THE STARK EFFECT OF THE HIGHER BALMER LINES IN STARS OF SPECTRAL TYPES A AND B

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ABSTRACT

The purpose of the investigation was to compute the number of Balmer lines which should be visible in a stellar spectrum on the following assumptions: (1) the stellar atmosphere consists of hydrogen only; (2) the static Stark effect is the only agent of line broadening; (3) the stellar atmosphere is uniformly stratified in gravitational equilibrium.

For the probability distribution of the field strength in an ionized gas, Holtsmark's theoretical results were used, as modified by Verweij. For the Stark pattern of the higher Balmer lines a simplifying assumption was introduced, suggested by Pannekoek. Eddington's equations of radiative transfer were integrated numerically for a number of wave lengths between 3660 A and 3750 A and for the following atmospheres: surface temperatures 10,080°, 12,600°, 16,800°, and at each temperature surface gravities 10^2, 10^4, 10^6, 10^8. The values of the coefficient of continuous absorption for hydrogen were taken from published computations by Pannekoek.

The theoretical results were compared with three groups of observational data: bright stars, eclipsing binaries, and two white dwarfs. The white dwarfs seem to conform to the theoretical expectations; but here the data on mass, radius, and temperature are more uncertain than for the two other classes of objects. It was found that in a large number of stars of the first two groups more hydrogen lines can be observed than are predicted by theory. In many of these stars the effective surface gravity seems to be only one one-hundredth of the dynamical value, or less. This agrees with results of Schalén and of Pannekoek and Reesinck.

Several possibilities are investigated which might account for the discrepancy found. Radiation pressure and electron scattering may, in a few cases, have some effect but not to an extent sufficient to explain all the disagreement. In the temperature range considered, the absorption by negative hydrogen ions is negligible. It is suggested that stellar atmospheres are not built according to the model, which assumes equilibrium and stratification under the influence of gravitation and radiation pressure.

1. It has been recognized for a long time that the Balmer lines of hydrogen in stellar spectra cannot be observed as far as the theoretical limit at 3647 A. The transition from line spectrum to continuum has been found to occur around wave length 3700 A in main-sequence stars, and the Balmer continuum extends to the red of 3647. It is generally assumed that the lines are broadened by Stark effect, which causes the lines to overlap and finally to merge into a continuum.

The present investigation is an extension of work by Verweij[1] and by Pannekoek.[2] Verweij has treated the Stark effect of the first four Balmer lines in great detail and has computed line profiles for a wide range of values of surface gravity and surface temperature. Pannekoek has given a simplified treatment for the higher Balmer lines.

The aim of the work presented in the following sections is to derive, from theoretical considerations, the number of Balmer lines visible, in its dependence on surface temperature and surface gravity. Three basic assumptions underlie the computations: (1) the atmosphere consists of hydrogen only; (2) the static Stark effect is the only broadening agent; and (3) the stellar atmosphere is uniformly stratified in gravitational equilibrium.

The following problems will be dealt with successively: (1) What is the strength of the electrical field active in a gas mixture of neutral atoms, ions, and electrons? (2) In what way is each Balmer line affected by it? (3) How does the field change in successive layers of the atmosphere? (4) What is the integrated effect on the observed lines?

2. In this section the first of these problems is considered. In an atmosphere consisting of neutral atoms, ions, and electrons in thermal motion, the electric field at any

[1] *Pub. Astr. Inst. U. Amsterdam*, No. 5, 1936. [2] *M.N.*, 98, 701, 1938.

›int is not constant but changes continuously. The problem is to find the statistical stribution of the field strength as a function of density.

As a first approximation we can suppose that the field at a point is caused by the ›arest neighbor only and that the effects of the more distant particles are negligible. handrasekhar[3] has given a simple derivation for the probability distribution of the field rength in that case. Since the distance by which the Stark components are displaced ith respect to the original line is proportional to the field strength, the probability stribution of the field strength yields that of the line width $\Delta\lambda$. The latter distribution then given by $e^{-(\Delta\lambda_1/\Delta\lambda)^{3/2}}d(\Delta\lambda_1/\Delta\lambda)^{3/2}$, in which $\Delta\lambda_1$ depends on the line under con- leration and on temperature and pressure. A probability distribution of this form as first given by Russell and Stewart[4] and has been applied by Elvey and Struve[5] in ıe computation of the contour of $H\gamma$ due to Stark effect.

