DERIVATION OF FUNDAMENTAL ASTRONOMICAL CONSTANTS FROM THE OBSERVATIONS OF EROS DURING 1926–1945

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Since its discovery by G. Witt at Berlin in 1898, the planet Eros has been the subject of considerable effort directed toward obtaining improved values of the mass of the earth-moon system and of the solar parallax. At its perihelion, it approaches the earth very closely and affords a satisfactory determination of the solar parallax by the trigonometric method. The dynamical method, however, affords an opportunity to determine not only the solar parallax, which comes indirectly from the mass determination, but also a more precise determination of the masses of the four inner planets, corrections to the orbital elements of the earth, and the determination of the equinox and equator. All these results, however, can be obtained only at the expense of the huge computational task of deriving an accurate orbit which is rigorously dynamical and at the same time in satisfactory agreement with the observations. The planet suffers large perturbations by the earth, but it is also considerably affected by Venus and even Mars and Mercury. All the gravitational effects, no matter how small they may be, are greatly magnified in the observations because of the close approaches of Eros to the earth and therefore they may be determined with greater precision. On the other hand, this magnification makes it more difficult to obtain a satisfactory agreement between the computed positions and the observations, but if an agreement can be obtained it is well worth all the effort.

The initial representation of the observations from 1893 to 1903, including prediscovery observations in 1893, 1894, and 1896, was obtained by the discoverer, G. Witt,[1] who used special perturbations of the elements by the planets Venus to Saturn. The result, which represented the sixteen normal positions within $\pm 3''.8$ with the exception of one outstanding residual of $+10''.2$ in right ascension, afforded the first determination of the mass of the earth-moon system from the orbit of Eros, and the corresponding solar parallax:

$$1 : m_{\oplus+☾} = 328882 \pm 986,$$
$$\pi_\odot = 8''.794 \pm 0''.009.$$

E. Noteboom[2] extended the work to include the opposition of 1914 and also general perturbations by Mercury, Uranus and Neptune. He finally represented his twenty-four normal positions from 1893 to 1914 within $\pm 4''.0$ and with the mean error $\pm 1''.05$ for unit weight, i.e., one average normal position. His results for the mass of the earth-moon system and the corresponding solar parallax were:

$$1 : m_{\oplus+☾} = 328370 \pm 102,$$
$$\pi_\odot = 8''.799 \pm 0''.001.$$

In 1933 G. Witt[3] published his final results for the orbit of Eros, including a barycentric ephemeris for the important perihelion opposition of 1930–31. He used thirty-four normal places from 1893 to 1931, including four provisional positions during 1930–31. The representation of the observations was not very satisfactory; the mean error was $\pm 1''.36$ for the average normal position and the maximum residuals were as large as $\pm 4''.7$. The value obtained for the mass of the earth-moon system was in close agreement with that found by Noteboom. His results were

$$1 : m_{\oplus+☾} = 328390 \pm 103,$$
$$\pi_\odot = 8''.799 \pm 0''.001.$$

A number of circumstances led G. Stracke to the decision in 1938 to undertake a completely new and independent computation of the perturbations of Eros to a higher degree of accuracy than had been done heretofore. Witt's final elements did not give a satisfactory representation. The accurate special perturbations in the elliptic elements had been continued until the opposition of 1937–38 by H. von Schelling, but even so, an empirical correction of $+0''.00644$ to the mean motion was required to bring Witt's final elements into agreement with the observations.

Stracke chose Witt's elements for the epoch 1931 January 18.0 U.T. as the basis for his computations and he chose to continue the use of the method of the variation of elliptic elements. By his work on the planet (887) Alinda he had succeeded in proving that this method can give a highly satisfactory result even for a very eccentric orbit. His goal was an accuracy of $0''.0001$ for the values of the functions in the single integrals and $0''.00001$ in the double integral. Therefore it became necessary to use smaller intervals and more significant figures. Over the interval from 1930 to 1938, Stracke[4] obtained a satisfactory representation of eleven normal positions in four oppositions with the mean error only $\pm0''.50$. This agreement continued throughout 1940 and 1942. However, after computing the motion of Eros backward from 1930 to 1924, Stracke was deeply disappointed that he could not obtain the same good representation; the residual in 1926 was $-1''.5$. He also computed the accurate perturbations forward to 1945, but after his death the observations in the 1944–45 opposition showed residuals up to $+5''$. Thus Stracke's very accurate perturbations were not able to represent the observations over the longer arc from 1926 to 1945 with any better agreement than had been obtained by Noteboom and Witt.

It is necessary to mention that for the computation of the perturbations by the earth, Stracke always used Witt's value for the mass of the earth-moon system which undoubtedly was the best one available. For example, the use of Newcomb's value $1:m_{\oplus+☾} = 329390$ would produce large residuals, up to about $10''$ in the closer approaches, even after only a relatively short interval of time. Stracke did not include a further correction of the earth's mass in his solutions, he tried to obtain a better representation by correcting the six orbital elements of Eros only. It was his intention to continue the accurate special perturbations backwards to 1898, and only after having done this and after the re-reduction of the places of all the comparison stars to the homogeneous system of the FK3 did he intend to use all the available observational material of Eros for the determination of masses of the planets and other problems. In the representation of the observations and for the computations of his ephemerides of Eros, Stracke applied to the positions of the earth the corrections to the solar coordinates derived by A. Kahrstedt[5] from the observations of the sun. On the computation sheets there is a last note from Stracke's hand, that the large residuals of 1926 cannot be caused by any uncertainty in the orbit of Eros. Stracke's final perturbations for Eros are slightly different from those published,[4] due to another recomputation which he made of the perturbations especially by the earth.

These are the elements which Stracke finally adopted for Eros:

Epoch and osculation 1931 January 18.0 U.T.

$$M = 0^\circ\ 35'\ 8''.489$$
$$\omega = 117\ 56\ 28.908$$
$$\Omega = 303\ 46\ 53.749$$
$$i = 10\ 49\ 45.914$$
(ecliptic and equinox 1930.0)
$$\varphi = 12\ 52\ 44.310$$
$$n = 2015''.292474$$

The present investigation is based upon the same considerations which influenced Stracke: the unsatisfactory representation of the observed motion of Eros could not originate in any defect so far as the precision of the computed perturbations is concerned. It appeared that in the present state of the Eros work only a general solution, including corrections to the masses of the inner planets and to the elements of the orbit of the earth, would produce a satisfactory agreement with the observations. A more extensive use of the observations in the close perihelion approach of 1930–31, combined with the observations in other rather close approaches, would actually enable one to obtain improved values of considerable weight for the masses of earth-moon system and of Venus, and probably even for some other fundamental constants. It must also be recognized that since the work of H. Spencer Jones[6] concerning the corrections to the longitudes of sun and moon in connection with the variable rotation of the earth we now are able, at least for the decades since 1900, to apply well-known corrections to the system of Universal Time in order to refer the observations to the so-called Newtonian Time or dynamical time which satisfies the equations of celestial mechanics.[7] It would be inconsistent to apply corrections only to the coordinates of the sun, as Stracke did, because the actual source of these corrections must affect the coordinates of Eros also. Therefore this general solution incorporates the use of Newtonian Time throughout.

It was possible to utilize the excellent perturbations of Eros by all the major planets from 1926–1945 which had been computed by Stracke. He derived these by successive approximations, alternating between differential corrections to the elements and recomputations required by the effects of these corrections, until the values became final. The following values of the masses of the disturbing planets were used:

Planet	Mass
Mercury	1:6000000
Venus	1: 408000
Earth + Moon	1: 328390
Mars	1:3093500
Jupiter	1: 1047.35
Saturn	1: 3501.6
Uranus	1: 22869
Neptune	1: 19314

As a check on the integrations, Stracke integrated the perturbations of the elements separately for each disturbing planet and compared the sum of these with the integration of the perturbation of each element as given by the sum of the components of the disturbing forces. This separation of the effect of each planet facilitated the derivation of the coefficients needed for the improvement of the masses.

The interval for the integration was usually 10 days, for Mercury it was 5 days; and during the close approach from 1930 October 12 to 1931 March 21 a 4-day interval was used for all the disturbing planets. A 5-day interval was used for all the disturbing planets from 1937 October 24 to 1938 April 12. Therefore from the zero epoch, 1931 January 18.0, until 1945 April 15 the total number of the integration intervals, n, was about 1040 for Mercury. Corresponding to the expression given by D. Brouwer[8] the probable error due to the rounding-off errors in the summation for the double integral would be equal to $\pm 0.1124\, n^{3/2}$, expressed in units of the last decimal place carried. This unit is $0''.00001$; hence it can be assumed that due to the double summation a probable error of $\pm 0''.038$ in the mean longitude may arise only for a very few of the normal places which are very distant from the epoch. The contribution from the other planets is much less, and the entire amount cannot exceed $\pm 0''.05$. This is in good agreement with the comparison between the two independent integrations which were carried out for the sum of all of the disturbing forces, and for each disturbing planet separately. The combined results of the eight separate summations differ from the values of the integration in which the disturbing forces were combined by less than the amounts which are indicated for the probable errors due to the summation effect. Because the 1944–45 approach is as small as 0.4 astronomical units, the probable uncertainty of our residuals, so far as the accumulation effect in the computation of the perturbations is concerned, even in this opposition is limited to about $\pm 0''.1$, the largest coefficient for dM_0 in the equations of condition being about $+2.5$ in this case. Since the final results give a representation of the average normal position with the probable error $\pm 0''.24$, this preliminary investigation of the probable error due to the accumulation effect makes certain that the computational accuracy of the perturbations is high enough to avoid any serious systematic error in our final results.

