A COMET MODEL. I. THE ACCELERATION OF COMET ENCKE

FRED L. WHIPPLE
Harvard College Observatory
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ABSTRACT

A new comet model is presented that resolves the chief problems of abnormal cometary motions and accounts for a number of other cometary phenomena. The nucleus is visualized as a conglomerate of ices, such as H_2O, NH_3, CH_4, CO_2 or CO, (C_2N_2?), and other possible materials volatile at room temperature, combined in a conglomerate with meteoric materials, all initially at extremely low temperatures ($<50°$ K). Vaporization of the ices by externally applied solar radiation leaves an outer matrix of nonvolatile insulating meteoric material. Quantitative and qualitative study shows that heat transfer through thin meteoric layers in a vacuum is chiefly by radiation, that the heat transfer is inversely proportional to the effective number of layers, and that an appreciable time lag in heat transfer can occur for a rotating cometary nucleus. Because of the time lag, such a cometary nucleus rotating in the "forward" sense will emit its vaporized ices with a component toward the antapex of motion. The momentum transfer from the kinetic velocity of the emitted gas will propel the nucleus in the forward sense, reduce the mean motion, and increase the eccentricity of the orbit. Such orbital effects occur for Comet D'Arrest; the mean daily motion of Comet Wolf I also appears to be decreasing.

Retrograde rotation can produce an acceleration in mean motion and a decrease in eccentricity, as observed for Comet Encke. If the decelerating force component is taken as one-quarter its maximum theoretical value, the present observed acceleration in the mean motion of Comet Encke can be produced by a loss of 0.002 of its mass per revolution. The corresponding mass loss for Comet D'Arrest is 0.005. For both comets the observed changes in eccentricity are obtained if the force acts proportionately to the solar energy flux but is cut off at a solar distance of about 2 A.U.

A second paper (Part II) soon forthcoming will be concerned with the physical problems of comet structure, loss of meteoric and gaseous material, and correlations with observed meteoric phenomena.

A. INTRODUCTION

The recent and valuable contribution by A. J. J. van Woerkom[1] on the origin of comets has strengthened the growing confidence in the concept that comets are ancient members of the solar system. He has shown that the process of replenishment of the periodic comets from the extremely long-period comets via capture by Jupiter appears to be the most plausible, if not the only plausible, process for maintaining the supply of periodic comets. The remaining question as to the statistical stability of a solar family of comets with semimajor axes up to about 10,000 A.U. has been studied by E. Öpik.[2] The losses to such a system by the gravitational action of passing stars is not serious over a period of even 3×10^9 years. Only a close approach of a passing star to the sun would remove the sun from control of a large fraction of the comets. The loss by stellar attraction for individual comets with periods up to a million years is statistically unimportant over such a long interval of time; but comets with longer periods will suffer statistical increases in perihelion distance that will tend appreciably to place them beyond Jupiter's reach.

There still remains, however, a discrepancy of approximately 20 between van Woerkom's calculated number of Jupiter "captures" of long-period comets into short-period orbits (one per 650 years) and the estimated loss of three periodic comets per century. This discrepancy might be removed by assuming a greater number of comets with peri-

[1] *B.A.N.*, **10**, 445, 1948; see also H. N. Russell, *A.J.*, **33**, 49–61, 1921; and H. A. Newton, *Mem. Nat. Acad. Sci. Washington*, **6**, 1, 1893.

[2] *Proc. Amer. Acad. Arts and Sci.*, **67**, 169, 1932. In a recent and extremely important paper, J. H. Oort (*B.A.N.*, **11**, 91, 1950) has independently expanded the work of Öpik to demonstrate that an extended cloud of comets about the sun would be sufficiently stable, yet disturbed enough by passing stars to provide comets for Jupiter's capture after 3×10^9 years. The postulated cloud extends about 1 parsec about the sun.

helion in the neighborhood of Jupiter, by establishing more certainly the rate of disintegration of comets, or by accepting G. Fayet's[3] suggestion that, statistically, several short-period comets arise from each parent-comet.

In any case, the lifetime of a short-period comet must lie generally in the range of from 3000 (one hundred comets being lost at a rate of three per century) to possibly 60,000 years. Probably more important is the number of small perihelion passages that can be weathered by a comet—of the order of several hundreds, at least, for perihelion distances as small as 0.5–1.0 A.U. For considerably greater perihelion distances the number probably increases to several thousand, thus permitting comets with periods up to 10^6 years to persist throughout all or most of the past history of the solid earth.

Even though parts of the preceding discussion are somewhat conjectural, we must certainly accept the conclusion that individual short-period comets cannot exist indefinitely in their present orbits and also that they must previously have existed at great distances from the sun, where their temperature throughout remained at extremely low values. K. Wurm[4] has discussed certain effects of such low temperatures in cometary phenomena. N. T. Bobrovnikoff[5] and P. Swings[6] have pointed out that certain possible parent-compounds, such as CO_2, H_2O, and NH_3, may be responsible for the observed radicals, such as CO, OH, and NH, in cometary spectra; Swings has suggested that these materials would exist in the solid state within the nuclei of comets.

In the present discussion I propose to investigate the possibility that the molecules responsible for most of the light of comets near perihelion arise primarily from gases long frozen in the nuclei of comets. Furthermore, I propose that these primitive gases constitute an important, if not a predominant, fraction of the mass of a "new" or undisintegrated comet.

On the basis of these assumptions, a model comet nucleus then consists of a matrix of meteoric material with little structural strength, mixed together with the frozen gases—a true conglomerate. Since no meteorites are known certainly to arise from cometary debris, we know very little about the physical structure of the meteoric material except that the pieces seem generally to be small. Hence we assume that the larger pieces are perhaps a few centimeters in radius and the smallest are perhaps molecular. As a convenience in terminology, the term "ices" will be used in referring to substances with melting points below about 300° C and "meteoric material" to substances with higher melting points.

Our only chemical knowledge of the meteoric material comes from the spectra of meteors,[7] which tell us that Fe, Ca, Mn, Mg, Cr, Si, Ni, Al, and Na, at least, are present. Physically the meteoric material is strong enough to withstand some shock in the atmosphere, but more than 3 per cent of the Harvard photographic meteors are observed to break into two or more pieces. A much larger percentage show flares in brightness, an indirect evidence of breaking. The high altitude of the disappearance of the photographic Giacobinid meteors of October 9, 1946, as observed by P. M. Millman and analyzed by L. Jacchia and Z. Kopal,[8] suggests that those meteoric bodies may have been unusually fragile or porous. It is difficult to defend the hypothesis that, as a whole, the bodies producing photographic meteors possess great physical rigidity or strength.

A careful determination of the relative abundances of the primitive ices in the nucleus of a comet and their physical properties will require an exhaustive study of the theory of

[3] *Bull. astr.*, **28**, 168, 1911.

[4] *Mitt. Hamburger Sternw., Bergedorff*, Vol. **8**, No. 51, 1943.

[5] *Rev. Mod. Phys.*, **14**, 164–178, 1942.

[6] *Ann. d'ap.*, **11**, 124, 1948.

[7] See, e.g., P. M. Millman, *Harvard Ann.*, **82**, 113, 1932, and **82**, 149, 1935; F. G. Watson, *Between the Planets* (Philadelphia: Blakiston Co., 1941), p. 108.

[8] Private communication.

cometary spectra and related phenomena, including evolutionary hypotheses. Only a few comments will be made here. The observed gases CH, CH^+, CH_2, CO, NH, NH_2, OH, and OH^+ can be accounted for by four possible parent-molecules of great stability, viz., CH_4, CO_2, NH_3, and H_2O. Photodissociation appears capable of producing the various radicals from these parent-molecules, although Wurm prefers CO instead of CO_2. Only the very important observed C_2, N_2^+, and CN molecules are unaccounted for above. The choice of C_2N_2 as a parent-molecule does not seem desirable because the dissociation of C_2N_2 is exothermic; nevertheless, I shall include it in the present discussion for lack of a better substitute (C_2H_2, N_2, HCN?). Quite possibly some of the radicals can exist permanently at very low temperatures.

The metals Na, Fe, Ni, and Cr—all observed in meteor spectra—have been observed in comet spectra at small solar distances. They require the presence of metals or, more generally, meteoric materials in molecular or atomic forms within the comet nucleus.

Some physical data for five of the possible parent-gases are given in Table 1.