Holtsmark[6] has treated the problem in a more general way, taking into account the ›ntributions to the field of all the charged particles in the gas. Holtsmark's results cannot › expressed in a closed analytical form but are evaluated from series expansions. Ver- eij[1] improved on one of these, and his values have been used in the present work.

It should be pointed out here that the quantity for which the statistical distribution actually computed is not the field strength itself but its ratio β to a standard field rength F_0. The quantity $\beta = F/F_0$, where $F_0 = 2.61\, n^{2/3}e$, in which n is the number of ıarged particles per cubic centimeter and e the charge of the electron. $F_0 = [1.87]$ $P/T)^{2/3}$, in which the brackets indicate the logarithm on the base 10 of the coefficient.

Hence, once the probability distribution $W(\beta)$ of β is known, we can, for every tem- ›rature and pressure, compute the distribution of the field strength. $W(\beta)$ satisfies the ›lation

$$\int_0^{\infty} W(\beta)\, d\beta = 1 \, .$$

3. This section deals with the Stark effect caused by the electrical field discussed ›ove. After Stark had, in 1913, discovered the effect of a strong electrical field on the ›ht emitted by atoms, theoretical treatments of the phenomenon were given by Som- erfeld,[7] Epstein,[8] and many others, first on the basis of the classical Bohr theory and ter by the application of wave-mechanical methods. Both methods agree in giving, for e first-order effect in hydrogen,

$$\Delta E = \frac{3h^2}{8\pi m e}\, n\,(n_2 - n_1)\, F \qquad \text{or} \qquad \Delta E = Cn\,(n_2 - n_1)\, F\,, \tag{1}$$

ere ΔE is the change in energy of the level of principal quantum number n; n_1 and n_2 parabolic quantum numbers, integers, subject to the conditions $0 \leqslant n_1, n_2 \leqslant n - 1$. e quantity that determines the possible transitions is $m = n - n_1 - n_2 - 1$. Only ısitions $\Delta m = 0, \pm 1$, are allowed. It is clear from the formula for ΔE that, for each itive value of ΔE, there is a corresponding negative value. Consequently, the pattern ines originating from the combination of two levels, both split by Stark effect, will ›ymmetrical around the position of the undisturbed line. In all computations the fine ıcture of each hydrogen line has been disregarded.

'he largest positive value which ΔE in equation (1) can have for any value of n oc- › for $n_2 = n - 1$, $n_1 = 0$, which yields $\Delta E = CFn\,(n - 1)$. Similarly, for $n = 2$, the est negative value of ΔE is $-2CF$. In both cases $m = 0$, and hence the extreme ponents, arising from a transition between these states, does exist. It follows that

Rev. *Mod. Phys.*, No. 1, 1943.

ıp. *J.*, **59**, 204, 1924.

ıp. *J.*, **72**, 277, 1930.

[6] *Ann. d. phys.*, **58**, 576, 1919.

[7] *Quantummechanics*, chap. ii, § 2.

[8] *Ann. d. phys.*, **50**, 489, 1916.

the maximum extension of the Stark pattern for the nth Balmer line at either side o the unperturbed line is determined by

$$\Delta E_{2n} = CF\{n(n-1)+2\}. \quad (2$$

Since

$$\Delta E_{2n} = h\Delta\nu$$

and

$$\Delta\nu = \frac{-c}{\lambda^2}\Delta\lambda ,$$

we obtain, by inserting the values of the constants and the expression for the wav length of the Balmer line,

$$\begin{aligned}\Delta\lambda_m &= 0.00256\left(\frac{n^2}{n^2-4}\right)^2\{n(n-1)+2\}F \\ &= s_n\beta F_0 ,\end{aligned} \quad (3)$$

where s_n is defined by equation (3) (cf. Pannekoek, *loc. cit.*). The quantity $\Delta\lambda_m$ is th maximum extension of the Stark pattern, expressed in angstroms.

