity is present from the beginning of the universe; but if, as we are now supposing, the condensation starts from a perturbation at some later time t_0, we are interested in model B only for $t > t_0$ so there is no reason why we should not consider models, such as B' in Figure 1, in which $\varepsilon \neq 0$. In the following, therefore, we shall take as equations of the perturbed model (3.1)—(3.4) with $\varepsilon \neq 0$.

[1] Of course, the model ceases to apply as the singular state is approached because it is no longer permissible to ignore the pressure. However, to get a rough estimate of the time of condensation this approximation is sufficient.

Let us denote the initial perturbations by

$$\alpha = (\varrho_0^* - \varrho_0)/\varrho_0, \qquad \beta = (\dot\varrho_0^* - \dot\varrho_0)/\dot\varrho_0,$$

where the suffix 0 means the value at $t = t_0$. Then from (3.3), (3.4), (3.7) and (3.8) we find

$$1 + \alpha = n^6 \sin^6 \tfrac{1}{2}\, \xi_0 \operatorname{cosec}^6 \tfrac{1}{2}\, \eta_0\,, \tag{3.13}$$

$$1 + \beta = n^9 \sin^9 \tfrac{1}{2}\, \xi_0 \cos \tfrac{1}{2}\, \eta_0 \operatorname{cosec}^9 \tfrac{1}{2}\, \eta_0 \sec \tfrac{1}{2}\, \xi_0\,, \tag{3.14}$$

where

$$n^3 = K/K^*. \tag{3.15}$$

From (3.13) and (3.14) we have

$$\frac{1+\beta}{(1+\alpha)^{3/2}} = \frac{\cos \frac{1}{2}\, \eta_0}{\cos \frac{1}{2}\, \xi_0}\,. \tag{3.16}$$

Eliminating η_0 between (3.13) and (3.16), we find

$$n^2 = (1+\alpha)^{1/3} \left\{ 1 + \left[1 - \frac{(1+\beta)^2}{(1+\alpha)^3} \right] \cot^2 \tfrac{1}{2}\, \xi_0 \right\}. \tag{3.17}$$

This equation is exact; if, however, we suppose that α and β are so small that their squares and products may be neglected, it gives

$$n^2 = 1 + \tfrac{1}{3}\, \alpha + (3\, \alpha - 2\, \beta) \cot^2 \tfrac{1}{2}\, \xi_0. \tag{3.18}$$

The time t_B of the final singular state of model B is given by

$$\begin{aligned} t_B &= 2\,\pi\, K^* - \varepsilon \\ &= n^{-3}\, t_A - \varepsilon\,. \end{aligned} \tag{3.19}$$

Since $t_A\,(= 2\,\pi\, K)$ is known from (3.11) it remains only to find ε in order to calculate t_B from (3.18) and (3.19). We can determine ε from (3.3) and (3.7):

$$t_0 = K(\xi_0 - \sin \xi_0) = K^*(\eta_0 - \sin \eta_0) - \varepsilon\,. \tag{3.20}$$

Remembering that ξ_T, the present value of ξ, is about 1.15, so that ξ_0 must be less than this, and supposing that $|\alpha|$ and $|\beta|$ are small, say less than 1/10, we find from an approximate solution of (3.20) (together with (3.13)) that ε is negligible compared with $2\,\pi\, K^*$, so that this expression may be taken as the time of condensation, reckoned from $t = 0$. Since from (3.11),

$$2\,\pi\, K = 5.8 \times 10^{10} \text{ yrs.},$$

we find, using (3.12) and (3.15), that for a condensation in $r < b$,

$$n^3 > \frac{2\,\pi\, K}{T} = 27\,. \tag{3.21}$$

Equations (3.21) and (3.18) give a minimum value for $3\,\alpha - 2\,\beta$ at $\xi = \xi_0$ to produce a condensation. It is easy to verify that an initial increase in the density tends to produce a condensation, and so does an increase in the rate of change of density. Decreases in ϱ and $\dot{\varrho}$ tend to produce rarefactions.