The observational material used in this investigation for the important perihelion opposition 1930–31 was generously provided by H. Spencer Jones and his collaborators from their work in determining the solar parallax trigonometrically.[9] They had reduced nearly 3000 photographic positions to a homogeneous system of comparison stars and derived the tabular errors (O − C) from Witt's ephemeris. These were smoothed so as to give a curve which represented in the best possible way the whole observed motion of Eros in the sky. From this curve normal positions at intervals of ten days from 1930 October 10 to 1931 April 28 were transmitted to Stracke specifically for the dynamical attack on Eros. This gave 21 normal places which were used during this opposition; Stracke had used only six.

Then Stracke's final elements and perturbations were used for the computation of the residuals. The rectangular solar coordinates were taken without any empirical correction from the *American Ephemeris and Nautical Almanac*. The normal places during the 1930–31 opposition were derived for 0^hU.T., which is the same instant as $(0^h + \Delta t)$ Newtonian Time; in other words, the coordinates of Eros and the sun were derived for a time which is Δt greater than the observed U.T. No corrections were applied to the elements of Eros at the osculating epoch because of this change in the time; these were allowed to be absorbed in the corrections which were derived in the general solution. The principal contribution of this nature amounted to about $-0''.55$ in the mean longitude. For all the other normal places the observed positions were corrected for the motion in right ascension and declination during a time interval $-\Delta t$. This gives the normal place for 0^h Newtonian Time. Table I gives

TABLE I. CORRECTIONS FROM ASTRONOMICAL TO NEWTONIAN TIME

Epoch	Δt	Epoch	Δt	Epoch	Δt
1925.5	$0^d.000265$	1932.5	$0^d.000271$	1939.5	$0^d.000276$
1926.5	0.000269	1933.5	0.000271	1940.5	0.000280
1927.5	0.000267	1934.5	0.000275	1941.5	0.000290
1928.5	0.000266	1935.5	0.000276	1942.5	0.000290
1929.5	0.000266	1936.5	0.000274	1943.5	0.000296
1930.5	0.000275	1937.5	0.000274	1944.5	0.000304
1931.5	0.000270	1938.5	0.000278		

the values of Δt = Newtonian Time − Universal Time. The values until 1932.5 are taken from *Astronomical Papers of the American Ephemeris*, Vol. 11, Part I, p. 41; for the transmission of the values since 1933.5 I wish to thank Mr. G. M. Clemence.

Sixteen observed normal positions were formed for the oppositions 1926 (1), 1928 (1), 1933 (3), 1935 (3), 1937–38 (3), 1940 (1), 1942 (1) and 1944–45 (3). Most of these could be taken over directly from Stracke's work[4] or from his computation sheets without any change. A revision of

TABLE II. OBSERVED NORMAL POSITIONS AND RESIDUALS

Date U.T.	α	δ	Equinox	√p α	√p δ	Δα cos δ	Δδ
1926 July 4.0	$19^h17^m22^s.63$	-31°15'32".2	1926.0	4	4	-2".45	-1".56
1928 Sept. 10.0	21 57 7.60	+ 4 55 19.2	1928.0	3	3	-1.69	-1.87
1930 Oct. 10.0	5 55 54.769	+45 39 48.25	1930.0	5	5	-1.11	+0.28
Oct. 20.0	6 33 50.629	+47 2 52.22	1930.0	5	6	-0.93	+0.73
Oct. 30.0	7 13 9.096	+47 44 20.75	1930.0	5	5	-1.11	+1.10
Nov. 9.0	7 52 44.833	+47 35 10.31	1930.0	6	7	-1.24	+1.62
Nov. 19.0	8 31 5.127	+46 27 33.71	1930.0	6	7	-1.20	+1.56
Nov. 29.0	9 6 19.363	+44 14 9.98	1930.0	6	7	-1.23	+2.35
Dec. 9.0	9 36 57.848	+40 46 16.76	1930.0	6	7	-1.20	+2.63
Dec. 19.0	10 1 40.021	+35 53 51.51	1930.0	6	6	-2.02	+3.28
Dec. 29.0	10 19 14.611	+29 22 49.71	1930.0	8	10	-1.87	+3.68
1931 Jan. 8.0	10 29 2.915	+20 58 34.39	1930.0	8	10	-2.35	+3.51
Jan. 18.0	10 30 33.063	+10 45 45.05	1930-0	16	15	-3.42	+3.74
Jan. 28.0	10 24 0.240	- 0 24 23.78	1930.0	17	15	-4.17	+3.29
Feb. 7.0	10 11 28.952	-10 48 24.92	1930.0	17	15	-4.02	+2.37
Feb. 17.0	9 56 25.148	-18 44 23.01	1930.0	14	13	-4.53	+2.02
Feb. 27.0	9 43 15.825	-23 36 24.07	1930.0	14	13	-3.59	+1.86
Mar. 9.0	9 35 25.002	-25 54 2.47	1930.0	14	10	-2.87	+1.82
Mar. 19.0	9 34 0.339	-26 24 52.57	1930.0	12	10	-2.49	+1.83
Mar. 29.0	9 38 55.207	-25 52 45.91	1930.0	10	10	-1.79	+1.42
Apr. 8.0	9 49 6.528	-24 50 57.72	1930.0	8	7	-1.59	+0.99
Apr. 18.0	10 3 19.370	-23 39 12.76	1930.0	7	6	-1.31	+0.83
Apr. 28.0	10 20 35.900	-22 31 17.46	1930.0	7	6	-0.87	+0.81
1933 Mar. 26.0	18 31 0.86	-36 29 32.4	1925.0	2	2	-1.53	+0.31
May 31.0	19 17 11.34	-37 37 23.7	1925.0	2	2	-0.97	-1.09
Aug. 14.0	17 36 23.08	-26 59 9.1	1925.0	2	2	-0.56	-0.64
1935 July 12.0	22 35 32.36	- 5 31 15.4	1935.0	4	4	-0.87	-0.91
Aug. 25.0	21 38 4.54	- 2 25 29.3	1935.0	7	7	-0.16	-0.41
Nov. 9.0	21 18 42.25	- 2 8 49.3	1935.0	3	3	+0.18	-0.26
1937 Nov. 3.0	4 22 21.61	+57 32 42.2	1937.0	4	4	-2.61	-0.08
1938 Jan. 14.0	4 15 21.98	+39 40 22.4	1938.0	6	6	-3.33	-0.24
Feb. 23.0	6 4 30.25	+10 3 48.3	1938.0	7	7	-0.62	+1.42
1940 July 30.0	16 50 50.77	-30 10 32.4	1940.0	6	6	-0.12	-0.81
1942 Aug. 13.0	21 11 22.68	- 9 21 7.5	1942.0	7	7	+0.50	-0.15
1944 Sept. 15.0	1 56 27.28	+36 27 36.3	1944.0	4	4	+2.84	+0.93
Nov. 30.0	0 25 28.57	+39 15 8.4	1944.0	3	3	+2.68	+3.94
1945 Feb. 2.0	3 1 7.40	+24 18 30.7	1945.0	2	2	+2.53	+0.66

the first normal position in 1933, due to the elimination of some outstanding residuals, produced the correction $+^s.02$ to the right ascension. The normal positions for 1940 and 1944–45 had to be derived by comparison of the observations with the ephemerides already computed by G. Stracke. For all of these normal places the tabular positions of Eros were computed anew and independently for the formation of the residuals (O – C). Table II gives the observed normal positions and the residuals for the equations of condition. These residuals already include the reduction to Newtonian Time, computed by means of the rate of change in the corresponding comparison ephemerides. The (O – C)'s are different from those obtained from Stracke's ephemerides, due to the fact that we introduced the Newtonian Time and omitted any arbitrary corrections to the sun's coordinates.

At first, weights p were assigned, corresponding to the number of observations used for the formation of each position. But these weights finally were not applied. If they had been applied, the influence of all the other oppositions outside of 1930–31 would become too small in the solution. With 21 positions in 1930-31, the actual influence of this important approach is sufficiently taken care of, without the application of weights. The final solution is based on weight unity for each of the 74 equations of condition. In Table II the $\sqrt{p}$ is given only to represent approximately the relative accuracy of the different normal places.