TABLE 1*

PROPERTIES OF CERTAIN MOLECULES

	MOLECULE				
	CH_4	CO_2	NH_3	C_2N_2	H_2O
Melting point (° K)	90	217	198	239	273
Heat of fusion (cal/gm)	50	45	108	....	80
Boiling point at 1 atm. (° K)	111	195	240	252	373
Heat of vaporization from solid (cal/gm)	188+	138+	435+	103++	670+
Vapor pressure at 191° K (atm.)	45.8	0.74	0.038	8.0×10^{-3}	3.7×10^{-7}

* Sources: *International Critical Tables of Numerical Data* (New York: McGraw-Hill Book Co., 1933) and the *Handbook of Chemistry and Physics* (Cleveland: Chemical Rubber Pub. Co., 1949).

As our model comet nucleus approaches perihelion, the solar radiation will vaporize the ices near the surface. Meteoric material below some limiting size will blow away (Part II, forthcoming) because of the low gravitational attraction of the nucleus and will begin the formation of a meteor stream. Some of the larger or denser particles may be removed by shocks (see below), but the largest particles or matrix will remain on the surface, to produce an insulating layer. After a short time (probably in the geologic past for all known comets) the loss of gas will be reduced materially by the insulation so provided.

A comet such as Encke's, if made of a nonvolatile solid, would come eventually to a temperature of approximately 140° K at aphelion.[9] Thus CH_4 would melt and vaporize quickly, while the other ices of Table 1 would vaporize more slowly. At perihelion temperatures, all the substances in Table 1 would be gaseous, even under high pressures. The quasi-equilibrium state arising from a slow external heating of the extremely cold ices can be visualized qualitatively as follows: at the base of the meteoric layer, only the ice with the lowest vapor pressure will still remain; hence this layer will consist only of "rotten" ice (H_2O) and meteoric material; the next layer will contain, in addition, C_2N_2 (if present), etc.

It is important to note that practically all heat reaching the ice (H_2O) at temperatures above about 180° K will be used in vaporization. The H_2O vapor that does not escape the comet will condense on the cooler layers beneath, producing evaporation of these materials.

[9] M. G. J. Minnaert, *Kon. Ned. Akad. Wet. Amsterdam*, **50**, 826, 1947.

The outer icy layers will arrive at a quasi-equilibrium state in the order of the vapor pressures of the ices at low temperatures. The thicknesses of the various layers will depend upon the temperature gradient, which, in turn, will depend upon the effective heat conductivities of the layers and the temperature of the outer layer. It will be shown below that the effective conductivities of such a conglomerate must be very low. The deep interior of the cometary nucleus will remain extremely cold, not only because of the low heat conductivity but also because the available heat will be used in vaporization, an extremely effective cooling mechanism in a vacuum. Hence the comparative rates of escape of the various primitive gases from the nucleus will depend primarily not upon their physical-chemical properties but upon their abundances. Some second-order effects may occur because of the variation of temperature gradient in the upper layers with the external heating; but such effects should not be manifest, for example, in correlations of cometary emission spectra with age except in the most extreme stages of disintegration.

The weakening of the upper layers of the icy core by selective vaporization of the ices may be expected to produce cometary activity of considerable intensity, especially near the sun. The surface gravities of cometary nuclei are certainly extremely low; hence surprisingly weak structures can persist over rather large areas of the nucleus. At irregular intervals collapses must occur. The heated meteoric material will then fall into the ices and produce rapid vaporization. The dust and smaller particles held in the upper layers will be shaken out and blown away, so that insulation produced by this material will be much reduced. Solar heat, consequently, will be much more effective in vaporizing the ices in the pit until equilibrium is again established. Such "cave-ins" might spread over appreciable areas. Other effects might occur if "pockets" of an ice with low melting points exist within an ice of higher melting point. Phenomena of mildly explosive, jet, or cracking types may occur, forcing out pieces of material much larger than those carried normally by the outgoing gas. Hence the type of nuclear activity that is observed for large comets with small perihelion distances would be expected from this type of comet model.

If the primitive ices constitute a large percentage of the total mass, the comet truly disintegrates with time. Its actual substance vaporizes; the surface gravity decreases; and, finally, all activity ceases as the last of its ice reservoir is exhausted. The observed sequence of phenomena in dying comets is entirely consistent with this picture. In the later stages, only a very small nucleus of the largest meteoric fragments remains (note the asteroid Hidalgo as a possible example).

The period of rotation of a comet with a single spheroidal nucleus would generally remain constant with age, so that the comet might dissipate slowly and uniformly. If, however, the nucleus were multiple or irregular in shape, the vaporization of ices could materially affect the rotation. Suppose, for example, a part of the surface were nearly in a plane passing through the center of gravity of the nucleus, while the remaining surface were generally smooth and approximately oval in shape. Meteoric material would fall from the vertical surface, exposing it to the full action of sunlight. Hence the excess of gas evolved from this surface would exert a force moment on the nucleus as a whole.

The effect of the resulting rotation, depending upon the initial circumstances, might easily produce rotational instability, permitting the sun's tidal action to complete the splitting of the nucleus. If the larger parts of the separated nucleus were unstable, the comet might disappear quickly. On the other hand, the pieces might be large enough to persist for a long period of time as individual comets. In fact, the phenomenon of splitting has occurred for several comets and has been followed by disappearance in some cases, but not in others. Either possibility may be expected on the basis of the present comet model, depending upon the mass, shape, and rotation of the nucleus.

It is clear that the answers to certain of the problems concerning the proposed comet model can best be determined in the laboratory rather than by theory. A pertinent experiment would involve the making of conglomerates of the various ices and meteoric materials, submitting them to small pressures and then observing the vaporization when the conglomerates were exposed to radiation from one surface, the other surfaces remaining insulated or refrigerated. An exhaust pump could simulate the conditions of low gas pressure in space, while even the action on meteoric material might be studied by utilizing fine powders to compensate the model for the large terrestrial gravity.

Even without such experiments, however, the suggested model of a comet nucleus is subject to a number of tests, both by theory and by observation. Certain phases of the problem will be discussed in the following sections, and the conclusions will be applied to demonstrate possible mechanisms for the observed acceleration of the mean motion of Comet Encke (and similar phenomena for other comets) and to explain in more detail the ejection of meteoric material from comet nuclei.

B. THE PROBLEM OF HEAT TRANSFER

It is obvious that the total solar radiation falling upon our model comet nucleus would be sufficient, in a relatively short time, to melt and vaporize quite sizable masses. In 1 year at 1 A.U. from the sun, a layer of ice some 4 meters thick would be vaporized from the surface of a spherical body, if all the solar radiation were absorbed. A much thicker layer of the other ices in Table 1 would be lost. Hence a cometary nucleus, 1 km in diameter and made of such ices, would scarcely persist for a hundred perihelion passages within 1 A.U. Also Minnaert[10] has shown that the temperature rise within the solid nucleus of a periodic comet is relatively rapid if the material conducts heat like ordinary stone. The surface layers of meteoric material, however, will greatly reduce the rate of heat transfer as compared to that of a solid body, and vaporization will maintain a low internal temperature. Let us consider, therefore, the likely processes of heat transfer.

If the meteoric layer is a coarse aggregate, very poorly cemented, direct conduction of heat by solids will be very slow because of the small areas of contact between discrete particles on the surface of the nucleus. The coefficient of heat conduction for the compact solid may be reduced by the order of ten thousand times, making this form of heat transfer negligible except within the particles themselves.

Heinrich Hertz[11] demonstrated that, for a steel sphere pressed by its own weight under the earth's gravity against a rigid horizontal steel plane, the radius of the circle of pressure contact, β, is related to the radius of the sphere, s, approximately as follows: $\beta/s = 10^{-3} \times s^{1/3}$ (mm). Since the surface gravity of a cometary nucleus is several orders of magnitude less than that of the earth, since the meteoric layer will be relatively thin, and since irregular particles should have much the same area of contact as spheres and the same order of elasticity as steel, it can be seen easily that the areas of contact for the transmission of heat among the particles may be less than 10^{-4} the cross-section of the solid material.

Further confirmation of this argument is provided by laboratory measures of heat conductivity through powders at various air pressures. The work of M. Smoluchowski[12] on powders with grain sizes between 0.0003 and 0.03 cm, and of W. G. Kannuluik and L. H. Martin[13] on similar powders has been reviewed by A. F. Wesselink.[14] The conductivities in pressures as low as 0.05 mm of *Hg* are in the neighborhood of 3×10^{-6} cal cm^{-1} $sec^{-1}(°C)^{-1}$, as compared to typical conductivities of 0.14 for iron and 0.005 for sandstone. Wesselink, whose problem of radiation from the lunar surface closely parallels the heat-conduction problems of this discussion, concludes that heat transfer by

[10] *Ibid.*

[11] "Contact of Elastic Solids," from *Misc. Papers*, trans. D. E. Jones and G. H. Schott (1896), p. 159.