Because the number of components increases greatly with n, the simplifying assump tion, suggested by Pannekoek,[2] was made, that each absorption line is spread out int a band of uniform absorption of width $2\Delta\lambda_m$ and so that the total absorption of th band is equal to the total absorption of the original Balmer line. If a_n' is the total atomi absorption in the realm of the nth line, then the atomic absorption in the band caused b any strength F of the field amounts to

$$\frac{a_n'}{2\, s_n\beta F_0} = a_n''$$

per unit of wave length. This a_n'' has the weight $W(\beta)$. The total atomic absorption a distance $\Delta\lambda$ from the line center, originating from that line, is then found by integratio of all the contributions arising from fields which spread the line as far as $\Delta\lambda$ or farther Hence a_n is equal to

$$\begin{aligned}a_n &= \int_{\beta_{\Delta\lambda}}^{\infty}\frac{W(\beta)\, a_n'}{2\, s_n\beta F_0}\, d\beta \\ &= \frac{a_n'}{2\, s_n F_0}\int_{\beta_{\Delta\lambda}}^{\infty}\frac{W(\beta)\, d\beta}{\beta} ,\end{aligned}$$

in which the lower limit, $\beta_{\Delta\lambda}$, is that value of β which is just large enough to spread th line to a width $\Delta\lambda$ from the line center.

In order to find the function

$$U(\beta) = \int_{\beta}^{\infty}\frac{W(\beta)\, d\beta}{2\beta} ,$$

$W(\beta)/\beta$, as given by Verweij,[1] was plotted on a sufficiently large scale to permit o graphical integration. Table 1 gives the values of $U(\beta)$.

The quantity called a_n' should now be considered more carefully. This quantity stand for the rate at which one atom absorbs energy in the realm of a Balmer line. This amount is, by a well-known relation, equal to

$$\frac{\pi e^2}{mc} f_{2n} , \quad \text{or to} \quad \frac{\pi e^2}{mc}\cdot\frac{\lambda^2}{c}\, 10^{-8}\cdot f_{2n} ,$$

f the energy is expressed per unit of frequency or per unit of wave length (angstroms), espectively. In these formulae, f_{2n} is the oscillator strength for the transition from the second to the nth level. The quantities f_{2n} have been computed by Menzel and Pekeris.[9] For the transition between the n'th and the nth level, f is given by the formula

$$f_{nn'}=\frac{2^6}{3\sqrt{3}\pi}\,\frac{1}{\varpi_{n'}}\,\frac{1}{\left[\left(\frac{1}{n'}\right)^2-\left(\frac{1}{n}\right)^2\right]^3}\left|\frac{1}{n^3}\,\frac{1}{n'^3}\right|g\,,$$

where $\varpi_{n'}$ = weight of the n'th level and g = correction factor of the order 1. A very mportant point which is brought out by the treatment as given by Menzel and Pekeris s the equivalence of the discrete and continuous spectra or between the bound-bound,

TABLE 1

β	$U(\beta)$	β	$U(\beta)$	β	$U(\beta)$
0.0	0.287	1.9	0.0928	4.6	0.0091
0.1	.286	2.0	.0839	4.8	.0080
0.2	.283	2.1	.0758	5.0	.0071
0.3	.278	2.2	.0684	6	.0041
0.4	.271	2.3	.0617	7	.0027
0.5	.262	2.4	.0557	8	.0018
0.6	.252	2.5	.0502	9	.0014
0.7	.240	2.6	.0454	10	.0011
0.8	.228	2.7	.0411	11	.0008
0.9	.215	2.8	.0373	12	.0006
1.0	.202	2.9	.0338	13	.0005
1.1	.188	3.0	.0310	14	.0004
1.2	.174	3.2	.0260	15	.0003
1.3	.161	3.4	.0220	16	.0003
1.4	.148	3.6	.0187	17	.0002
1.5	.135	3.8	.0160	18	.0002
1.6	.124	4.0	.0238	19	.0002
1.7	.113	4.2	.0120	20	.0002
1.8	0.1024	4.4	0.0104	30	0.0001
				40	....

ound-free, and free-free transitions. The only change which the above formula undergoes for bound-free transitions is the substitution of ik for n, where k is no longer limited o integer values, and a similar change in the expression for g. It follows, then, that for arge n there is a smooth transition from the line spectrum to the continuous spectrum, vith the absorption coefficient proportional to $1/\nu^3$. In the region of the higher Balmer ines, from a certain line number on, which depends on the value of F_0, the lines are pread out by the Stark effect so strongly as to form one continuous band of absorption. This band merely acts as an extension of the Balmer continuum into the region to the red f the theoretical Balmer limit at 3647. The absorption in this continuum varies as $1/\nu^3$.