As to the individual observations which are used for the formation of the positions from 1930 to 1938, the necessary references and details can be found in the papers of G. Stracke[4] and H. Spencer Jones.[9] The position for 1926 represents the average residuals of 13 observations[10] made in Algiers and Johannesburg from 1926 June 4 to July 18. For the normal position in 1928, 10 observations[11] from Algiers, Kasan and Uccle from 1928 July 27 to September 18 were used. The arithmetic mean of the residuals referred to a comparison ephemeris was derived in the same way as for 1926. The individual residuals were always small, never exceeding a few seconds of arc, so that no serious second-order effect could arise. The position for 1940 is based on 45 observations made in La Plata from 1940 May 28 to September 26, transmitted by letter to G. Stracke, and 2 observations from Tokyo. The average of the (O−C)'s from Stracke's ephemeris for 1940 was $+0^s.03$, $-0''.5$. For 1942 a large amount of observational material was obtained at very many observatories. The normal corresponds to the mean residual $+0^s.04$, $+0''.3$ of 65 observations from 1942 May 28 to October 12, referred to the ephemeris of G. Stracke, the observations having been transmitted to him by letters. It is unnecessary to give here all the references to the numerous places where these observations have since been published. But it may be mentioned that 23 observations[12] made at the Lick Observatory in 1942 which are not used here have practically the same average residuals as those which are used, namely, $+0^s.03$, $+0''.4$. The last observations used in this work were made at the Lick Observatory in 1944–45 and published by J. M. Vinter Hansen.[13] These are extremely valuable because with residuals as large as $+5''$ they show very clearly the necessity of including in the general solution something more than only the six orbital elements of Eros. From the 22 observations from 1944 August 16 to 1945 February 22, which so far seem to be the only observations in this opposition, three normal positions were derived.

In Table III the osculating elements of Eros for the time of each normal place are collected.

TABLE III. OSCULATING ELEMENTS OF EROS

Date Newtonian Time	M	ω	Ω 303°+	i 10°49'+	φ 12°52'+	a	Equinox
1926 July 4.0	151°59'12".245	177°52'43".527	45'27".118	45".668	48".672	1.4581790	1926.0
1928 Sept. 10.0	239 12 3.697	177 55 1.537	46 18.008	39.124	36.105	1.4581132	1928.0
1930 Oct. 10.0	304 36 45.291	177 55 45.609	47 6.389	42.256	47.539	1.4581153	1930.0
Oct. 20.0	310 12 41.911	177 55 42.708	47 5.773	42.279	48.574	1.4581254	1930.0
Oct. 30.0	315 48 37.746	177 55 40.483	47 4.896	42.335	49.620	1.4581353	1930.0
Nov. 9.0	321 24 32.429	177 55 39.401	47 3.736	42.440	50.639	1.4581444	1930.0
Nov. 19.0	327 0 25.545	177 55 40.023	47 2.257	42.624	51.531	1.4581514	1930.0
Nov. 29.0	332 36 16.823	177 55 42.765	47 0.485	42.917	52.182	1.4581552	1930.0
Dec. 9.0	338 12 6.174	177 55 47.811	46 58.546	43.342	52.456	1.4581547	1930.0
Dec. 19.0	343 47 53.707	177 55 55.092	46 56.668	43.905	52.175	1.4581488	1930.0
Dec. 29.0	349 23 39.527	177 56 4.575	46 55.104	44.582	51.015	1.4581351	1930.0
1931 Jan. 8.0	354 59 24.009	177 56 16.171	46 54.084	45.315	48.482	1.4581100	1930.0
Jan. 18.0	0 35 8.488	177 56 28.908	46 53.749	45.914	44.310	1.4580721	1930.0
Jan. 28.0	6 10 54.808	177 56 41.490	46 53.756	46.067	38.920	1.4580259	1930.0
Feb. 7.0	11 46 44.296	177 56 52.985	46 53.185	45.614	33.313	1.4579806	1930.0
Feb. 17.0	17 22 37.002	177 57 3.023	46 51.380	44.752	28.458	1.4579436	1930.0
Feb. 27.0	22 58 31.995	177 57 11.596	46 48.567	43.834	24.688	1.4579169	1930.0
Mar. 9.0	28 34 28.319	177 57 18.792	46 45.385	43.075	21.868	1.4578986	1930.0
Mar. 19.0	34 10 25.284	177 57 24.799	46 42.359	42.529	19.757	1.4578865	1930.0
Mar. 29.0	39 46 22.257	177 57 30.056	46 39.763	42.179	18.170	1.4578791	1930.0
Apr. 8.0	45 22 19.089	177 57 34.709	46 37.734	41.970	16.887	1.4578748	1930.0
Apr. 18.0	50 58 16.072	177 57 38.498	46 36.218	41.858	15.780	1.4578721	1930.0
Apr. 28.0	56 34 13.098	177 57 41.609	46 35.089	41.803	14.839	1.4578706	1930.0
1933 Mar. 26.0	87 20 19.522	177 59 44.164	41 50.779	37.467	14.474	1.4578940	1925.0
May 31.0	124 17 59.590	177 59 28.675	41 48.957	37.696	12.598	1.4578722	1925.0
Aug. 14.0	166 17 43.314	177 59 27.412	41 50.353	37.006	3.824	1.4579144	1925.0
1935 July 12.0	196 32 6.338	177 59 45.153	49 40.580	40.147	11.861	1.4578664	1935.0
Aug. 25.0	221 10 45.847	177 59 26.371	49 40.184	39.930	11.800	1.4578793	1935.0
Nov. 9.0	263 44 36.280	177 58 58.115	49 38.245	39.502	12.063	1.4579155	1935.0
1937 Nov. 3.0	309 35 33.770	178 2 56.959	50 51.843	46.111	39.489	1.4578456	1937.0
1938 Jan. 14.0	349 54 31.342	178 3 21.318	51 26.639	49.507	55.354	1.4579595	1938.0
Feb. 23.0	12 17 55.151	178 3 41.403	51 25.711	48.719	68.736	1.4580785	1938.0
1940 July 30.0	149 19 39.493	178 6 2.568	52 4.483	38.404	26.169	1.4581251	1940.0
1942 Aug. 13.0	205 47 33.054	178 6 2.579	53 5.407	42.744	31.778	1.4581843	1942.0
1944 Sept. 15.0	273 25 29.493	178 7 5.951	54 43.283	41.947	25.663	1.4581733	1944.0
Nov. 30.0	315 57 50.381	178 7 31.810	54 32.060	41.951	29.865	1.4581666	1944.0
1945 Feb. 2.0	351 47 17.439	178 7 38.885	55 21.410	42.609	39.658	1.4582379	1945.0

The next big step was the computation of the coefficients for the equations of condition. Besides the elements of Eros, the following unknowns were to be included for a differential correction in the least-squares solution: the obliquity of the ecliptic ϵ, the mean longitude l'', the eccentricity e'' and the perihelion π'' for the orbit of the earth, the masses $m_{☿}$, $m_{♀}$, $m_{\oplus+☾}$ and $m_{♂}$ of the four inner planets, and the corrections to the equinox, $-\Delta\alpha_0$, and to the equator, $-\Delta\delta_0$. From the large number of normal places in the perihelion approach of 1930–31 one may expect to obtain relatively good results for the corrections to the elements of the earth. Practically, the results for e'', π'' and l'' refer to the osculating elements of the earth in 1930–31 and the corrections $\Delta\alpha_0$ and $\Delta\delta_0$ to the system of the Eros comparison stars for the opposition 1930–31,[15] due to the overweighting of this opposition in the solution.

For the correction of the orbital elements of Eros the method for the variation of the elliptic elements in the "Form I" given by Bauschinger[16] was used. In this method the corrections to the ecliptical elements, ω, Ω, and i, are transformed to the differentials

$$\begin{aligned} dp &= \sin i \sin \omega \, d\Omega + \cos \omega \, di \\ dq &= \sin i \cos \omega \, d\Omega - \sin \omega \, di \\ ds &= \cos i \, d\Omega + d\omega, \end{aligned}$$

which are independent of the ecliptical and equatorial systems. For the coefficients of the four elements l'', ϵ, e'' and π'' of the earth the equations and tables given by S. Newcomb and J. Meier[17] were used. No attempt was made to include a correction to the radius vector R of the earth, because the mean motion of the earth cannot actually be affected by the results, especially after the introduction of Newtonian Time which is supposed to have absorbed the apparent secular acceleration in the longitude of the sun. Nevertheless, it was necessary to compute the coefficients $d\alpha/dR$ and $d\delta/dR$, because they were needed for the computation of the coefficients for the corrections to the masses of the disturbing planets.

For the improved value of any one of the four masses involved, let

$$m = m_0(1 + \vartheta), \quad \text{or} \quad dm = m_0\vartheta.$$

Then the perturbations of Eros computed with the mass m_0 ought to be corrected by multiplication with the factor $(1 + \vartheta)$, or by addition of ϑ times their value. Therefore, the required term in the α-equations of condition is

$$\frac{\partial\alpha}{\partial e}\frac{de}{dm}\,dm,$$

and this becomes

$$\frac{\partial\alpha}{\partial e} \times (\text{perturbation in } e) \times \vartheta.$$

The factor $\partial\alpha/\partial e$ is already known, since it is the coefficient for the correction of the element e. The coefficient of ϑ is then obtained by taking the sum over all the six elements: $\partial\alpha/\partial e$ multiplied by the perturbation in e.