From (3.17) we have approximately

$$\tan^2 \tfrac{1}{2}\,\xi_0 = (3\,\alpha - 2\,\beta)/(n^2 - 1);$$

imposing now the condition (3.21) that a condensation shall have formed by the present time, and once again taking $|\alpha|$, $|\beta| < \frac{1}{10}$, we find

$$\tan \tfrac{1}{2}\,\xi_0 < \tfrac{1}{4},$$

whence

$$\xi_0 < \tfrac{1}{2}$$

and

$$t_0 < 2 \times 10^8 \text{ yrs}. \qquad (3.22)$$

Hence if the condensation formed as a result of a small perturbation, this perturbation must have taken place earlier than 2×10^8 years after the initial singular state. At times later than this, condensations cannot be initiated except by perturbations of order 1/10 or greater.

Considering now values of t_0 less than (3.22), we find from (3.18) that the minimum perturbation in the density α (with $\beta = 0$) at time t_0 required to produce a condensation by the present time T is given approximately by

$$\alpha = \tfrac{1}{3}\,(n^2 - 1)\tan^2 \tfrac{1}{2}\,\xi_0 \sim \tfrac{1}{12}\,(n^2 - 1)\,\xi_0^2,$$

where $n^3 = 27$. Using (3.7) we find

$$\alpha \sim \frac{1}{12}\,(n^2 - 1)\left(\frac{6\,t_0}{K}\right)^{2/3},$$

and substituting for n and for K

$$\alpha \sim 5 \times 10^{-7}\, t_0^{2/3}. \qquad (3.23)$$

Thus 1000 years after the start of the universe a perturbation of about 5×10^{-5} is required to produce a condensation; and after 10^8 years the perturbation must be 1/10. If one takes $\alpha = 0$, then values of β of the same order are required.

From the above it is clear that the nebulae could have been produced by fairly small perturbations in the density in the early history of the universe. Of course, the simplification of putting the pressure equal to zero would not be permissible at the earliest stages. At what stage this does become admissible depends on the hypotheses made about the singular state, which determines the amount of radiation subsequently present.

The mechanism of condensation is worth noticing. A region where the density is higher than the average expands independently of the rest of the universe, and runs through its expansion and contraction more quickly. The motion of matter inside a comoving sphere is not influenced by the matter outside. This recalls a result of BONDI (1947), who proved this for a more general model than that considered here.

It should also be noticed that in the contraction occurring during the later stages of the condensation process the velocities of the condensing matter may become large. This is shown by the fact that as the model B tends towards its singular state at t_B, $\dot{R}^*$ tends to infinity. Now it was shown by HOYLE (1951) that deviation from spherical symmetry in a contracting mass can produce rotation. Thus the condensation process might explain the rotation of the nebulae, and the large velocities involved might account for the presence of turbulent matter in them. On this theory, the turbulence must have appeared during the formation of the nebulae, and not before.

Although small, the perturbations required to produce nebulae are, however, very much too large to have been caused by the random fluctuations of groups of molecules on ordinary gas theory, which gives for a collection of N molecules of an ideal gas

$$\frac{\delta \varrho}{\varrho} = N^{-1/2}. \tag{3.24}$$

The mass of a nebula is about 10^{44} gm. so if it is composed of hydrogen, $N = 3 \times 10^{67}$ and $\delta\varrho/\varrho \sim 10^{-34}$. Thus on this mechanism the formation of the nebulae would be vastly improbable. Alternatively, if one calculates from (3.24) the probable number of particles involved in a fluctuation of magnitude, say 10^{-5}, one finds that the number is 10^{10}. Thus on ordinary statistical theory the mass of the condensations would be a minute fraction of one gram.