The steps in the computation of the absorption arising from the broadened Balmer nes are as follows:

1. Compute at each λ under consideration the $\Delta\lambda$ for the lines which contribute.
2. From $\Delta\lambda = \beta F_0 s_n$, find β for each value of F_0 used.
3. Find $U(\beta)$ from Table 1.
4. Multiply $U(\beta)$ by f_{2n}/s_nF_0. A factor of 10^{-4}, occurring in the quantities f_{2n}, was dded to the other multiplicative constants.

[9] *M.N.*, **96**, 77, 1935.

5. Sum the contributions of all the lines for which the value mentioned in 4 is appreciable (e.g., larger than 0.01); this gives the final absorption a. This sum is found to become a constant, independent of the wave length and of F_0 in the region where the lines overlap completely.

The factor $\pi e^2/mc \cdot \lambda^2/c \cdot 10^{-12}$ was omitted throughout these preliminary computations. It was assumed to be constant in the short region of wave lengths considered and was taken care of as explained in section 5.

TABLE 2

LOG a AS A FUNCTION OF λ AND LOG F_0

λ	LOG F_0							Remarks
	−1	0.70−1	0.30	0.70	1.00	1.40	2.00	
3659.42...	0.11	0.08	0.08	0.08	0.08	0.08	0.08	H 33
63.41...	0.32	0.08	.08	.08	.08	.08	.08	H 29; 1
68.35...	0.32−1	0.06	.08	.08	.08	.08	.08	
73.76...	0.80	0.16	.08	.08	.08	.08	.08	H 23
75.25...	0.73−2	0.94−1	.085	.085	.085	.085	.085	
76.37...	0.90	0.23	.085	.085	.085	.085	.085	H 22
77.74...	0.54−2	0.85−1	.085	.085	.085	.085	.085	
79.36...	1.00	0.31	.085	.085	.085	.085	.085	H 21
81.10...	0.30−2	0.60−1	.085	.085	.085	.085	.085	
82.81...	1.11	0.42	.09	.09	.09	.09	.09	H 20
84.92...	0.15−2	0.40−1	.07	.09	.09	.09	.09	
86.83...	1.22	0.52	.10	.09	.09	.09	.09	H 19
89.14...	0−2	0.17−1	.04	.09	.09	.09	.09	
91.56...	1.34	0.64	.14	.09	.09	.09	.09	H 18
3697.15...	1.46	0.77	.21	.09	.09	.09	.09	H 17
3700.62...	....	0.72−2	.82−1	.08	.09	.09	.09	
03.86...	1.60	0.90	.31	.09	.09	.09	.09	H 16
07.60...	....	0.50−2	.60−1	.05	.095	.10	.10	
11.97...	1.74	1.04	.45	.13	.105	.10	.10	H 15
16.84...	....	0.22−2	.31−1	.96−1	.08	.10	.10	
21.94...	1.89	1.19	.59	.23	.105	.10	.10	H 14
34.37...	2.05	1.36	.76	.37	.14	.105	.105	H 13
34.95...	0.26−1	0.50	.68	.36	.13	.11	.11	
37.70...	....	0.40−2	.46−1	.03	.06	.11	.11	
40.20...	....	0.84−3	.86−2	.60−1	.96−1	.11	.11	
41.70...	....	....	.71−2	.47−1	.00	.11	.11	
44.70...	....	0.83−3	.91−2	.64−1	.03	.11	.11	2
46.20...	....	0.23−2	.26−1	.95−1	.12	.11	.11	
48.70...	0.23−2	0.37−1	.40	.44	.24	.11	.11	3
50.15...	2.23	1.53	0.93	0.54	0.26	0.12	0.11	H 12