That is all that is necessary to obtain the coefficients of the correction to the mass of earth + moon, because the position of the earth in its orbit will not be affected by a differential change of its own mass. A correction in the mass of Mercury, Venus or Mars will change not only the perturbations caused by these planets in the orbit of Eros, but also the disturbed positions of the earth, as computed from Newcomb's Tables of the Sun, and in this second way affect the positions of Eros as observed from the earth. In order to take care of this, for each of these three disturbing planets from Newcomb's Tables of the Sun the perturbations in the longitude and in the radius vector of the earth, counted from the zero epoch 1931 January 18.0, were computed, also, using the factors given by Newcomb,[14] the secular variations of e'' and π'' from the same epoch. The four values of these perturbations in l'', R, e'' and π'', multiplied by the unknown mass correction factor ϑ, have to be inserted in the equations of condition in the same way as before. Therefore in this case the summation includes ten terms instead of six. For these computations the above-mentioned coefficients $d\alpha/dR$ and $d\delta/dR$ are necessary, in spite of the fact that they are not needed later for any orbital correction dR of the earth. To obtain the four different ϑ's for the mass corrections in seconds of arc, the same as the other unknowns, the computed coefficients must be multiplied by $\sin 1''$. The equinox correction is $-\Delta\alpha_0$ and the equator correction $-\Delta\delta_0$. Then the coefficients of $\Delta\alpha_0$ are $\cos\delta$ in the α-equations and zero in the δ-equations, and for $\Delta\delta_0$ the coefficients are zero in the α-equations and unity in the δ-equations.

Table IV contains all the computed coefficients for the 16 unknowns in the 74 equations of condition. In order to bring all the coefficients to the same order of magnitude, the mean motion correction dn of Eros is changed to $100\,dn$ and the four mass factors ϑ are replaced by $\vartheta\,10^{-4}$. No

TABLE IV. EQUATIONS OF CONDITION

No.	Date Newtonian Time	ΔM_0	$10^2\Delta n$	$\Delta\varphi$	Δs	Δp	Δq	$10^{-4}\vartheta_{\oplus+☾}$
				Right Ascensions				
1	1926 July 4.0	+1.488	- 24.575	+1.176	+2.230	-0.229	-0.688	+ 57.622
2	1928 Sept. 10.0	+1.459	- 12.870	-2.180	+2.043	+0.780	-0.918	+ 26.618
3	1930 Oct. 10.0	+1.978	- 0.982	-2.570	+1.386	-0.295	-0.054	+ 1.761
4	Oct. 20.0	+2.164	- 0.845	-2.401	+1.442	-0.480	-0.148	+ 1.768
5	Oct. 30.0	+2.340	- 0.650	-2.090	+1.487	-0.670	-0.301	+ 1.727
6	Nov. 9.0	+2.507	- 0.398	-1.631	+1.524	-0.844	-0.518	+ 1.636
7	Nov. 19.0	+2.681	- 0.099	-1.039	+1.573	-0.978	-0.798	+ 1.501
8	Nov. 29.0	+2.903	+ 0.232	-0.347	+1.665	-1.049	-1.134	+ 1.328
9	Dec. 9.0	+3.231	+ 0.582	+0.418	+1.843	-1.040	-1.518	+ 1.117
10	Dec. 19.0	+3.748	+ 0.950	+1.242	+2.162	-0.939	-1.940	+ 0.858
11	Dec. 29.0	+4.539	+ 1.336	+2.132	+2.680	-0.731	-2.385	+ 0.603
12	1931 Jan. 8.0	+5.635	+ 1.744	+3.099	+3.417	-0.400	-2.816	+ 0.179
13	Jan. 18.0	+6.925	+ 2.141	+4.097	+4.299	+0.052	-3.151	0.000
14	Jan. 28.0	+8.030	+ 2.434	+4.944	+5.078	+0.573	-3.261	+ 0.300
15	Feb. 7.0	+8.468	+ 2.524	+5.412	+5.423	+1.051	-3.072	+ 1.189
16	Feb. 17.0	+8.076	+ 2.399	+5.412	+5.215	+1.389	-2.652	+ 2.236
17	Feb. 27.0	+7.106	+ 2.132	+5.074	+4.613	+1.571	-2.148	+ 2.919
18	Mar. 9.0	+5.952	+ 1.841	+4.609	+3.881	+1.641	-1.672	+ 3.132
19	Mar. 19.0	+4.875	+ 1.586	+4.154	+3.195	+1.644	-1.265	+ 3.032
20	Mar. 29.0	+3.976	+ 1.392	+3.772	+2.629	+1.610	-0.928	+ 2.807
21	Apr. 8.0	+3.267	+ 1.258	+3.474	+2.191	+1.550	-0.647	+ 2.582
22	Apr. 18.0	+2.719	+ 1.172	+3.239	+1.863	+1.471	-0.414	+ 2.410
23	Apr. 28.0	+2.297	+ 1.121	+3.051	+1.620	+1.374	-0.219	+ 2.310
24	1933 Mar. 26.0	+0.672	+ 5.943	+1.394	+0.935	-0.265	-0.109	+ 11.572
25	May 31.0	+1.300	+ 11.791	+1.642	+1.910	-0.412	-0.532	+ 25.117
26	Aug. 14.0	+1.047	+ 9.253	+1.107	+1.556	-0.029	-0.185	+ 21.340
27	1935 July 12.0	+0.976	+ 16.477	-1.087	+1.441	+0.181	-0.948	+ 36.060
28	Aug. 25.0	+1.420	+ 23.720	-1.617	+2.070	+0.534	-1.029	+ 52.657
29	Nov. 9.0	+0.609	+ 10.119	-1.118	+0.900	+0.572	-0.333	+ 22.847
30	1937 Nov. 3.0	+3.909	+ 97.732	-5.725	+2.986	+0.181	+0.053	+221.636
31	1938 Jan. 14.0	+6.293	+158.389	-6.582	+4.125	+0.110	+0.380	+230.205
32	Feb. 23.0	+4.250	+108.100	-1.708	+2.497	+0.225	-0.629	+ 29.799
33	1940 July 30.0	+1.182	+ 40.577	+1.712	+1.668	-0.003	-0.008	- 28.857
34	1942 Aug. 13.0	+1.380	+ 58.369	-1.034	+2.073	+0.318	-1.044	- 68.019
35	1944 Sept. 15.0	+1.930	+ 96.896	-3.823	+1.921	+0.735	-0.292	-124.317
36	Nov. 30.0	+2.076	+103.810	-4.770	+1.977	+1.338	+0.606	-156.224
37	1945 Feb. 2.0	+1.869	+ 94.459	-3.121	+1.254	+0.130	+0.554	-204.150
				Declinations				
38	1926 July 4.0	+0.506	- 8.521	+0.574	+0.736	+0.703	+2.110	+ 19.779
39	1928 Sept. 10.0	+0.971	- 8.184	-1.696	+1.263	-1.325	+1.560	+ 16.885
40	1930 Oct. 10.0	-0.664	+ 0.784	+1.280	-0.605	-1.902	-0.350	- 0.763
41	Oct. 20.0	-1.018	+ 1.003	+1.774	-0.856	-1.957	-0.603	- 0.966
42	Oct. 30.0	-1.451	+ 1.203	+2.261	-1.140	-1.968	-0.883	- 1.158
43	Nov. 9.0	-1.959	+ 1.361	+2.681	-1.453	-1.934	-1.186	- 1.314
44	Nov. 19.0	-2.532	+ 1.452	+2.968	-1.787	-1.859	-1.518	- 1.398
45	Nov. 29.0	-3.150	+ 1.453	+3.055	-2.132	-1.753	-1.896	- 1.375
46	Dec. 9.0	-3.784	+ 1.341	+2.881	-2.474	-1.613	-2.354	- 1.220
47	Dec. 19.0	-4.388	+ 1.093	+2.386	-2.793	-1.422	-2.937	- 0.928
48	Dec. 29.0	-4.868	+ 0.674	+1.500	-3.041	-1.126	-3.676	- 0.606
49	1931 Jan. 8.0	-5.077	+ 0.048	+0.170	-3.139	-0.642	-4.526	- 0.179
50	Jan. 18.0	-4.887	- 0.778	-1.544	-3.031	+0.087	-5.269	0.000
51	Jan. 28.0	-4.387	- 1.685	-3.349	-2.790	+0.968	-5.503	- 0.175
52	Feb. 7.0	-3.953	- 2.460	-4.817	-2.644	+1.727	-5.047	- 0.646
53	Feb. 17.0	-3.825	- 2.959	-5.714	-2.701	+2.168	-4.137	- 1.321
54	Feb. 27.0	-3.844	- 3.177	-6.079	-2.826	+2.339	-3.196	- 2.069
55	Mar. 9.0	-3.779	- 3.185	-6.051	-2.862	+2.377	-2.422	- 2.705
56	Mar. 19.0	-3.558	- 3.052	-5.754	-2.764	+2.367	-1.821	- 3.112
57	Mar. 29.0	-3.216	- 2.830	-5.288	-2.561	+2.332	-1.344	- 3.278
58	Apr. 8.0	-2.819	- 2.564	-4.737	-2.301	+2.275	-0.950	- 3.253
59	Apr. 18.0	-2.416	- 2.279	-4.160	-2.022	+2.193	-0.616	- 3.100
60	Apr. 28.0	-2.039	- 1.997	-3.598	-1.750	+2.086	-0.333	- 2.870
61	1933 Mar. 26.0	+0.292	+ 2.370	+0.591	+0.335	+1.159	+0.477	+ 4.980