[12] *Bull. Acad. Sci. Cracovie, A*, 1910, p. 129, and *A*, 1911, p. 548.

[13] *Proc. R. Soc. London, A*, **141**, 144, 1933.

[14] *B.A.N.*, **10**, 351, 1948.

radiation becomes important *in vacuo* for particles of the order of 0.05 cm and larger, at temperatures in the neighborhood of 300° K.

Conduction of heat by gases must be small for our cometary nucleus. Where the mean free path of molecules is greater than the dimensions of the interstices between the meteoric particles, the conduction will increase with the density of the gas, but the heat conducted will be negligible. Interplanetary space in the neighborhood of the earth's orbit contains less than 10^3 electrons per cubic centimeter[15] and therefore probably a comparable number of atoms, since all are presumably ionized. Mean free paths are large compared even to the cometary nucleus. The only gas capable of carrying appreciable heat would be that vaporized within the nucleus, again at a relatively low density. Hence, for the moment, we can neglect gas conduction of heat as also negligible; but the subject must be reinvestigated when accurate information is available as to the quantity of gas involved.

The scattering of sunlight among the meteoric particles will fall off exponentially with depth, and, for particles of low probable albedo, cannot contribute greatly to heat transfer.

We are left, therefore, with the radiative transfer of heat from layer to layer via the normal low-temperature radiation and absorption by the meteoric particles, as the most likely mechanism for the transfer of solar energy from the surface of the nucleus to the icy core.

To gain an idea as to the properties of such radiation transfer in a vacuum, let us visualize a situation that will represent crudely the equilibrium conditions.[16] Suppose we substitute for the meteoric layers ideal flat layers of gray-body material with zero thickness, complete opacity, zero heat capacity, a low-temperature albedo of A_1, and a solar-radiation albedo of A_0. Suppose also that these layers, 1, 2, . . . , i, . . . , n, are infinite parallel planes, separated by unspecified but finite distances. Let the normal energy flux per unit area absorbed by both sides of the surfaces be represented by F_i and the corresponding radiation from each of the two sides be represented by E_i.

For boundary conditions we postulate that layer 1 absorbs the normal flux of solar radiation $(1 - A_0)F_0$ and that layer n, on the side away from the incoming radiation, is connected to a heat sink at constant temperature T_n, which absorbs excess energy at a rate Q per unit area and time, where

$$F_\eta - E_\eta = Q \,. \tag{1}$$

For all the other layers,

$$F_i = 2E_i \qquad (i \neq n) \,. \tag{2}$$

Layer 1 absorbs the flux $(1 - A_0)F_0$ from the sun, $E_2\,(1 - A_1)(1 + A_1^2 + A_1^4 + \ldots)$ after reflections from layer 2, and $E_1\,(1 - A_1)(A_1 + A_1^3 + A_1^5 + \ldots)$ by reflection of its own radiation from layer 2.

If we let A be the corrected albedo after multiple reflections, given by

$$1 - A = (1 - A_1)\,(1 + A_1^2 + A_1^4 + \ldots)\,, \tag{3}$$

then

$$A = (1 - A_1)\,(A_1 + A_1^3 + A_1^5 + \ldots)\,. \tag{4}$$

[15] F. L. Whipple and J. Gossner, *Ap. J.*, **109**, 380, 1949.

[16] This simplified solution, derived independently by the author, embodies the basic viewpoint given by Wesselink (*loc. cit.*).

Hence, from equations (2), (3), and (4) we derive the equilibrium equations for the layers 1, i, and n, noting that the ith layer receives its reflected radiation from two sides:

$$2E_1 = (1 - A_0) F_0 + AE_1 + (1 - A) E_2 , \tag{5}$$

$$2E_i = (1 - A) E_{i-1} + 2 AE_i + (1 - A) E_{i+1} \qquad (i \neq 1, n) , \tag{6}$$

and

$$F_n = (1 - A) E_n - 1 + AE_n . \tag{7}$$

Since no heat is taken up by the system except at the nth layer, the quantity Q is given by

$$Q = (1 - A_0) F_0 - E_1 , \tag{8}$$

leading to the following relation, from equation (1):

$$F_n = Q + E_n = (1 - A_0) F_0 - E_1 + E_n . \tag{9}$$

From equations (2), (5), (6), (7), and (9) we can now write the n equations of equilibrium in the form:

$$\frac{2 - A}{1 - A} E_1 - E_2 = \frac{1 - A_0}{1 - A} F_0 , \tag{10_1}$$

$$E_1 - 2E_2 + E_3 = 0 , \tag{10_2}$$

. . . .

$$E_{i-1} - 2E_i + E_{i+1} = 0 , \tag{10_i}$$

. . . .

$$E_{n-2} - 2E_{n-1} + E_n = 0 , \tag{10_{n-1}}$$

$$\frac{1}{(1 - A)} E_1 + E_{n-1} - E_n = \frac{1 - A_0}{1 - A} F_0 . \tag{10_n}$$

Since we have established boundary conditions by F_0 and E_n, there are only $n - 1$ unknowns in these n equations. Hence the determinant must be zero; the sum of the last $n - 1$ equations of (10), is, indeed, identical with the first.

We can solve for E_1 from the sum of the $n - 1$ equations $(10_1) + (10_3) + \ldots (i - 2)(10_i) + \ldots (n - 2)(10_n)$, with the following results:

$$E_1 = \frac{n - 1}{n - A} (1 - A_0) F_0 + \frac{1 - A}{n - A} E_n ; \tag{11}$$

and, by equation (8),

$$Q = \frac{1 - A}{n - A} [(1 - A_0) F_0 - E_n] . \tag{12}$$

By substitution in the successive equations (10), we find

$$E_i = \frac{n - i}{n - A} (1 - A_0) F_0 + \frac{i - A}{n - A} E_n . \tag{13}$$

We see that Q, the flux transferred to the heat sink, approaches zero for large values of n, even though no heat is retained by the intermediate layers. It can be shown easily that this result holds even in the limiting case when $A = A_1 = 0$, a situation where the albedo is zero and the layers absorb all the radiation incident upon them.

For large values of n, we can differentiate equation (13) for the gradient of E_i with layer number i, obtaining,

$$\frac{dEi}{di} = \frac{-1}{n-A}[(1-A_0)F_0 - E_n]. \tag{13a}$$

Hence the gradient in emission is uniform with layer number. This shows that the emission falls uniformly from layer 1 (if $[1 - A_0]F_0 > E_n$) to the heat sink and that the emission gradient is inversely proportional to the number of layers. The total heat transfer is thus proportional to the emission gradient. It can be shown that the temperature gradient is not linear with layer number.

The maximum value of the surface temperature of the nucleus of the comet is of considerable interest. Its value, obtained from equation (11) and Stefan's law, is given by

$$T_1^4 = \frac{(n-1)(1-A_0)}{(n-A)(1-A_1)}\frac{F_0}{\sigma} + \frac{1-A}{n-A}T_n^4. \tag{14}$$

We can derive a maximum value of T_1 by setting $n = \infty$ in equation (14). The result is

$$T_1\ (\text{max.}) = \left(\frac{1-A_0}{1-A_1}\frac{F_0}{\sigma}\right)^{1/4}. \tag{15}$$

If A_0 and A are small, approximately 0.1, and if we equate them, we can derive the well-known relation for the maximum temperature[17] of the comet model:

$$T_1\ (\text{max.}) = 390^\circ r^{-1/2}(\cos\theta)^{1/4}, \tag{16}$$

where r is the solar distance in astronomical units and θ is the zenith distance of the sun from a point on the sunlit side of the comet.

Equation (16), which agrees well with the measurements by E. Pettit[18] of the temperature (370° K) of the lunar surface at the subsolar point, leads to relatively high values of the temperature. The melting point of water (normal pressure), for example, is reached at the great solar distance, $r = 2$.

One notes that, in this idealized problem of heat transfer by radiation, the equilibrium condition is attained instantly. In the actual comet model, the heat capacity of the layers will introduce a time lag. Hence a solution for the accurate temperature distribution in the insulating layers with variable insolation becomes more complex.