1. At log F_0 = 0.20−1, log a = 0.15.
2. At log F_0 = 1.20, log a = 0.09.
3. At log F_0 = 0.48, log a = 0.50.

The computations were carried through for the values F_0 = 0.1, 0.5, 2, 5, 10, 25, 10(Results are given in Table 2 and Figures 1 and 2. It is seen that the absorption in th center of the Balmer line is very high for small F_0 and decreases for larger values, unt for very large F_0(log F_0 = 2) even H 12 at 3750 merges into the continuum (Fig. 1 For wave lengths between two lines the situation is just the opposite: for small F_0 no lin is drawn out far enough to contribute any absorption, but for larger F_0 the absorptio increases until it reaches the same limiting value as at the line centers (Fig. 2). Point nearer to the line center in the wings of the line may show a combination of the tw

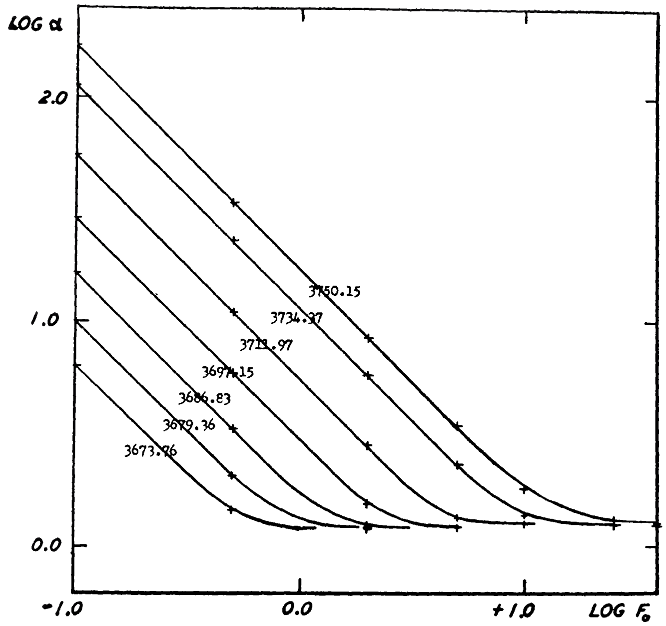

FIG. 1.—Absorption coefficient α in the centers of Balmer lines as a function of the normal field-strength F_0.

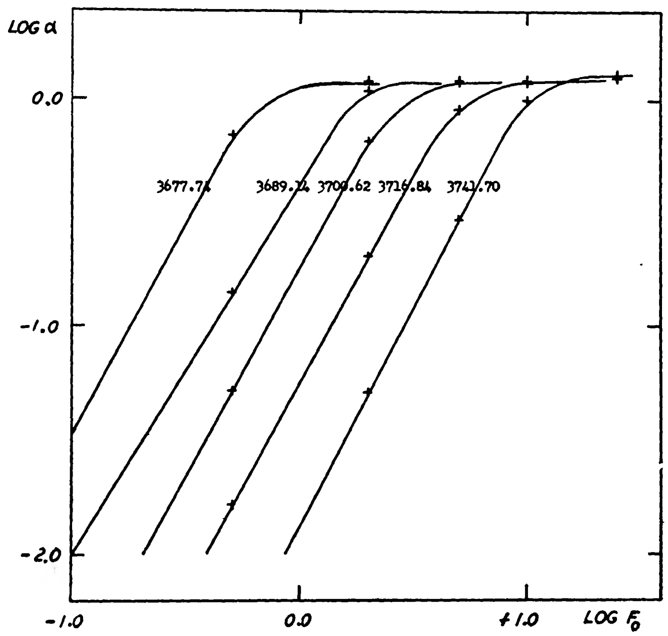

FIG. 2.—Absorption coefficient α between successive Balmer lines as a function of the normal field-rength F_0.

effects, passing through a maximum. In those cases a few additional values of a were computed, in order to make graphical interpolation possible. The plotting on a logarithmic scale made the curves very nearly linear, with only a small region of curvature.