TABLE IV. EQUATIONS OF CONDITION (*continued*)

No.	$10^{-4}\vartheta_{♀}$	$10^{-4}\vartheta_{♂}$	$10^{-4}\vartheta_{☿}$	$\Delta l''$	$\Delta\varepsilon$	$\Delta e''$	$e''\Delta\pi''$	$\Delta\alpha_{\circ}$	$\Delta\delta_{\circ}$
				Right Ascensions					
1	- 14.263	- 3.837	-0.737	-1.541	-0.202	+0.216	-3.080	+0.855	0.000
2	- 7.607	- 0.397	-0.436	-1.280	-0.107	+2.164	-1.449	+0.996	0.000
3	- 0.367	- 0.198	-0.004	-0.655	+0.004	+1.093	+2.210	+0.699	0.000
4	- 0.229	- 0.202	-0.019	-0.794	-0.064	+0.894	+2.723	+0.681	0.000
5	- 0.073	- 0.202	-0.025	-0.945	-0.211	+0.541	+3.227	+0.672	0.000
6	+ 0.085	- 0.193	-0.031	-1.088	-0.436	+0.045	+3.686	+0.674	0.000
7	+ 0.218	- 0.177	-0.029	-1.239	-0.725	-0.569	+4.049	+0.689	0.000
8	+ 0.305	- 0.156	-0.025	-1.434	-1.042	-1.243	+4.332	+0.716	0.000
9	+ 0.335	- 0.136	-0.018	-1.714	-1.338	-1.955	+4.610	+0.757	0.000
10	+ 0.323	- 0.110	-0.008	-2.075	-1.587	-2.691	+4.951	+0.810	0.000
11	+ 0.262	- 0.083	+0.002	-2.606	-1.763	-3.481	+5.428	+0.871	0.000
12	+ 0.157	- 0.049	+0.004	-3.272	-1.842	-4.398	+6.080	+0.934	0.000
13	0.000	0.000	0.000	-4.059	-1.809	-5.415	+6.685	+0.982	0.000
14	- 0.193	+ 0.064	+0.004	-4.784	-1.645	-6.455	+7.454	+1.000	0.000
15	- 0.367	+ 0.143	+0.017	-5.221	-1.361	-7.259	+7.609	+0.982	0.000
16	- 0.480	+ 0.214	+0.038	-5.160	-0.994	-7.496	+7.076	+0.947	0.000
17	- 0.487	+ 0.264	+0.056	-4.613	-0.622	-7.108	+5.992	+0.916	0.000
18	- 0.418	+ 0.291	+0.067	-3.808	-0.301	-6.290	+4.678	+0.900	0.000
19	- 0.299	+ 0.288	+0.064	-2.978	-0.044	-5.302	+3.484	+0.896	0.000
20	- 0.174	+ 0.269	+0.037	-2.253	+0.150	-4.384	+2.514	+0.900	0.000
21	- 0.062	+ 0.246	+0.009	-1.683	+0.299	-3.601	+1.756	+0.907	0.000
22	+ 0.040	+ 0.222	-0.001	-1.239	+0.407	-2.958	+1.248	+0.916	0.000
23	+ 0.124	+ 0.199	+0.001	-0.910	+0.483	-2.467	+0.856	+0.924	0.000
24	+ 2.996	+ 0.070	+0.117	+0.034	-0.005	-0.034	-1.067	+0.804	0.000
25	+ 7.100	+ 0.202	+0.454	-1.275	-0.203	-0.469	-2.706	+0.792	0.000
26	+ 4.479	+ 0.194	+0.272	-0.618	+0.029	+0.087	-1.549	+0.891	0.000
27	+ 7.884	+ 2.328	+0.407	-0.665	-0.376	+0.927	-1.207	+0.995	0.000
28	+ 10.208	+ 3.522	+0.908	-1.290	-0.215	+1.858	-1.799	+0.999	0.000
29	+ 5.206	+ 1.516	+0.393	-0.017	+0.197	+0.526	-0.681	+0.999	0.000
30	+ 52.938	+15.878	+1.836	-4.014	+0.487	+5.941	+5.754	+0.537	0.000
31	+108.241	+25.376	+3.400	-3.975	+0.970	+2.532	+8.747	+0.770	0.000
32	+ 71.992	+16.941	+4.510	-1.610	-0.012	-0.687	+4.485	+0.985	0.000
33	+ 19.081	+ 3.533	+1.084	-0.739	+0.117	-0.217	-1.762	+0.864	0.000
34	+ 33.671	+ 2.456	+2.278	-1.284	-0.256	+1.546	-2.055	+0.987	0.000
35	+ 53.895	+ 2.859	+1.894	-1.362	-0.084	+2.934	+0.178	+0.804	0.000
36	+ 40.260	+ 3.324	+3.068	-1.418	+1.178	+3.696	+1.003	+0.774	0.000
37	+ 28.507	+ 3.380	+3.741	-0.174	+0.482	+1.718	+1.517	+0.911	0.000
				Declinations					
38	- 4.857	- 1.470	+0.256	-0.096	+1.283	-0.184	-0.394	0.000	+1.000
39	- 5.244	- 0.209	-0.281	+0.185	+0.292	+1.187	-0.248	0.000	+1.000
40	+ 0.423	+ 0.097	+0.006	+1.111	-0.392	-1.108	+0.071	0.000	+1.000
41	+ 0.473	+ 0.116	+0.016	+1.247	-0.683	-1.354	-0.241	0.000	+1.000
42	+ 0.484	+ 0.131	+0.025	+1.396	-1.012	-1.562	-0.673	0.000	+1.000
43	+ 0.453	+ 0.142	+0.033	+1.552	-1.369	-1.688	-1.225	0.000	+1.000
44	+ 0.388	+ 0.143	+0.036	+1.711	-1.758	-1.688	-1.860	0.000	+1.000
45	+ 0.296	+ 0.135	+0.035	+1.852	-2.184	-1.523	-2.549	0.000	+1.000
46	+ 0.191	+ 0.115	+0.028	+1.944	-2.658	-1.163	-3.210	0.000	+1.000
47	+ 0.082	+ 0.089	+0.015	+2.030	-3.197	-0.598	-3.743	0.000	+1.000
48	- 0.002	+ 0.057	0.000	+1.975	-3.777	+0.185	-3.984	0.000	+1.000
49	- 0.030	+ 0.020	-0.002	+1.426	-4.295	+1.102	-3.717	0.000	+1.000
50	0.000	0.000	0.000	+0.844	-4.523	+2.056	-2.871	0.000	+1.000
51	+ 0.068	- 0.004	-0.003	+0.309	-4.191	+2.886	-1.623	0.000	+1.000
52	+ 0.150	- 0.002	-0.010	+0.165	-3.312	+3.565	-0.628	0.000	+1.000
53	+ 0.233	- 0.012	-0.022	+0.451	-2.255	+4.112	-0.137	0.000	+1.000
54	+ 0.288	- 0.037	-0.036	+0.877	-1.336	+4.454	+0.070	0.000	+1.000
55	+ 0.298	- 0.070	-0.049	+1.182	-0.638	+4.511	+0.172	0.000	+1.000
56	+ 0.257	- 0.097	-0.053	+1.304	-0.117	+4.317	+0.388	0.000	+1.000
57	+ 0.182	- 0.114	-0.037	+1.275	+0.281	+3.891	+0.634	0.000	+1.000
58	+ 0.094	- 0.122	-0.014	+1.164	+0.583	+3.389	+0.850	0.000	+1.000
59	+ 0.006	- 0.123	-0.005	+1.017	+0.806	+2.845	+1.002	0.000	+1.000
60	- 0.063	- 0.118	-0.007	+0.865	+0.960	+2.313	+1.064	0.000	+1.000
61	+ 1.149	+ 0.153	+0.050	-0.360	+0.073	-0.392	-0.014	0.000	+1.000

TABLE IV. EQUATIONS OF CONDITION (*continued*)