Let us now investigate roughly the radiative conductivity between finite layers of matter. A differentiation of Stefan's radiation equation, connecting rate of energy transfer, E, and temperature, T, for a gray body gives

$$dE = 4\sigma(1-A_1)T^3 dT. \tag{17}$$

For layers, each of thickness l, the effective conductivity, K_l, in calories per centimeter per second per 1° C is, for infinite conductivity of the material,

$$K_l = 4\sigma(1-A_1)T^3 l, \tag{18}$$

where

$$\sigma = 1.36\times 10^{-12}\ \text{cal cm}^{-2}\ \text{sec}^{-1}\ (^\circ\text{C})^{-4}.$$

At temperatures of 200°, 273°, and 500° K, the conductivity (if $1 - A_1 = 0.9$) becomes $K_1 = 0.00004\,l$, $0.00010\,l$, and $0.00061\,l$, respectively. For comparison, the conductivity of ground cork in air at room temperature is 0.00012. Hence the radiative con-

[17] See, e.g., Minnaert, *loc. cit.*

[18] *Ap. J.*, **91**, 408, 1940.

ductivity of layers of thickness 1 cm is like that of ground cork, a good heat insulator. Since the conductivity of the solid meteoric material is presumably much larger, it will not affect appreciably the radiative conductivity of the layers, so long as their individual thickness does not exceed a few centimeters.

Equation (18), for $l = 0.026$ cm and $T = 418°$ K, yields a conductivity of $K = 9 \times 10^{-6}$ cal cm^{-1} sec^{-1} (° C)$^{-1}$. The value is in rough agreement with Smoluchowski's measures, mentioned above, for the coarsest powders at $T = 45°$ C. It appears that the conductivity (or areas of contact) for very fine powders is greater than would be suggested by Hertz's theory.

With the value of K_l given by equation (18) we can now compute the thickness, L, of the material required to transmit a fraction, χ, of the incident solar radiation to the bottom of the layers, if the total temperature differential is ΔT. The incident energy flux is $F_0(1 - A_0)/n^2$, so that,

$$\chi F_0 (1 - A_0) = \frac{K_l \Delta T}{L}. \tag{19}$$

For $r = 1$ A.U. and $T_n = 200°$ (a generous value), we find

$$L = \frac{K_l}{\chi} \times 7 \times 10^3. \tag{20}$$

Hence, for the temperature range from 200° to 500° K, L/l varies from $0.3/\chi$ to $4/\chi$. If we accept a 1 per cent efficiency ($\chi = 0.01$) in solar heat transfer, the number of layers varies between 30 and 400. This result checks the general order of magnitude of equation (12), which would give $n = L/l$, or about $1/\chi$.

C. TIME LAGS IN HEAT TRANSFER

The question of time lags in heat transfer through the meteoric layers of a rotating cometary nucleus is of great importance in the discussion of the acceleration of Comet Encke. Since the classical theories of periodic heat flow are not applicable to the general case of a rotating cometary nucleus if heat transfer is primarily by radiation, this problem must be investigated by methods of numerical integration. Such an investigation is under way and will be presented in a later paper. Nevertheless, some fragmentary information can be obtained from classical heat theory if constant values of the conductivity and diffusivity are assumed.

The diffusivity, h^2, in units of square centimeters per second, is given by

$$h^2 = \frac{K_l}{C_l \rho_l}, \tag{21}$$

where C_l and ρ_l are, respectively, the effective specific heat and the density of the meteoric layers, including interstices. We may adopt $\rho_l = 2$ gm/cm^3 for an aggregate of stony-iron particles and a corresponding specific heat, $C_l = 0.15$ cal/gm.

It is apparent from equations (17) and (18) that the rate of transport of heat from the heated surface to the interior of a cold body will increase much more rapidly with temperature for a given temperature gradient if the transport is by radiation rather than by conduction through a solid. Hence, if we adopt a conductivity corresponding to the radiative value at the surface temperature, we shall overestimate the inward heat flux at the maximum of the cycle and underestimate the time lags within the cometary nucleus.

With these limitations in mind, let us adopt the value of K_l from equation (18) and the maximum surface temperature from equation (16) to derive the following expression for h^2:

$$h^2 = 9.67 \times 10^{-4} l \,(r_{\text{A.U.}})^{-3/2}. \tag{22}$$

For meteoric layers of thickness $l = 1$ cm and a solar distance $r = 1$ A.U., the value of h^2 is 0.001 cm^2 sec^{-1}. As a comparison, an accepted typical value of the diffusivity of average soil is 0.005. It appears that we have adopted a rather large value for h^2 in view of the fact that the conglomerate meteoric material may itself conduct heat very poorly. The consequence of this large value of h^2 will be to reduce further the calculated amount of the lag in heat transfer to the icy core.

The classical theory for heat transfer through a "semi-infinite solid" with a periodic temperature variation at one surface and a heat sink of constant temperature at the other surface is complicated, involving a Fourier expansion.[19] Considering the general roughness of the present solution, we can better apply the "cold-wave" approximation as used by L. R. Ingersoll and O. J. Zobel.[20] Here the solution for the heat transfer in an infinite solid with a simple periodic temperature variation at a plane surface is applied to a solid not completely in equilibrium under the variation. The application appears rather satisfactory in the case of measured and calculated temperatures at moderate depths in the earth at the end of meteorological "cold waves" of a few days' duration.

For a periodic variation in surface temperature, of the form T (surface) $= \sin (2\pi/P)t$, where P is the period, the lag in temperature maximum[21] at a depth x from that at the surface is $x/2h\sqrt{P/\pi}$. The instant of maximum inflow of heat at any depth precedes the maximum temperature by one-eighth of a period.[22] Hence the lag in maximum heat flux at a depth $x = nl$ is given by the equation

$$\frac{\text{Flux lag}}{\text{Period}} = \frac{nl}{2h\sqrt{\pi P}} - \frac{1}{8}. \tag{23}$$

After the substitution of h from equation (22), the lag becomes

$$\frac{\text{Flux lag}}{\text{Period}} = 9.1n\,(r_{\text{A.U.}})^{3/4}\sqrt{\frac{l}{P}} - \frac{1}{8}, \tag{24}$$

where n is dimensionless, l is expressed in centimeters, and P in seconds.

The quantity n represents the depth, measured in layers of meteoric material, and, from equation (12), its inverse represents roughly the proportion of the heat flux at the surface transmitted to the nth layer. Hence equation (24) shows that the lag in the time of flux maximum from the time of maximum surface temperature attains the greatest fraction of the comet's period of rotation when (a) the number of meteoric layers is large, corresponding to a slow rate of evaporation of the ices; (b) the meteoric material is coarse; (c) the solar distance is great; and (d) the period of rotation is short.

Even though the constants and exponents in equation (24) will be changed by a more rigorous analysis, the writer believes that the qualitative conclusions in the preceding paragraph will still be valid for heat transfer by radiation. The lag measured from the instant of noon will be greater than the lag given by equation (24) because of a lag in the attainment of maximum surface temperature; but this effect probably will be considerably smaller than the one-eighth period correction.[23] For extremely small values of n the total lag in maximum rate of vaporization after the instant of noon will become negligible.

[19] See, e.g., H. S. Carslaw, *Mathematical Theory of the Conduction of Heat in Solids* (New York: Macmillan Co., 1921), p. 68.

[20] *An Introduction to the Mathematical Theory of Heat Conduction* (Boston: Ginn & Co., 1913), p. 40.

[21] See Ingersoll and Zobel, *loc. cit.*

[22] See, e.g., Wesselink, *loc. cit.*

[23] The corresponding lag in temperature for the moon during and after eclipse was only a few minutes (see Pettit, *loc. cit.*). The particle sizes, as shown by Wesselink, are less than 0.03 cm, however.

Numerical application in equation (24) shows that the lag in maximum rate of heat transfer after maximum surface temperature is zero for a cometary nucleus rotating in a period 6 days, at a depth of ten layers, for meteoric particles 1 cm thick at a solar distance of 1 A.U. Since the surface temperature is taken as 390° K and the temperature of the top of the ice layer must be in the neighborhood of 200° K, it is apparent that our adopted coefficient of conductivity is much too high and that an appreciable lag should occur under these circumstances if more accurate calculations were made. The actual conductivity near the ice layer would be appreciably less than is assumed in the derivation of equation (24). It should be noted also that equation (24) applies to the equatorial zone of a cometary nucleus of zero obliquity to the ecliptic. Other latitudes will be subject to more lag. If the obliquity were appreciable, further corrections would be needed.

Nevertheless, we now have some basis for estimating the general order of magnitude of time-lag effects to be expected in the vaporization of ices in the rotating model cometary nucleus.

D. THE LUMINOSITY LAW

With regard to the theory that all cometary gases result from desorption, a view brilliantly supported by Levin,[24] one effect might possibly provide a critical test between the hypothesis that the gases are replenished at great solar distances and the hypothesis that gases are stored in the nucleus. Carbon atoms captured from interplanetary space would have been subjected to the direct effects of cosmic radiation, hence should show an appreciable quantity of the isotope C_{14}. On the other hand, C_{14} atoms, if protected sufficiently well for long periods of time (much greater than 5700 years) by the outer layers of the cometary nucleus would disintegrate and not appear in cometary spectra. The isotope ratio C_{14}/C_{12} is normally so small, however, that measures of C_{14} from the spectra of comets would require impossibly long exposures at the high dispersion necessary.