4. Condensations in open universes.

For an open (ever-expanding) universe the line-element is (2.9) with $k = -1$ or 0. I shall deal only with the case where $k = -1$; the case $k = 0$ is similar, and the same conclusions apply. Taking $k = -1$, the model is

$$ds^2 = -[R(t)]^2 \, [(1 + r^2)^{-1} \, dr^2 + r^2 \, (d\theta^2 + \sin^2\theta \, d\Phi^2)] + dt^2, \, (A) \tag{4.1}$$

where

$$\begin{aligned} R &= K(\cosh u - 1), \\ t &= K(\sinh u - u), \\ 8\pi\varrho &= 6\,K/R^3, \end{aligned} \tag{4.2}$$

K being a positive constant. Using the same method as for the closed model, we adopt (4.1) for $r > a$, and for $r < b$, we take

$$ds^2 = -[R^*(t)]^2 \, [(1 + r^2)^{-1} \, dr^2 + r^2 \, (d\theta^2 + \sin^2\theta \, d\Phi^2)] + dt^2, \, (B) \tag{4.3}$$

where

$$R^* = K^* (\cosh v - 1),$$
$$t + \varepsilon = K^* (\sinh v - v),$$
$$8 \pi \varrho^* = 6 K^*/R^{*3}.$$

As contemporary data we may take

$$\varrho = 10^{-30}\,\text{gm./cm}^3.,$$
$$\dot{R}/R = 2 \cdot 8 \times 10^{-10}\,(\text{yrs.})^{-1},$$

which give

$$K = 1.3 \times 10^7\,\text{yrs.},$$
$$T = 3.5 \times 10^9\,\text{yrs.}, \tag{4.4}$$
$$u_T = 6.3,$$

T and u_T being the present values of t and u.

If we compare the models (A) and (B) above, we find curves like those shown in Figure 2, where in (B) we have taken $\varepsilon = 0$, as it can be shown that for small perturbations ε must be too small to affect significantly the time of condensation. Both models start from a singular state at $t = 0$, but there is no later singular state, and in both the density tends to zero as $t \to \infty$. To estimate the development of condensation we have to study how $(\varrho^* - \varrho)/\varrho$ changes with time. The present average density of a nebula is about 10^{-24}gm./cm^3. so that the present value of $(\varrho^* - \varrho)/\varrho$ is about 10^6. Taking, as in § 3,

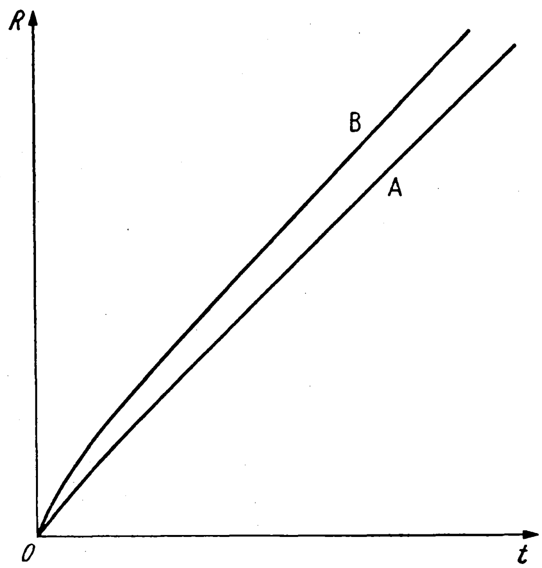

Fig. 2. Condensation in open models

$$n^3 = K/K^*$$

and

$$\alpha = (\varrho_0^* - \varrho_0)/\varrho_0, \qquad \beta = (\dot{\varrho}_0^* - \dot{\varrho}_0)/\dot{\varrho}_0,$$

the problem now is to find the value of n, and hence of α and β at given time t_0 of the perturbation, which is required to give a value of 10^6 for $(\varrho^* - \varrho)/\varrho$ at the present time T.

As in § 3 we find

$$n^2 = 1 + \tfrac{1}{3}\alpha - (3\alpha - 2\beta)\coth^2 \tfrac{1}{2} u_0,$$

u_0 being the value of u at time t_0. Let us now consider the value of $(\varrho^* - \varrho)/\varrho$ at some time t later than the time t_0 of perturbation. For our purposes ϱ^*/ϱ will be a sufficient approximation, and we have

$$\frac{\varrho^*}{\varrho} = n^6 \frac{\sinh^6 \frac{1}{2} u}{\sinh^6 \frac{1}{2} v} \tag{4.5}$$