No.	Date Newtonian Time	ΔM_0	$10^2\Delta n$	$\Delta\varphi$	Δs	Δp	Δq	$10^{-4}\vartheta_{\oplus+☾}$
				Declinations (cont.)				
62	1933 May 31.0	+0.597	+ 5.047	+1.014	+0.788	+1.244	+1.605	+ 11.348
63	Aug. 14.0	+0.139	+ 1.306	+0.062	+0.212	+0.275	+1.744	+ 3.445
64	1935 July 12.0	+0.610	+ 10.210	-0.581	+0.908	-0.305	+1.594	+ 22.806
65	Aug. 25.0	+0.835	+ 14.167	-1.155	+1.174	-0.949	+1.828	+ 31.512
66	Nov. 9.0	+0.508	+ 8.737	-0.974	+0.652	-1.050	+0.610	+ 18.940
67	1937 Nov. 3.0	-0.068	- 0.710	+1.125	-0.340	-2.719	-0.799	- 5.330
68	1938 Jan. 14.0	+1.364	+ 35.116	-0.032	+0.836	-1.459	-5.017	+ 47.262
69	Feb. 23.0	-2.232	- 58.111	-1.501	-1.434	+1.267	-3.534	- 17.671
70	1940 July 30.0	+0.010	+ 0.346	0.000	+0.016	+0.648	+1.759	+ 0.731
71	1942 Aug. 13.0	+0.740	+ 31.334	-0.688	+1.096	-0.603	+1.979	- 35.476
72	1944 Sept. 15.0	+0.601	+ 30.540	-1.158	+0.484	-1.919	+0.762	- 39.716
73	Nov. 30.0	+2.450	+124.115	-3.694	+1.898	-2.287	-1.036	-189.692
74	1945 Feb. 2.0	+0.228	+ 11.301	-0.773	+0.166	-0.545	-2.320	- 26.337

TABLE V. NORMAL EQUATIONS (UNIT = 0.1)

ΔM_0	$10^2\Delta n$	$\Delta\varphi$	Δs	Δp	Δq	$10^{-4}\vartheta_{\oplus+☽}$	$10^{-4}\vartheta_{♀}$
+8767	+ 33208	+ 3089	+ 5894	+ 161	+ 197	+ 15944	+ 17524
	+1051747	−31553	+25158	−2433	− 106	− 132441	+562344
		+ 7288	+ 2006	− 849	− 41	− 1031	− 18059
			+ 4104	+ 77	+ 216	+ 10808	+ 13023
				+1320	− 271	+ 1820	− 940
					+3269	+ 41	− 21
						+2447986	+ 90642
							+320702

$10^{-4}\vartheta_{♂}$	$10^{-4}\vartheta_{☿}$	$\Delta l'$	$\Delta\epsilon$	$\Delta e'$	$e''\Delta\pi''$	$\Delta\alpha_0$	$\Delta\delta_0$	
+ 3707	+ 894	− 4544	+ 676	− 4793	+ 6666	+1095	− 601 =	− 4381
+102403	+29942	−19176	+ 884	+17386	+30770	+6776	+1825 =	+ 3391
− 3260	− 844	− 1207	− 348	− 6238	+ 698	+ 229	− 388 =	− 2121
+ 2608	+ 683	− 3080	+ 483	− 2992	+ 4046	+ 796	− 389 =	− 2921
− 122	− 59	− 424	+ 346	+ 429	+ 469	+ 111	− 31 =	− 212
+ 48	+ 35	+ 725	+2176	+ 555	− 1003	− 330	− 472 =	− 650
+ 77641	− 9622	−15606	+2895	+ 1093	+23225	+ 919	−1690 =	−40384
+ 62031	+15725	−10363	+ 744	+ 9875	+17767	+3471	+ 846 =	− 121
+ 14081	+ 2818	− 2256	+ 239	+ 1635	+ 4201	+ 632	+ 69 =	− 1034
	+ 929	− 470	+ 45	+ 447	+ 758	+ 200	+ 56 =	+ 164
		+ 2742	+ 109	+ 2252	− 3746	− 643	+ 235 =	+ 2192
			+1768	+ 400	− 74	− 113	− 390 =	− 804
				+ 6402	− 3146	− 475	+ 322 =	+ 2739
					+ 7431	+ 795	− 165 =	− 3182
						+ 277	0 =	− 451
							+ 370 =	+ 419

distinction is to be made between Δn, $\Delta\phi$, etc., used in this table and dn, $d\phi$, etc., used in the text.

The coefficients of the normal equations for the 16 unknowns were very conveniently computed with punched cards on an IBM 602-A Calculating Punch. After the solution was obtained, it was also possible to compute the residual of each equation of condition on this machine. Thus it was practicable to compute two solutions of the problem: the first with different weights for the different normal places, and the second with equal weights for all the normal places. The first solution was not retained because of the poorer distribution of the residuals and lower weights for the masses.

The normal equations for the second solution, multiplied by the factor 10 for convenient tabulation, are given in Table V. They were solved by means of a desk calculator. As to the determinateness of the solution, only the longitude of the perihelion of Eros was derived with a relatively large uncertainty. This lack of determinateness was overcome by adopting for ds the value which corresponds to putting dl'' equal to zero. The adopted value of ds lies within the range of values specified by its own probable error.

TABLE IV. EQUATIONS OF CONDITION (*continued*)

No.	$10^{-4}\vartheta_{♀}$	$10^{-4}\vartheta_{♂}$	$10^{-4}\vartheta_{☿}$	$\Delta l''$	$\Delta\varepsilon$	$\Delta e''$	$e''\Delta\pi''$	$\Delta\alpha_0$	$\Delta\delta_0$
				Declinations (cont.)					
62	+ 2.988	+ 0.283	+0.201	-0.466	+1.162	-0.498	-0.498	0.000	+1.000
63	+ 0.375	+ 0.073	+0.038	+0.439	+0.631	-0.098	+0.114	0.000	+1.000
64	+ 5.059	+ 1.541	+0.253	-0.061	+0.936	+0.396	-0.383	0.000	+1.000
65	+ 6.428	+ 2.093	+0.538	+0.173	+0.622	+0.866	-0.446	0.000	+1.000
66	+ 4.457	+ 1.334	+0.334	+0.100	-0.595	+0.502	+0.037	0.000	+1.000
67	- 0.561	- 0.449	-0.059	+0.872	-1.262	-1.331	+0.883	0.000	+1.000
68	+ 23.191	+ 5.162	+0.735	-2.097	-3.989	-1.111	+3.047	0.000	+1.000
69	- 38.269	- 9.080	-2.374	-0.064	-1.515	+1.365	-0.602	0.000	+1:000
70	+ 0.199	+ 0.200	+0.010	+0.465	+0.844	-0.155	+0.346	0.000	+1.000
71	+ 18.609	+ 1.635	+1.225	+0.122	+0.818	+0.608	-0.542	0.000	+1.000
72	+ 16.520	+ 1.100	+0.596	+0.684	+0.176	+0.210	+0.754	0.000	+1.000
73	+ 47.061	+ 3.960	+3.628	-1.345	-1.694	+1.784	+2.828	0.000	+1.000
74	+ 3.250	+ 0.250	+0.454	-0.708	-1.505	+0.147	+0;841	0.000	+1.000

The solution furnished the following results and probable errors.

Improved masses of the four inner planets:

$$1:m_{☿} = 6120000 \pm 43000$$
$$1:m_{♀} = 408645 \pm 208$$
$$1:m_{\oplus+☾} = 328452 \pm 43$$
$$1:m_{♂} = 3110000 \pm 7700.$$

The corrections to the elements of Eros are:

$$dM_0 = +0''.506 \pm 0''.336$$
$$ds = -1.319 \pm 0.531$$
$$dp = +0.061 \pm 0.021$$
$$dq = +0.122 \pm 0.018$$
$$d\varphi = -0.226 \pm 0.063$$
$$dn = +0''.000634 \pm 0''.000011,$$

and correspondingly the improved elements of Eros are obtained.

Epoch and osculation 1931 January 18.0
Newtonian Time

$$M = 0° \; 35' \; 8''.995$$
$$\left.\begin{aligned}\omega &= 177 \; 56 \; 28.213\\ \Omega &= 303 \; 46 \; 53.113\\ i &= 10 \; 49 \; 45.849\end{aligned}\right\} 1930.0$$
$$\varphi = 12 \; 52 \; 44.084$$
$$n = 2015''.293108.$$

The corrections obtained to the elements of the earth are:

$$dl'' = 0''.00 \pm 0''.19$$
$$d\epsilon = -0.43 \pm 0.11$$
$$de'' = -0.22 \pm 0.11$$
$$e''d\pi'' = -0.33 \pm 0.18,$$

and the corrections for equinox and equator of all the positions:

$$\Delta\alpha_0 = +0''.30 \pm 0''.07$$
$$\Delta\delta_0 = -0.12 \pm 0.19.$$

The sum of the squares of the residuals was brought down from 310.30 to 7.55, but a considerable fraction of the diminution is misleading because a large part of the residuals has been produced artificially in the first place by the change from Universal to Newtonian Time.