The process of adsorption must, nevertheless, play a role in the processes of the proposed comet model. The side of the comet turning away from the sun will cool rapidly to extremely low temperatures. The gases vaporized by the in-moving heat wave will therefore tend both to be absorbed and even possibly to condense on the cold surfaces of the outer meteoric particles. Note that all comets must rotate to some extent with respect to the direction of the sun, if for no other reason than because of libration.

A quantity of extreme importance in Levin's theory is the exponent of r in the equation for the total luminosity of a comet. N. T. Bobrovnikoff,[25] from an extremely careful study of the luminosities of forty-five comets, derived a value of 3.32 ± 0.16 for the exponent, differing appreciably from the value 2, appropriate to a purely reflecting body of fixed dimension. He finds no indication of a phase-angle effect—evidence that direct reflection from solid particles may not play an important role in cometary luminescence. The observed fluctuations in the exponent are enormous from comet to comet and even for a given comet during a single approach to the sun. Bobrovnikoff finds a maximum value of 11.4, not well determined, and a minimum value of -11 (for Comet Westfal, which disappeared as it approached perihelion). For comets with the exponent less than the mean, the median value is 2.4, and for those with the exponent greater than the mean, the median is 4.4.

If the light from our comet model is to arise essentially from re-emission by gases generated proportionally to insolation and if the lifetimes of the radiating gases are both short and independent of solar distance, then the predicted exponent would be 4.0. Various factors, however, complicate enormously this simple prediction. The contribution to the comet's luminosity from light directly reflected by the nucleus should vary as $1/r^2$, reducing the exponent somewhat, but not greatly, for comets with strong molecular spectra. The reduction of the gaseous output below proportionality with insola-

[24] *Russian A.J.*, **21**, 48, 1943.

[25] *Contr. Perkins Obs.*, Nos. 15 and 16, 1941–1942.

tion at great solar distances should increase the exponent, particularly in the zone between 1 and 2 A.U. Quite large values should be expected in this transition region between reflection and gaseous re-radiation.

Another very important factor in the luminosity exponent arises from the finite lifetimes of the radicals, particularly C_2 and CN conspicuous near the nucleus. Wurm[26] has calculated that these molecules have short lifetimes, proportional to r^2. Hence their contribution to the luminosity law should vary as $1/r^2$ if their production is assumed to be proportional to the insolation.

The problem of the exponent of r to be expected from a gas with a long lifetime and from the dust escaping the nucleus becomes rather difficult. The exponent depends upon the rate of escape of the gases and dust from the region of the nucleus and upon the precise method of measurement of the comet luminosity, as well as upon the value of the lifetime of the gas. Since all these factors may vary while the comet is under observation, the problem is nearly insoluble at the present phase of cometary theory.

If the lifetime of the gas is short compared with the period of observation of the comet, but long compared to the time of escape from the head, and if the period of escape depends upon a solar repulsive force proportional to r^{-2}, then time of escape should be proportional to r. The luminosity law would then vary as $1/r^3$. A somewhat similar argument applies to light reflected from escaping fine dust.

The law to be predicted from the present comet model depends upon these various factors properly weighted. Any estimate would have to be made for the particular circumstances of an individual comet. An average value of about 3 for the exponent appears reasonable, allowing some correction for the transition from reflection to gaseous re-emission. The exponent should be much larger in the transition region. The fact that the exponent should be in the general neighborhood of Bobrovnikoff's measured value may be taken as slightly encouraging to the present model.

Particularly is this true in view of the large variations in the exponent that are both observed and expected from this type of model. Vsessviatsky's[27] result of a mean exponent of 4.12 ± 0.24, obtained from observations of forty-six comets, is in poorer agreement with the model. There are reasons, however, to rely more on Bobrovnikoff's results because he applied carefully determined systematic corrections to the observations to reduce them to a uniform system of telescope aperture and observing conditions.

Twelve of the forty-five comets studied by Bobrovnikoff pass perihelion in the solar distance range from 1.0 to 1.5 A.U. The mean value (weighted as recommended by Bobrovnikoff) of the luminosity exponent for these twelve comets is 4.5, conspicuously greater than for comets of greater or smaller perihelion distances. The exponent is below 3.32 for only three of the twelve comets. We may conclude that the transition region must occur at a solar distance just below 1.5 A.U.; observations of cometary spectra are in rough agreement with this conclusion.

E. THE ACCELERATIONS OF COMET ENCKE AND OTHER COMETS

The systematic increase in mean motion and the decrease in the eccentricity of Encke's Comet, conspicuous during the first half of the nineteenth century and smaller but definite since then, has long been a subject of speculation. Theories of a resisting medium stumble upon the lack of other evidence for the medium, upon the variation in the rate of acceleration with time, and upon the lack of uniformity or nonoccurrence of such effects for other comets. We will first discuss some of the observations, both for Comet Encke and for other comets.

F. Tisserand[28] quotes V. Asten as having obtained for Encke's Comet a value of

[26] *Loc. cit.*

[27] *Russian A.J.*, 2, Part 3, 68, 1925.

[28] *Traité de mécanique céleste* (Paris: Gauthier-Villars, 1896), IV, 226.

$+0\overset{''}{.}1044$ for $\Delta\mu$, the change per revolution in the mean daily motion. The corresponding change in the eccentricity was $\Delta\phi = -3\overset{''}{.}68$. These values applied for the interval from 1819 to 1865, but the secular changes appeared to be inappreciable from 1865 to 1871. For the period 1871–1894, Backlund[29] used $\Delta\mu = +0\overset{''}{.}0677$ and $\Delta\phi = -2\overset{''}{.}39$. In the interval 1894–1904 he found that the values $\Delta\mu = +0\overset{''}{.}0486$ and $\Delta\phi = -1\overset{''}{.}69$ led to a good representation of the observations for five oppositions, but for the opposition of 1908 the values $\Delta\mu = +0\overset{''}{.}0126$ and $\Delta\phi = -0\overset{''}{.}42$ gave best results. For the interval from 1918 to 1934 Matkiewicz[30] found that the changes were quite dependent upon the details of analysis. He reports that Backlund had been able to fit the observations from 1901 to 1914 with $\Delta\mu = +0\overset{''}{.}0375$. Matkiewicz gives, as one set of solutions for $\Delta\mu$, the values $+0\overset{''}{.}046$ for 1918–1924, $+0\overset{''}{.}020$ for 1921–1928, and $+0\overset{''}{.}097$ for 1924–1931. For the period 1924–1934 he obtained $\Delta\mu = +0\overset{''}{.}044$. At times he adopts the relation $\Delta\phi = -30\Delta\mu$, in rough agreement with Asten's results and Backlund's published values, but his least-squares solutions for the elements suggest that the value is numerically too high in the interval 1918–1931.

From these results we cannot escape the following conclusions: first, that the mean motion of Encke's Comet is generally accelerated with time and, second, that the acceleration is variable and does not certainly become negative at any time. The eccentricity appears to suffer a concomitant diminution, less well determined but generally giving $\Delta\phi = -30\Delta\mu$, or smaller, numerically. Averaged over longer intervals of time, the mean value of $\Delta\mu/\mu$ per revolution is about $+9.7 \times 10^{-5}$ for 1819–1865 and about $+4.2 \times 10^{-5}$ for both the intervals 1865–1901 and 1901–1934. The corresponding values of Δe are -9.4×10^{-6} and -4×10^{-6}, respectively. For comparison purposes an acceleration of 9.7×10^{-5} corresponds to 2.7 hours per period, an effect that is relatively large and cumulative over successive periods.

Tisserand found that Asten's measured relation between $\Delta\mu$ and $\Delta\phi$ is consistent with an acceleration of the comet produced by a resisting medium in which the resistance varies as some direct power of the velocity and an inverse power of the solar distance. He found that the ratio is not very sensitive to the actual powers, over the range from 1 to 5 in velocity and 2 to 4 in solar distance.