and

$$\sinh v - v = n^3 (\sinh u - u). \tag{4.6}$$

As u, v tend to infinity (t very large), (4.6) gives approximately

$$e^v = n^3 \, e^u, \tag{4.7}$$

and from (4.5) and (4.7) we have

$$\varrho^*/\varrho = n^{-3}, \tag{4.8}$$

so that if n is small a condensation will eventually form. However, at the present time $u = 6.3$, so that the approximation (4.7) is not valid if n is, say, about 1/100. For this value of u and for small n we may approximate (4.6) by

$$\tfrac{1}{6} v^3 = \tfrac{1}{2} n^3 e^u,$$

so that

$$v = 3^{1/3} n \, e^{\frac{1}{3} u}. \tag{4.9}$$

Substituting (4.9) into (4.5) we find approximately

$$\frac{\varrho^*}{\varrho} = n^6 \frac{e^{3u}}{v^6} = \frac{1}{9} e^u.$$

With $u = 6.3$ this gives $\varrho^*/\varrho \sim 60$, so that the amount of condensation would not have been sufficient, even if n is small. (It is clear from (4.8) that if n is not small, no significant condensation would have formed.) Thus the mechanism suggested by Figure 2 cannot have been responsible for the formation of the nebulae.

Another mechanism of condensation in the open universe is represented by taking for the line-element B (inside $r = b$) a portion of a closed model, such as (3.1). As explained in § 2, this is permissible provided that the solution for the transition zone $b < r < a$ satisfies the boundary conditions. If instead of (4.3) we take (3.1), we have

$$\frac{\varrho^*}{\varrho} = n^6 \frac{\sinh^6 \frac{1}{2} u}{\sin^6 \frac{1}{2} \eta},$$

$$\eta - \sin \eta = n^3 (\sinh u - u), \tag{4.10}$$

and, approximately,

$$n^2 = (1 + \alpha)^{1/3} [-1 + (3\,\alpha - 2\,\beta) \coth^2 \tfrac{1}{2} u_0], \tag{4.11}$$

where once again ε is neglected since the perturbations are small. As in § 3, the time of formation of the condensation may be identified roughly with the time of the singular state of the model (3.1) which occurs at $\eta = 2\pi$. Therefore, from (4.10) we have the following condition for a condensation to have formed before the present time

$$2\pi < n^3 (\sinh u_T - u_T);$$

substituting $u_T = 6.3$, we find

$$n^3 > 0.024. \tag{4.12}$$

From (4.11) and (4.12) we find, if $|\alpha|$, $|\beta| < 1/10$, that approximately

$$\coth^2 \tfrac{1}{2} u_0 > 2.2$$

whence

$$t_0 < 1.1 \times 10^7 \text{ yrs.} \tag{4.13}$$

Thus the perturbation must have taken place very early in the history of the universe.

If we consider perturbations at times earlier than (4.13), we find from (4.11), taking $\beta = 0$, that the minimum perturbation in the density at time t_0 needed to produce a condensation by time T is roughly

$$\alpha = \tfrac{1}{3}(1 + n^2) \tanh^2 \tfrac{1}{2} u_0,$$

where n is given by (4.12). Using (4.2) and (4.4) we find that for small t_0, the minimum perturbation α necessary is given by

$$\alpha \sim 5 \times 10^{-6}\, t_0^{2/3}. \tag{4.14}$$

Comparing (3.23) and (4.14) we see that, as would be expected, the perturbation required to cause a condensation in the open model is considerably larger than that needed in the closed model at the same time. In both cases the mechanism of the condensation process is the same — that is, in the condensing region there is eventually a contraction and approach to a singular state. In both cases also the perturbation required to produce nebulae is much larger than could be expected from the random fluctuations of the ordinary kinetic theory of gases.

5. *Condensations in* LEMAÎTRE's *model*

In the models considered in the previous section I have taken the cosmological constant, Λ, to be zero. If Λ is not zero there is a great diversity of models which can be chosen to satisfy the three fundamental observed data mentioned in the Introduction. Of these, two have been studied in some detail: the Eddington-Lemaître model, which arises from a perturbation of the static Einstein universe, and the Lemaître model, in which Λ is positive and which expands without limit from a point source. Both these models have been thought to offer favourable possibilities for explaining the formation of the nebulae. I do not intend in this paper to discuss the Eddington-Lemaître model, but I shall deal briefly with the formation of condensations in the Lemaître model.