With the coefficient given by de Sitter[18] for the relation between the solar parallax and the mass of earth + moon, the solar parallax is found to be

$$\pi_{\odot} = 8''.79835 \pm 0''.00039.$$

This is an unqualified confirmation of the results of Noteboom and Witt, and in addition, the accuracy is increased. This value of the solar parallax is, of course, in disagreement with the trigonometric result $\pi_{\odot} = 8''.790 \pm 0''.001$ which H. Spencer Jones[9] derived from the observations of Eros in 1930–31. There is the additional fact that the trigonometric result obtained from the close approach in 1930–31 disagrees beyond the limits of the assigned probable errors from the corresponding determination by Hinks from the close approach of Eros in 1900–01. On the other hand, all the dynamical results mentioned in this paper, Witt, Noteboom, Witt, Rabe, even though they have been derived from different observed arcs in the interval from 1893 to 1945, are in excellent agreement within their assigned probable errors.

The present result for the mass of earth + moon comes from material which, so far as the computed motion of Eros is concerned, is entirely different and independent of the work formerly done by Noteboom and Witt. Also the observational material is almost entirely different and independent.

The normal positions are represented by the solution with the final residuals given in Table VI, together with the distance ρ from the earth.

Almost all of the residuals are smaller than $\pm 0''.5$, and the probable error of one average position comes out to be

$$\epsilon = \pm 0''.243$$

for the average distance $\rho = 0.496$. This corresponds to a probable error of $\pm 0''.12$ at unit distance. The probable errors of the above results for the 16 unknowns were computed with $\epsilon = \pm 0''.243$. The satisfactory representation of the observed motion of Eros for a longer interval of time, undoubtedly is due to the inclusion of corrections for the planetary masses. For example, in the representation of the right ascension for 1938 January 14, the mass correction for Venus contributes $-3''.53$ to the residual $\Delta\alpha \cos\delta$. Mars produces up to $2''.8$ and even Mercury up to $1''.8$ in the residuals by the corrections to their masses. This makes it apparent why the results have such large weights. In the approach of 1930–31 the effect of the corrections to the elements of the earth on the residuals is considerable. The correction $d\epsilon$ produces up to $1''.9$ in the declinations, de'' up to $1''.7$ and $e''d\pi''$ up to $2''.9$ (for 1938 January 14) in the right ascensions.

TABLE VI. FINAL RESIDUALS OF EROS

Date	$\Delta\alpha \cos\delta$	$\Delta\delta$	ρ
1926 July 4	$-0^s.042$	$-0''.12$	0.748
1928 Sept. 10	−0.004	−0.48	0.691
1930 Oct. 10	−0.002	−0.03	0.655
Oct. 20	+0.018	+0.12	0.585
Oct. 30	+0.012	+0.14	0.520
Nov. 9	+0.006	+0.26	0.460
Nov. 19	+0.009	−0.23	0.406
Nov. 29	+0.006	+0.10	0.355
Dec. 9	−0.013	−0.09	0.309
Dec. 19	−0.029	+0.11	0.268
Dec. 29	+0.007	+0.16	0.232
1931 Jan. 8	+0.015	−0.15	0.203
Jan. 18	−0.009	+0.23	0.183
Jan. 28	−0.012	+0.21	0.174
Feb. 7	+0.018	−0.26	0.178
Feb. 17	−0.028	−0.32	0.192
Feb. 27	0.000	−0.30	0.216
Mar. 9	+0.005	−0.17	0.248
Mar. 19	−0.007	+0.08	0.285
Mar. 29	+0.010	−0.04	0.327
Apr. 8	+0.001	−0.17	0.375
Apr. 18	+0.006	−0.06	0.428
Apr. 28	+0.024	+0.14	0.488
1933 Mar. 26	−0.072	+0.61	1.166
May 31	−0.013	−0.12	0.770
Aug. 14	+0.017	−0.27	0.999
1935 July 12	−0.033	+0.04	0.938
Aug. 25	+0.052	+0.61	0.726
Nov. 9	+0.028	+0.05	1.113
1937 Nov. 3	+0.015	−0.47	0.409
1938 Jan. 14	−0.026	−0.38	0.215
Feb. 23	+0.035	+0.66	0.284
1940 July 30	−0.014	−0.47	0.934
1942 Aug. 13	−0.002	+0.14	0.751
1944 Sept. 15	+0.011	+0.33	0.668
Nov. 30	+0.010	+0.18	0.403
1945 Feb. 2	−0.041	+0.19	0.464

In view of the general importance of the results, the question may arise whether or not a new computation of the perturbations of Eros by the earth would exhibit a second-order effect that could change these results, especially for $m_{\oplus + \text{☾}}$.

In the method used for the computations of the perturbations, in view of the small corrections to the basic elements, the disturbing forces practically depend on the osculating, disturbed values of the longitude in the orbit and of the radius vector at each time. An accuracy of $0''.0001$ in the single integrals and of $0''.00001$ in the double integral is obtained, with certain exceptions in the perturbations of the perihelion, if the perturbations are computed with an accuracy of $0^\circ.001$ in the longitudes or of 0.00001 in the radius r or in $\log r$, except during the close approaches to the earth. For a certain time during these approaches an accuracy of $0^\circ.0001$ or 0.000001 was necessary.

Within the limits of computational accuracy the corrections to the elements a and e for Eros are negligible, because their influence can never exceed $0^\circ.0001$, and the same is true for the change in the inclination i. The longitude is affected by dL_0, dn, and even the mass corrections. The mean longitude contributes $-0''.55$, which is artificially caused by the change from Universal to Newtonian Time. The remaining changes are shown below.

Time	ΔL (Rabe-Stracke)
July 1926	$-1''.1$
approach 1930–31	$-0.3 \pm 0''.4$
approach 1937–38	+0.6
approach 1944–45	+1.9

These values, which change very slowly, are always below the limit for the accuracy of the ordinary computations. Considering then only the three approaches of Eros to the earth, for the most important one of these, 1930–31, the correction is practically zero within the limit of its uncertainty. The correction in 1937–38 just reaches the limit of the higher computational accuracy. Only the correction in 1944–45 in the limits of the $0^\circ.0001$ accuracy may produce a small real effect in the last 10 intervals used in the present investigation. In the most unfavorable case, for 1944 November 26, the following corrections to the components of the disturbing force by earth + moon are obtained:

$$dR = +0.000041\,R, \quad dS = +0.000019\,S, \quad dW = -0.000026\,W.$$

For this approach to 0.4 astronomical units the 4-day interval was not necessary. The 10-day variations of the perturbations by the earth may be changed at most by 0.00005 of their own values. The variations themselves are not greater than 2″ in the single integrals or 0.″2 in the double integral. In an integration over 10 intervals the possible errors are thus limited to 0.″001 in the single integrals and to 0.″0006 in the double integration. That means that no appreciable second-order effect within the probable accuracy of the results is present.

Another point which may be open to criticism is the question of the longitudes of the earth used for the computation of the perturbations. G. Stracke in his comparison of theory and observations always used Universal Time, and so a certain correction to the sun's mean longitude of about +1.″5 (cf. *Nautical Almanac* 1940, page 503) was to be applied. For the computation of the perturbations by the earth Stracke at first used uncorrected longitudes. In connection with a final revision of the perturbations especially for the approach of 1930–31, he later recomputed the perturbations by the earth with the correction +0.″065 (t − 1900) to the longitudes of the earth. These corrections now are considered too large. However the fact that Stracke computed the perturbations twice, with longitudes of the earth which in 1930–31 are different by 2″, affords the possibility of checking what effect may be caused by a systematic error of about +2″ − 1.″5 = +0.″5 in the earth's longitude. A comparison of the two different results for the longitude perturbations, with and without the longitude correction to the earth, gives differences so small that an error of $\Delta L_{\oplus}$ = +0.″5 in 1930–31 produces an effect in the longitudes of Eros which corresponds to a change of less than +0.″00001 in the mean daily motion of Eros. This amount is within the probable error of the final result for the mean motion of Eros.

All these considerations show convincingly that the perturbations of Eros have reached their definitive values, in so far as they are involved in this investigation.

The influence of the masses of the different disturbing planets on the residuals of Eros can easily be seen from the corresponding coefficients in the equations of condition, Table IV. If for any one of the four inner planets, p_α is the whole perturbation (counted from the zero epoch 1931 January 18.0) in α, and p_δ the perturbation in δ, then what is tabulated under $10^{-4}\,\vartheta$ for the corresponding planet, is

$$p_\alpha \cos\delta\; 10^4 \sin 1'' \quad \text{and} \quad p_\delta\; 10^4 \sin 1''.$$

It may be questioned how much the probable error for the masses obtained here, especially for the small mass of Mercury, may be affected by the uncertainty which is caused by the accumulation error in the double integration of the perturbations by Mercury. Therefore the following investigation was made.