Bobrovnikoff[31] points out that Comet Encke is by no means the only comet to show peculiar changes in its motion. Period Comet Wolf I, for example, suffers a *decrease* in mean motion of $4\overset{''}{.}2 \times 10^{-7}$ per day, according to the study by Kamienski[32] covering all the observations of the comet from 1884 to 1942. With the present (1942) value of $\mu = 428\overset{''}{.}2$, corresponding to a period of 8.3 years and a perihelion distance of 2.4 A.U., the value of $\Delta\mu/\mu$ per revolution is then -3.0×10^{-6}. In 1884, before the recent close approach to Jupiter, when $\mu = 524''$, $P = 6.8$ years, and $q = 1.57$, the corresponding value of $\Delta\mu/\mu$ was -2.0×10^{-6}. The deceleration in the mean motion of Comet Wolf I is relatively less than a tenth that for Comet Encke. Kamienski apparently does not require a secular change in the eccentricity to fit the observations. It is difficult to assess the significance of the small deceleration in the motion of Comet Wolf I as derived by Kamienski, particularly as a very close approach to Jupiter is included in the interval of the solution. Most unfortunately, many of the calculations were destroyed during the recent war.

Comet Wolf I is of particular interest because of its large decrease in magnitude at recent solar approaches. Bobrovnikoff[31] concludes that the comet had faded 2–3 mag. in 1934 and 1942 in comparison to its fairly constant "absolute magnitude" of 11 from 1884 to 1925. He also finds that the apparent diameter of the comet is relatively much greater after perihelion than before at the same solar distances—less than 4 A.U.

[29] *Mem. Acad. St. Petersburg*, Ser. 8, Vol. **22**, No. 2, 1911.

[30] *Bull. Obs. Poulkovo*, Vol. **14**, No. 6, 1935.

[31] *Pop. Astr.*, **56**, 130, 1948.

[32] *M.N.*, **106**, 267, 1947.

Another comet that has shown evidence of a change in motion is periodic Comet D'Arrest. A. W. Recht,[33] from extensive calculations, has found systematic changes per revolution amounting to $+1.51 \times 10^{-4}$ in a and $+1.69 \times 10^{-5}$ in e, over the eight observed solar approaches from 1851 to 1923. Since the period is about 6.7 years (the perihelion distance is 1.32 A.U.), the change in mean motion per revolution is $\Delta\mu/\mu = -6.4 \times 10^{-5}$, of the same order of magnitude as for Comet Encke but of opposite sign. The secular change in e is also of the opposite sign to that for Comet Encke and somewhat greater. Recht concludes that the deceleration is unquestionably real.

There are other comets whose motions do not seem to have conformed well to simple Newtonian theory. Examples are periodic Comets Brooks, Kopff, and Brorsen-Metcalf, while others could undoubtedly be found. Even Halley's Comet arrived late at perihelion in 1910 by 2.7 days.[34] This tardiness after 76 years seems small, but the corresponding value of $\Delta\mu/\mu$ is equal to the older value for Comet Encke, though of the opposite sign.

Hence we see that there are at least two well-authenticated examples of systematic changes in the mean motions of comets, one showing an acceleration (Encke) and one a deceleration (D'Arrest). Also there is suspicion of such effects in the motions of other comets. In both the two best examples the eccentricity also changes systematically, $\Delta\phi$ bearing a fairly constant ratio to $\Delta\mu/\mu$ but being opposite in sign. Both comets are intrinsically faint.

The proposed comet model provides a possible mechanism for accelerating or decelerating the motion of a rotating comet. The gas escaping from the nucleus will leave with a velocity corresponding roughly to the mean speed of the gas molecules at the temperature of the surface layer of the meteoric blanket. The momentum of the escaping gas will exert a force on the nucleus. If there is an appreciable lag between the time of gas escape and the meridian passage of the sun with respect to the rotating nucleus, this force will possess a component perpendicular to the radius vector of the comet's orbit. The force may act in any direction, depending upon the direction of the axis and the sense of rotation of the nucleus.

Suppose that the cometary nucleus is spherical, of mass M, and that it loses mass at a rate dM/dt with an average speed $\bar{v}$. If γ represents the dimensionless component of the force $\bar{v}dM/dt$ perpendicular to the radius vector and in the orbit plane, positive with respect to the motion of the comet, the acceleration to the cometary nucleus, S_T, in the same sense as γ, will be given by

$$S_T = +\frac{1}{M}\frac{dM}{dt}\bar{v}\,. \tag{25}$$

If ζ represents the dimensionless component of the total force along the radius vector outward from the sun, the net outward acceleration, S_R, is, correspondingly,

$$S_R = +\zeta\frac{1}{M}\frac{dM}{dt}\bar{v}\,. \tag{26}$$

Let us assume that the acceleration perpendicular to the orbital plane is zero in the case of a cometary nucleus rotating with its pole perpendicular to the ecliptic.

The average speed of a gas of molecular weight m and temperature T_g is given by

$$\bar{v} = \left(\frac{8kT_g}{\pi m}\right)^{1/2}, \tag{27}$$

[33] *A.J.*, **48**, 65, 1939.

[34] P. H. Cowell and A. C. D. Crommelin, *M.N.*, **71**, 320, 1911.

where k is the Boltzmann constant. The temperature to be applied in equation (27) can be adopted from equation (16), as a slightly high value. The zenith angle of the sun, θ, will depend both upon the time lag of the escaping gas and upon the mean latitude. Since $\cos\theta$ will enter equation (27) only to the one-eighth power, let us simply assume a more reasonable temperature than that of equation (16), $T = 300\ r^{-1/2}(^\circ\ \mathrm{K})$. We may then define η by the following, derived from equation (27):

$$\bar{v} = \frac{\eta}{(r_{\mathrm{A.U.}})^{1/4}} = \left(\frac{8 \times 300^\circ K}{\pi m}\right)^{1/2} \frac{1}{(r_{\mathrm{A.U.}})^{1/4}}. \tag{28}$$

If we adopt $m = 20 \times 1.661 \times 10^{-24}$ gm as typical of the gases in Table 1, we find $\eta = 0.56 \times 10^{-5}$ cm/sec.

Now the average velocity $\bar{v}$ may be assumed to be random in direction as the gas leaves any small area of the cometary nucleus. Hence the average velocity component normal to the surface can be shown to be $2\bar{v}/3$. If, now, the gas vaporizing from a hemisphere is produced proportionately to the effective insolation centered at the subsolar point, the average velocity of the gas perpendicular to the plane of the hemisphere is $4\bar{v}/9$. This value of the average velocity applies to a nonrotating cometary nucleus vaporized by sunlight. The small correction arising from the dependence of $\bar{v}$ upon the temperature at various parts of the nucleus is sufficiently allowed for in the low value of T_0 adopted previously. The effect in a rotating nucleus would be almost the same as for a stationary nucleus if the direction of the velocity component were corrected by the appropriate angle of lag. A minor error will arise, however, from the greater lag in the regions at high latitudes. We may neglect this effect and adopt the approximation that the maximum numerical value of both γ and ζ separately is 4/9. If the lag is zero, for example, $\gamma = 0$, and $\zeta = 4/9$.

The value of $(1/M)dM/dt$ for the comet must be found from observation. It will be nearly proportional to the solar flux at small and moderate solar distances. At great distances where the vapor pressure of ice (H_2O) becomes negligible (approximately 160° K) the effective conductivity of the meteoric layers will be reduced by the added insolation of the ice (H_2O) layer. It is not possible to generalize as to the solar distance at which this effect takes place. The upper limit is in the neighborhood of 4 A.U. (eq. [16]), but the meteoric layer may reduce the distance very appreciably. The amount of the effect will depend also upon the abundance of H_2O. Let us adopt $r_m = 3$ A.U. as the practical limit to the loss of mass for short-period comets. If we combine various factors involving the area and structure of the cometary nucleus and various radiation constants in a coefficient, ξ, we may write

$$\frac{1}{M}\frac{dM}{dt} = +\frac{\xi}{r^2}. \tag{29}$$

By the definitions of equations (25), (26), and (28) the accelerations become

$$S_T = \frac{\gamma}{\zeta} S_R = \frac{\xi\gamma\eta}{r^{9/4}} \tag{30}$$

Returning to perturbation theory and noting that the resistance and outward force are symmetrical with respect to perihelion, we write the standard equations[35] for the

[35] See, e.g., Moulton, *Introduction to Celestial Mechanics* (New York: Macmillan Co., 1923), p. 404.

perturbations in the elements a, e, and the longitude of perihelion, ϖ, in terms of the forces and true anomaly, v_1, noting that the terms in ($\sin v_1$) times (powers of r) cancel out,

$$\frac{1}{a}\frac{da}{dt}=\frac{2\sqrt{1-e^2}}{\mu r}S_T, \tag{31$_1$}$$

$$\frac{de}{dt}=\frac{\sqrt{1-e^2}}{\mu a^2 e}\left[\frac{ap}{r}-r\right]S_T, \tag{31$_2$}$$

$$\frac{d\varpi}{dt}=-\frac{\sqrt{1-e^2}}{\mu a e}\cos v_1 S_R, \tag{31$_3$}$$

where p is the parameter of the orbit.