If, instead of (1.1), one starts with the field equations

$$-8\pi T_{ik} = G_{ik} - \tfrac{1}{2} G g_{ik} + \Lambda g_{ik},$$

and takes the pressure to be zero, one finds that the following is a solution

$$ds^2 = -[R(t)]^2 [(1-r^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\Phi^2)] + dt^2, \quad (5.1)$$

where

$$\left(\frac{\dot R}{R}\right)^2 = \frac{1}{3}\Lambda - \frac{1}{R^2} + \frac{2K}{R^3}, \quad (5.2)$$

$$8\pi\varrho = 6K/R^3, \quad (5.3)$$

K being a positive constant. Equations (5.1)—(5.3) represent LEMAÎTRE's model in the pressure-free case. If one puts $\Lambda = 0$, it reduces to the ordinary closed point-source model (3.5).

The integration of (5.2) (which can be carried out with elliptic functions) was studied by DE SITTER (1931). Put

$$y = R/2K, \qquad \tau = t/2K;$$

then (5.2) gives

$$\left(\frac{dy}{d\tau}\right)^2 = \frac{1}{y} - 1 + \gamma y^2, \quad (5.4)$$

where

$$\gamma = \tfrac{4}{3} K^2 \Lambda. \quad (5.5)$$

With the help of (5.4) de SITTER classified the solutions of (5.2) as follows:

I. if $\gamma > 4/27$, there is only one solution, in which R increases from zero to infinity;

II. if $0 \leqq \gamma \leqq 4/27$, there are two solutions, in one of which R oscillates between 0 and R_1, and in the other R decreases from infinity to R_2 and then increases from R_2 to infinity, R_1 and R_2 being certain constants;

III. if $\gamma < 0$ there is one solution, in which R oscillates between 0 and R_1.

The Lemaître universe falls within Case I, and is represented by curve I in Figure 3, which is adapted from DE SITTER's diagram. The mechanism of condensation proposed by LEMAÎTRE (1933) is as follows. Let us suppose, as we did in sections 3 and 4, that the condensing region $r < b$ is represented by part of another homogeneous model; let us also suppose that in $r < b$ the initial density and velocity are such that the motion follows the oscillating solution of Case II. This requires $0 \leqq \gamma \leqq 4/27$. Then the matter in $r < b$ will not go on expanding indefinitely but will follow a curve such as IIa or IIb in Figure 3. In the former case ($\gamma = 4/27$) the proper radius of the region, and the density, tend to constant values as $t \to \infty$; and in the latter case there will eventually be a contraction of the space and the density will start to increase as the model approaches a singular state. In either case a condensation will form.

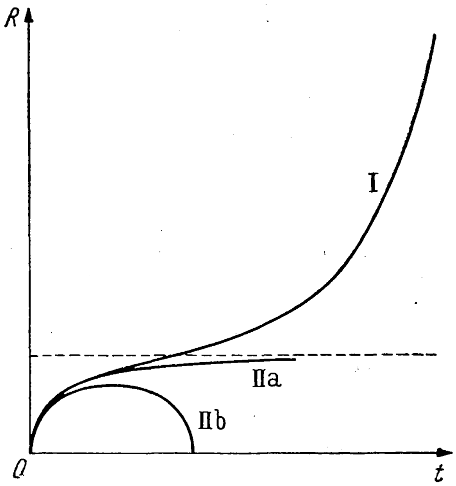

Fig. 3. Condensation in LEMAÎTRE'S model

In order to find the magnitude of the perturbations which would be required to cause condensation in this way, we need to estimate the values of α and β (defined in section 3) needed to cause an appropriate alteration in γ. From (5.3) and (5.5) we find

$$8\pi\varrho = \Lambda\,\gamma^{-1} y^{-3},$$

$$8\pi\dot{\varrho} = -\sqrt{3}\,\Lambda^{3/2}\,\gamma^{-3/2} y^{-4}\frac{dy}{d\tau}.$$