By means of the expression $\pm 0.1124\, n^{\frac{3}{2}}$ for the probable accumulation error in the double integral, n being the number of intervals, for each equation of condition the corresponding probable error in the mean longitude of Eros was computed. With the coefficients for dM_0, these values were transformed into the corresponding probable errors in α and δ. From those the probable errors in the coefficients for the mass determination of Mercury could be computed. These errors, expressed as fractions of the coefficients themselves, all came out between 0.0000 and ±0.0015, with an average value of ±0.0013.

The coefficients for Mercury are proportional to the mass of the planet. The basic value may be m_0. The effect of the accumulation of error may be interpreted as an uncertainty in the result for $m_☿$ in so far as the coefficients and the residuals are not computed with m_0, but effectively with an erroneous mass $m_0(1 \pm 0.0013)$. Therefore $m = m_0(1 + \vartheta_☿)$ must have the same factor of uncertainty. The probable error for $m_☿$ as derived from the least-squares solution, expressed in units of the mass itself, came out to be ±0.0068. The two uncertainty factors which may be combined by the rules of error computation are 1 ± 0.0068 and 1 ± 0.0013. The probable error of the result thus becomes ±0.0069 of its own value. The neglected accumulation errors from the single integrals can be cared for by increasing this to ±0.0070. This has the effect that the probable error for the reciprocal of $m_☿$, which first came out to be ±42000, changes its value to ±43000, as already given above in the collection of the results. It may be mentioned here that for the other planets the number of integration intervals is considerably smaller and the perturbations so much larger that the corresponding accumulation errors cannot produce any appreciable effect on the probable errors of the results for the masses.

In order to examine how sensitive the representation of the observations is to the adopted mass of Mercury, another investigation was performed. The sum of the squares of the final residuals is 7.55, if the residuals are expressed in units of 1″. By changing arbitrarily the value for the reciprocal of $m_☿$ positively and negatively by its own probable error, and inserting the changed values for the mass correction in the equations of condition, the new sums of the squares of the residuals were computed. They came out to be 9.44 and 9.29, respectively. By these considerable changes it is evident that the solution for the mass of Mercury must be relatively strong. This can be confirmed, in another way, by trying to solve the equations of condition again for only the one unknown $m_☿$, with the (O−C)'s obtained from the solution for the 16 unknowns with the result

$$\Delta\vartheta_☿ = +45'' \pm 252'',$$

i.e., practically zero, in addition to the first solution, which was represented by the result

$$\vartheta_☿ = -3969'' \pm 1404''.$$

The comparison star places of the observations have not been reduced uniformly to one homogeneous system. But the system of the catalogue of comparison stars used in 1930–31 for all of the observations is supposed to be very close to the system of the FK3, and the differences in the other oppositions between the other systems and the FK3 can not be expected to be so large that the derived normal positions would be affected by amounts comparable with the probable error $\pm 0\overset{''}{.}24$. It seems to be quite justified in this stage of the Eros work to neglect the higher-order effects which the reduction of the observations to one really homogeneous system may produce. A uniform reduction, which would entail a vast amount of work, would be desirable in order to obtain a still higher degree of precision, especially in connection with the extension of the perturbations and the comparison between theory and observations back to 1898.

After having completed the general solution for the time 1926–45, an additional solution to determine the coefficient of the lunar equation and the mass of the moon from the outstanding residuals of the close approach 1930–31 was attempted. For this purpose an ephemeris of the highest possible accuracy was necessary. Therefore not the 21 computed positions in 1930–31 as applied in this work were used, but the special ephemeris computed by G. Witt[3] for the trigonometric determination of the solar parallax and the mass of the moon. Witt's ephemeris was reduced to the basis of the present results by inserting for Eros the following element differences, Rabe−Witt:

$$\Delta M_0 = +0\overset{''}{.}351 \qquad \Delta i = +0\overset{''}{.}264$$
$$\Delta\omega = +1.468 \qquad \Delta\varphi = -0.978$$
$$\Delta\Omega = -1.172 \qquad \Delta n = +0.003356$$

in the equations of condition, together with the results for the other 10 unknowns obtained in the present investigation. The 21 tabular positions of Eros so computed can differ from those used previously in this article, partly on account of increased accuracy and smoothness, partly by certain small systematic differences from the perturbations used by Witt. The normal positions are the same as used in the earlier solutions of this article. They are supposed to fit on a smoothed curve, freed already to a considerable extent from the small irregular fluctuations caused by systematic errors in the comparison star places. A representation of the outstanding residuals in α by means of a power series, without taking care of the lunar equation, shows very clearly small residuals of periodic character, which change their signs always very close to the times for which the reduction R_α to the barycenter disappears.

It may be mentioned that H. Spencer Jones[9] in his determination of the lunar equation from the Eros observations in 1930–31 always dealt only with short arcs, eliminating so far as possible the bad influence of the irregular fluctuations caused by the star positions, and using individual normal positions as derived from the observational material for the different nights. So the determination from 21 smoothed positions in one solution is a quite different method. Because after this reduction the residuals are very small and smooth, they appear to be capable of a good representation by means of a power series. To eliminate the effect of the changing distance ρ of Eros, residuals and reduction to the barycenter, R_α, were multiplied by ρ and the resulting equations from the right ascensions are

$$\rho(\Delta\alpha \cos\delta) = a_0 + a_1T + a_2T^2 + a_3T^3 + d\rho(R_\alpha \cos\delta),$$

T being the time in units of 100 days from the epoch 1931 February 7.0, and d the unknown correcting factor of the lunar equation. The R_α's were taken from Witt's ephemeris.

The 21 residuals $\rho(\Delta\alpha \cos \delta)$ from 1930 October 10 to 1931 April 28, at intervals of 10 days, are listed as Residuals I in Table VII.

With the omission of the first one as too outstanding, the remaining 20 equations yield by a least-squares solution

$$d = -0.00080 \pm 0.00043.$$

The remaining residuals, listed as Residuals II, still show periodic fluctuations, but of a kind which can not be absorbed by the lunar equation.

TABLE VII. RESIDUALS BEFORE AND AFTER THE SOLUTION FOR THE LUNAR EQUATION

Residuals I	Residuals II	Residuals I	Residuals II
(+0″374)		+0″006	−0″002
+0.470	+0″013	−0.002	−0.003
+0.409	+0.004	0.000	+0.004
+0.348	−0.006	+0.004	+0.001
+0.302	−0.010	+0.030	−0.009
+0.233	−0.019	+0.065	−0.007
+0.193	−0.007	+0.115	−0.018
+0.163	+0.004	+0.208	−0.014
+0.127	+0.021	+0.319	+0.007
+0.085	+0.017	+0.453	+0.011
+0.035	−0.004		

The basic value of the constant of the lunar equation used in Witt's ephemeris is $L = 6''.4305$, and hence the present solution gives

$$L = 6''.4356 \pm 0''.0028.$$

This is in good agreement with the result obtained by H. Spencer Jones as revised by H. Jeffreys[19] from the observations of Eros in 1930–31, $6''.4378 \pm 0''.0018$. A combination of the two values

$$L = 6''.437 \pm 0''.002$$

may be used with the result for the solar parallax, $\pi_\odot = 8''.79835$, in the well-known relation for the mass of the moon, μ, in units of the earth's mass, giving

$$1:\mu = 81.375 \pm 0.026.$$

The declinations of Eros in 1930–31 were used in the same way for the determination of the lunar equation. The result, though in full agreement with the one from the right ascensions within the limits of its probable error, is not entitled to any consideration because its probable error is too large.

At the conclusion of this investigation it may be stated that the results show very clearly the significance of Eros for the precise determination of certain astronomical constants. No other well-observed planet is available to serve the same purpose. It will therefore be highly desirable to extend this work backward to 1898. By the use of the entire observational material and the reduction of the star places to a uniform system, results of still higher precision will be obtained.

It is my pleasure to record these acknowledgments. This present work would not have been possible without the excellent perturbations computed by G. Stracke during the last years of his life. A vast amount of work is invested in these computations, which formed the basis on which the present investigation could be built. I am very thankful to Prof. A. Kopff for placing at my disposal all of the computations by Stracke, Witt, and von Schelling, as collected in the "Astronomisches Rechen-Institut," in accordance with Stracke's wish that after his death I might continue with the Eros problem. In spite of the fact that the completion of this work to its fullest extent has still to be done, I hope with the present results to have partly fulfilled the obligation which I feel is due Stracke's wishes. The extent to which observational data have been borrowed from the work of H. Spencer Jones has already been acknowledged. To D. Brouwer and G. M. Clemence I express very sincerely my thanks for their interest and valuable discussions concerning the probable errors connected with the results. The examination of the various sources of errors and the second-order effects which might introduce systematic errors into the final results was inspired by these discussions. Finally I wish to thank Paul Herget, Director of the Cincinnati Observatory, very cordially for his supporting interest which made it possible for me to complete this work in a relatively short time. I am indebted to him also for assistance with the punched-card computations and his very appreciable help in preparing the manuscript for publication.

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