For a partial revolution of the comet from $-v_0$ to $+v_0$, the time integral, I, of any function, $f(v_1)$, becomes

$$I=\frac{2}{k\sqrt{p}}\int_0^{v_0} r^2 f(v_1)\,dv_1, \tag{32}$$

where k^2 is the gravitational constant for motion about the sun.

Hence we find, for the perturbations in the elements and for the loss of mass during one revolution,

$$\frac{\Delta a}{a}=\frac{4\xi\gamma\eta}{k\mu a^{1/2}}\int_0^{v_0} r^{-5/4}dv_1, \tag{33$_1$}$$

$$\Delta e=\frac{2\xi\gamma\eta}{k\mu a^{5/2}e}\int_0^{v_0} r^{-5/4}(ap-r^2)\,dv_1, \tag{33$_2$}$$

$$\Delta\varpi=-\frac{2\xi\zeta\eta}{k\mu a^{3/2}e}\int_0^{v_0} r^{-1/4}\cos v_1 dv_1 \tag{33$_3$}$$

and

$$\frac{\Delta M}{M}=\frac{2\xi v_m}{kp^{1/2}}, \tag{33$_4$}$$

where $\Delta\varpi$ is the longitude of perihelion, v_m corresponds to $r = 3$ A.U., and v_0 represents the limits to which the force S_T is operative.

The limit in true anomaly, v_0, beyond which the force S_T becomes zero, can only be guessed on the basis of physical arguments. From equation (24), the lag in heat transfer should be greater at great solar distances, presumably enhancing the tangential acceleration if the lag were small near perihelion. On the other hand, secondary processes other than the reduction in the rate of vaporization will tend to reduce the tangential acceleration. In particular, adsorption of gases in the meteoric layer will increase as the gas output decreases and as the general temperature level falls. Hence the gas vaporized on the night hemisphere may be carried around to the morning limb before being released. Condensation of gases other than H_2O may also occur, adding to the effect.

The ratio of $\Delta e/(\Delta\mu/\mu)$ provides, fortunately, a measure of the general region beyond which tangential acceleration is small. In the case of Comet D'Arrest, Recht's measure of this ratio corresponds almost exactly to a constant value of the perihelion distance, q. Tangential acceleration very near to perihelion would produce such a result, while acceleration over a large range in v would greatly reduce the relative value of Δe. Since $q = 1.32$, the upper limit of v_0 at $r = 2$ A.U. appears reasonable. The same limit gives consistent results for Comet Encke.

Let us assume, then, that the nucleus of Comet Encke rotates in a retrograde direction with a fairly small obliquity of the ecliptic. Its period must exceed 3 hours, the approximate period of a surface satellite about a sphere of density 1.5 gm/cm, for the weak nucleus to remain intact. If the period is less than a week (or possibly considerably longer) and the very stringent conditions of equation (24) are fulfilled, the gas will escape predominantly on the forward face, and a resisting force will oppose the motion.

The orbital elements needed for the present calculations are $a = 2.217$ A.U., $e = 0.847$, and $\mu = 1070''$ per day or 6.0×10^{-8} radians per second. The integrals, I, in equations (33_1)–(33_3) to a limit of $v_0 = 145°$ ($r = 2$ A.U.) and $v_m = 169°$ ($r = 3$ A.U.) are, respectively, $I_a = 5.8 \times (\text{A.U.})^{-5/4}$, $I_e = 6.2 \times (\text{A.U.})^{3/4}$, and $I_{\varpi} = 0.87\ (\text{A.U.})^{-1/4}$, while $v_m = 2.8$.

Since $\Delta a/a = -2\Delta\mu/(3\mu)$, the ratio of the secular changes in e and μ becomes

$$\Delta e = -\frac{\Delta\mu}{\mu}\frac{1}{3a^2 e}\frac{I_e}{I_a}. \tag{34}$$

The calculated value of Δe derived from $\Delta\mu/\mu = 9.7 \times 10^{-5}$ is -8.3×10^{-6}, to be compared with Asten's value -9.4×10^{-6}. The agreement could be improved by reducing v_0, but Matkiewicz did not fully confirm the older ratio of $\Delta e/(\Delta\mu/\mu)$, obtaining a somewhat smaller numerical value. If v_0 is permitted to run to π, including the entire orbit, the relative value of Δe is reduced to about half the adopted value.

By eliminating ξ from equations (33_1) and (33_4), we can solve for $\Delta M/M$ per revolution in terms of the ratio $\Delta\mu/\mu$. The result is

$$\frac{\Delta M}{M} = \frac{\Delta\mu}{\mu}\frac{\mu}{3\eta\gamma\sqrt{1-e^2}}\frac{v_m}{I_a} \times (1\ \text{A.U.})^{5/4}. \tag{35_1}$$

After substituting numerical values, we find that, for Comet Encke, per revolution,

$$\frac{\Delta M}{M} = -\frac{\Delta\mu}{\mu}\frac{4.9}{\gamma}. \tag{35_2}$$

If we adopt for γ, the component of the emitted gas in the orbit plane perpendicular to the radius vector, the conservative value of -0.1 and adopt the older maximum value of $\Delta\mu/\mu = 9.7 \times 10^{-5}$, we find, for the relative loss of mass per revolution,

$$\frac{\Delta M}{M} = 0.0048. \tag{35_3}$$

In other words, the comet need lose no more than one-half of 1 per cent of its mass per revolution to be accelerated at the maximum observed rate. At the lower present-day acceleration of $\Delta\mu/\mu = 4 \times 10^{-5}$, the mass loss need not exceed one-fifth of 1 per cent per revolution. With a uniform loss of radius corresponding to the maximum rate above, the comet could persist for some six hundred revolutions, or more than a thousand years. The magnitude over the past century and a half since discovery need not have changed appreciably (about 0.1 mag.).

Vsessviatsky[36] came to the conclusion that Comet Encke has faded by 1 mag. in the last 100 years. The conclusion is subject to some question. I have not been able to correlate variations in the brightness of the comet at various apparitions with variations in $\Delta\mu/\mu$. It appears doubtful that the general decrease in $\Delta\mu/\mu$ is associated physically with the suspected secular diminution in brightness, although such a correlation is still possible.

With regard to the more remote history of Comet Encke, particularly its association

[36] *Russian A.J.*, **4**, 298, 1927.

with the Taurid meteors,[37] the present hypothesis opens up an area of investigation too large for elaboration here. Of particular interest is the question whether Comet Encke was "captured" into a short-period orbit by Jupiter. Such a capture requires the aphelion distance of Comet Encke to have been reduced from Jupiter's distance to its present value of 4.1 A.U. A second study of the intricate relationships between the Taurid meteors and Encke's Comet is planned for the future.

In applying the above theory to Comet D'Arrest, equations (33) may be integrated directly by removal of the $r^{-1/4}$ term from under the integrals. The variation in $r^{-1/4}$ from perihelion, 1.32–2 A.U., is small. Pertinent orbital elements are $a = 3.55$ A.U., $e = 0.627$, and $\mu = 532''$ per day. The integrals in equations (33_1)–(33_3) to a limit of $v_0 = 83°$ ($r = 2$ A.U.) and $v_m = 117°$ ($r = 3$ A.U.) are, respectively, $I_a = 0.86 \times (\text{A.U.})^{-5/4}$, $I_e = 4.58 \times (\text{A.U.})^{3/4}$, and $I_\varpi = 0.88$, while $v_m = 2.0$.

With the same assumptions for Comet D'Arrest as for Comet Encke, except that the rotation of the comet must be direct, we find that, for $\Delta\mu/\mu = -6.4 \times 10^{-5}$, $\Delta M/M = 0.005$ and $\Delta e = +1.5 \times 10^{-5}$. Hence the comet needs to lose only one-half of 1 per cent of its mass per revolution, while the calculated increase in e per revolution is near the observed value of 1.7×10^{-5}.