After a calculation, these, together with (5.4), give

$$\alpha \equiv \frac{\delta\varrho}{\varrho} = -\frac{\delta\gamma}{\gamma} - 3\frac{\delta y}{y}, \tag{5.6}$$

$$\beta \equiv \frac{\delta\dot{\varrho}}{\dot{\varrho}} = \frac{\delta\gamma}{2\gamma}\left[\frac{3 - \frac{3}{y} - 2\gamma y^2}{\frac{1}{y} - 1 + \gamma y^2}\right] + \frac{\delta y}{y}\left[\frac{4 - \frac{9}{2y} - 3\gamma y^2}{\frac{1}{y} - 1 + \gamma y^2}\right]. \tag{5.7}$$

Eliminating δy between (5.6) and (5.7) we have

$$\frac{\delta\gamma}{\gamma} = 6\beta\left(\frac{1}{y} - 1 + \gamma y^2\right) + 2\alpha\left(-\frac{9}{2y} + 4 - 3\gamma y^2\right). \tag{5.8}$$

Examining the Lemaître model in the manner of DE SITTER, we find that to fit contemporary data we must take γ greater than 0.149. On the other hand, for LEMAÎTRE's method of condensation to work we need $\gamma \leqq 4/27 = 0.14815$. Thus the perturbations α and β must be responsible for a change in γ inside $r = b$ of the order of 0.001, so that $\delta\gamma/\gamma \sim 10^{-2}$. It is easily found from (5.8) that to produce such a value for $\delta\gamma/\gamma$ (except at times so early that the pressure-free model does not apply) the perturbations required are of the same order as those needed in the models considered in the previous sections.

A more comprehensive treatment of the effect of perturbations on the Lemaître model would involve the solution of equation (5.2) and an analysis similar to those of sections 3 and 4. However, in regard to the process of condensation, the only important difference in the model from those with $\Lambda = 0$ is the existence of the mechanism discussed above. Therefore there is no reason to suppose that the Lemaître model is more favourable than the others to the formation of condensations from very small perturbations.

6. *Conclusion*

The main conclusion of this paper that in the world-models of general relativity with zero cosmological constant the nebulae cannot have resulted from gravitational instability following perturbations of magnitude predicted by ordinary statistical theory. Although this result depends on the acceptance of a highly simplified model of a condensation, there seems no reason to suppose that a more realistic model would give radically different conclusions. It depends also on the use of the pressure-free model of the universe throughout, which means that the calculations apply to the later history of the universe; but as LIFSHITZ found that in models with pressure perturbations could not become large, it seems probable that only the pressure-free era is important for the formation of condensations.

To obtain a satisfactory theory of the formation of the nebulae it appears to be necessary either to use a model with a longer time-scale, or to find a source of larger fluctuations in density or velocity.

A longer time-scale would help the theory of condensations in two ways. On the one hand, it would give the small perturbations predicted by (1.2) time to grow, and on the other, it would, if long enough, permit the occurence of occasional large fluctuations which could grow into nebulae comparatively quickly. Models of general relativity with long time-scales require $\Lambda \neq 0$. An obvious candidate (but not the only one) is the Eddington-Lemaître model which starts from a perturbation of the Einstein universe. If, as is now thought possible, the heavy elements can have a contemporary origin, the main objection to this model (that it has no state of high density) is removed. Since the Einstein universe has an infinite past, there is plenty of time for small perturbations to grow, or for improbably large fluctuations to occur.

In regard to the second possibility, GAMOW (1952) has suggested that the large fluctuations required might have arisen from turbulence of the cosmic medium. This does not help us very much unless the origin of the turbulence can be explained. One might of course add to the mystery of the singular state by postulating primordial turbulence, but this would be only a last resort; and even then, it would have to be shown that the primordial turbulence could have persisted through the era of high pressure. It was shown in section 3 of this paper that high velocities can arise in the course of condensation, and it is possible that turbulence is a result rather than a cause of the condensation process. It seems preferable therefore to seek another reason for the large perturbations.

Another suggestion is one of TERLETSKY (1952) that ordinary gas theory may be quite inadequate to deal with very large masses of gravitating gas. TERLETSKY considers that such large masses are liable to much larger fluctuations than those predicted by (1.2). It may be that the large perturbations required for the formation of the nebulae can be accounted for by some development of TERLETSKY's theory.

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