For Comet Wolf I, the results depend upon the assumed orbital elements, which were changed markedly by Jupiter during the period of observation. The elements for 1912, $a = 3.6$ A.U. and $q = 1.59$, if $\Delta\mu/\mu = -2 \times 10^{-6}$, lead to $\Delta M/M = 1.4 \times 10^{-4}$ and $e = +6 \times 10^{-7}$. The elements for 1925, $a = 4.1$ A.U. and $q = 2.44$, if $\Delta\mu/\mu = -3 \times 10^{-6}$ when the integrations are carried to $r_0 = 3$ A.U., lead to $\Delta M/M = 2.3 \times 10^{-4}$ and $e = 1.2 \times 10^{-6}$. The cometary rotation, of course, must be direct. The calculated loss of mass per revolution is very small, only one- or two-hundredths of 1 per cent, while the predicted secular change of eccentricity, about one part in a million, would be undetectable. The fact that Kamienski did not find a secular change in e is therefore consistent with the present theory.

The predicted secular change in the direction of perihelion, $\Delta\varpi$ per revolution, from equation (33_3) is a very small quantity (of the order of a second of arc). It becomes zero as the integration is carried to $v_m = \pi$. It is of interest that the above-calculated losses of mass per revolution to explain the secular accelerations of Comets Encke, D'Arrest, and Wolf I do not appear excessive.

F. EFFECTIVE SOLAR ATTRACTION FOR COMETS

If comets are losing material in the manner proposed in the preceding discussion, there is little question that the component of ejection will be greater statistically along the radius vector toward the sun than normal to it. Certainly, this will be true whenever the time lag in vaporization corresponds to less than an eighth period of cometary rotation, when the rotation is extremely slow or when irregular ejection occurs because of "cave-ins." In the case of extremely great time lags the H_2O ice may tend to freeze on the night hemisphere of the nucleus, while all gases will be adsorbed to a greater or lesser extent on the night hemisphere. Hence, statistically, the cometary nuclei will tend to emit material toward the sun. The phenomenon of the sunward ejection of material from cometary nuclei has long been recognized for the bright comets. Early drawings show the effect strikingly.

The sunward component of the ejection momentum will reduce the solar attraction for a cometary nucleus and affect its orbital motion. We may adopt the rough approximation that the quantity of gas ejected toward the sun by vaporization is proportional to the solar radiation flux and, therefore, inversely proportional to the solar distance. If we neglect the small variation with solar distance of the average speed of the ejected gas particles, the resultant force on the cometary nucleus varies according to the in-

[37] F. L. Whipple, *Proc. Amer. Phil. Soc.*, **83**, 711, 1940.

verse-square law of solar distance. The effective radial force is represented in equation (26), if $\bar{v}$ of equation (28) is taken as constant so that an inverse-square law results. As a consequence, the present comet model predicts an effective reduction in the gravitational constant for comets. The effect of the repulsive force should be observable statistically in cometary orbits observed at single apparitions.

The reality of such a predicted decrease in the Gaussian constant, k^2, can be investigated by means of published definitive orbits for comets. The final least-squares solution for corrections to the elements can be made to include a seventh unknown, Δk^2, with relatively little effort. The statistical mean value of Δk^2 for a number of comets should be significantly negative if the present comet model is essentially correct.

The solution for Δk^2 from a definitive comet orbit is relatively simple because, in the observational equations in right ascension and declination, the differential coefficient for Δk^2 is equal to that for ΔT_0 (T_0 = time of perihelion passage) multiplied by $(T_0 - t)/k^2$, where t is the reduced time of observation. Hence the tabulated values of the perihelion-passage coefficients in the observation equations for the normal positions can be used to derive the coefficients for Δk^2. The least-squares solution can be repeated with the seventh unknown to yield the most probable value of Δk^2.

Since few comets are observed at very great distances from perihelion, the neglect of the variation in $\bar{v}$ with solar distance and the neglect of possible variations of vaporization from linearity with insolation are of little consequence in practice.

The Harvard graduate students listed below are now analyzing the definitive orbits of the following comets to determine the respective values of Δk^2: 1862 III, S. Hamid; 1882 II, F. Kameny; 1886 II, H. J. Smith; 1905 III, A. Hoag; and 1911 II, D. Murcray. The results will be published elsewhere. Preliminary results indicate that the true value of Δk^2 is often masked by a relatively large value of its probable error. Small comets that might give larger values Δk^2 are usually faint and observed for only a short time, while the large comets are well observed but show a small effect. In almost all cases the residuals in the observational equations are markedly reduced by the solution for Δk^2.

No significantly positive values of Δk^2 have been obtained for any of the five comets under investigation. The most significant value of Δk^2 has been obtained from the definitive orbit of Comet 1905 III (Giacobini) by S. Szelegowski.[38] Mr. Hoag finds that $\Delta k^2 = -0''.31 \pm 0''.10$ (m.e.), or $\Delta k^2/k^2 = -9 \times 10^{-5}$, and that the mean error of a single normal position is reduced by a factor of 2. Pertinent orbital elements for Comet 1905 III are $a = 37$ A.U., $e = 0.970$, $q = 1.114$ A.U., and $i = 40°$.

To evaluate $\Delta k^2/k^2$ in terms of the relative loss of cometary mass per period $\Delta M/M$, we may integrate equation (29) with respect to time within $r_m = 3$ a.u., or $v_m = 1.86$, obtaining the following relation with the coefficient, ξ:

$$\frac{\Delta M}{M} = \frac{2\,\xi\, v_m}{k\sqrt{p}}. \tag{36}$$

A comparison of the radial force, S_R, in equation (30) with the gravitational acceleration of the sun, and the evaluation of ξ from equation (36), lead to the following expression for $\Delta M/M$:

$$\frac{\Delta M}{M} = \frac{\Delta k^2}{k^2}\,\frac{2\,r^{\frac{1}{4}}\,v_m}{\zeta\eta\sqrt{p}}\sqrt{\frac{GM_s}{1\ \text{A.U.}}}, \tag{37}$$

where M_s is the mass of the sun and c.g.s. units apply except for p and r (A.U.).

If we adopt $r = 1.5$ A.U., $v_m = 1.86$, $\Delta k^2/k^2 = -9 \times 10^{-5}$, ζ as 0.3 near its maximum of 4/9, and $p = 2.2$ A.U. for Comet 1905 III, we find that $\Delta M/M = 0.04$ per revolution.

[38] *Acta Astr.*, Ser. a, **3**, 57, 1934.

The mass loss so derived may possibly seem rather great. On the other hand, it might still persist for fifty or more apparitions. Perhaps the comet, a rather faint one, is actually dying, or perhaps the value of $\Delta k^2/k^2$ derived for a comet from a single apparition is too much affected by systematic errors in the observations. Such errors may well arise if the cometary emanations systematically displace the apparent center of the comet from the nucleus.

It is clear that values of $\Delta k^2/k^2$ should be derived from orbits of periodic comets at more than one apparition. Such a solution, coupled with a determination of the systematic changes in period eccentricity and direction of perihelion, will yield the ratio of the acceleration components along and normal to the radius vector in the orbit plane, ζ and γ, respectively. It will also provide a check on v_m. Such a result would establish the value of $\Delta M/M$ fairly satisfactorily, even though the acceleration component normal to the orbit plane probably cannot be detected from the observations. Its predicted effects on the inclination and node are very small and are compounded in a complex fashion with the direction of the axis of rotation when the axis is not normal to the orbit.

Near the completion of the present manuscript, the writer's attention was called to the research by A. D. Dubiago,[39] entitled "On the Secular Acceleration of Comets." Secular accelerations are studied for the five comets Encke, Biela, Brooks, Winnecke, and Wolf I. A critical summary of the conclusions is presented. Of especial value are Dubiago's determinations of the accelerations for Comets Brooks and Winnecke. He then calculates the quantity $\bar{v}\Delta M/M$ to account for the observed accelerations by ejection of matter from the comets near perihelion. He rejects gaseous emission (as an insufficient mechanism) and favors a force based on the expulsion of solid particles. It is interesting to note that, among the six comets discussed by Dubiago and the writer, three exhibit a positive value of $\Delta\mu/\mu$ and three a negative value. If the accelerations arise as a consequence of rotation, the sense of the rotation appears to be at random.

Dubiago bases his calculations of mass loss on the theory developed by Bessel.[40] It was of great interest to the writer to learn that Bessel proposed[41] and later defended strongly the proposition that the acceleration of Comet Encke need not arise from a resisting medium. He argued, particularly from his beautiful drawings of jets from the nucleus of Halley's Comet,[42] that material streams out from a comet; if so, the orbital motion of the comet must be affected. Only the amount of the changes can be in question.

In the forthcoming second part of this paper the results of this first part will be interpreted in terms of the physical characteristics of certain comets; some of the pertinent information from observations of meteor streams will be introduced to augment the picture.

[39] *A.J. Soviet Union*, **25**, No. 6, 361, 1948.

[40] *A.N.*, **13**, 345, 1836.

[41] *Ibid.*, p. 3.

[42] *Ibid.*, p. 185.