4. We now proceed to the third question in section 1: the change of the field through the atmosphere. In order to establish the relation between optical depth and field strength, the relations should be known which connect optical depth and temperature, on the one hand, and optical depth and electron pressure, on the other. For the change of temperature with optical depth, the relation

$$T = T_0 \sqrt[4]{1 + C\tau} \tag{4}$$

was used, where

T_0 = Surface temperature,

T = Temperature at optical depth τ ,

$$\tau = -\int k \rho \, d r \,, \tag{5}$$

$$C = \frac{3}{2} \frac{\bar{k}}{k} \frac{1}{4} \frac{h\nu}{kT} \,,$$

in which

$\bar{k}$ = Mean coefficient of continuous absorption ,

k = Coefficient of continuous absorption, both per unit mass .

The results of Pannekoek's computations for $\bar{k}$ and k were used,[10] which are given in the form $\bar{k} = \bar{\kappa} P x$ and $k = \kappa P x$; $\bar{\kappa}$ and κ are tabulated as functions of temperature; P is the electron pressure, and x the degree of ionization.

The change of the electron pressure with optical depth can be computed from the following relations:

$$\frac{x}{1-x} P = 10^{-5040(13.53/T)} \cdot T^{5/2} 10^{-0.48} = K \text{ (Saha)} \,, \tag{6}$$

$$P = \frac{x}{1+x} p \,, \tag{7}$$

where p is the total pressure.

$$g \rho \, d r = - d p \,, \tag{8}$$

in which g is the surface gravity. This is the well-known hydrostatic equation. It follows from this last equation that

$$k g \rho \, d r = - k \, d p \quad \text{or} \quad g \, d\tau = k \, d p \,. \tag{9}$$

The relations (4)–(9) suffice to determine at each optical depth the value of T and P and hence of F_0 and a.

5. Finally, the foregoing has to be applied to the integration of the equations of radiative transfer. The usual Eddington formulae for the case of pure scattering were used:

$$\frac{dH}{d\tau} = J - E \,, \tag{10}$$

$$\frac{dJ}{d\tau} = q^2 H \,, \tag{11}$$

[10] *Pub. Astr. Inst. U. Amsterdam*, No. 4, 1935.

vhere

$$q^2 = 3\left(1+\frac{s}{k}\right). \tag{12}$$

[n these formulae

s = Line absorption coefficient (scattering coefficient) ,

$$J = \frac{1}{4\pi} \int I \, d\omega ,$$

$$H = \frac{1}{4\pi} \int I \cos \vartheta \, d\omega ,$$

I = Intensity of radiation ,

E = Intensity of black-body radiation at optical depth τ .

Jnder the assumptions made, equations (10) and (11) apply to each wave-length inter-val separately; hence the quantities appearing in these formulae should be defined ac-:ordingly.

The boundary condition for large optical depth is $J_\infty = E_\infty$. As boundary condition ıt the surface, the customary approximation $J_0 = 2H_0$ was used at first. Later some ›f the computations were repeated with the correct relation $J_0 = \sqrt{3}H_0$.[11, 12] These ıew computations indicated an increase in H_0 of about 4 per cent; all the results based ›n the first boundary condition were therefore increased by 4 per cent.

Because q^2 varies with optical depth in a way which cannot be given by a mathemati-:al expression, no analytical solution of equations (10) and (11) was possible. Instead, ıumerical integration in the way outlined by Verweij[1] was performed. For that purpose ι table of the relation q^2 versus τ is required for each wave length at which H_0 is to be omputed. In order to obtain such tables, a graph of the variation of log F_0 with τ was onstructed for each of the atmospheres considered, as indicated in section 4. For each ⁊ave length, then, log α could be read from the graphs of log α versus log F_0. On in-erting the constants, it was found that

$$\log \frac{s}{k} = 6.86 + \log \alpha + \log \frac{y_2}{K\kappa} .$$

.ere y_2 is the fraction of atoms in the second level, as given by Boltzmann's formula. or λ and κ the values at 3700 A were taken.

For the following twelve atmospheres computations were carried through:

$$\left.\begin{array}{ll} \theta_0 = 0.50 , & T_0 = 10{,}080^\circ , \\ \theta_0 = 0.40 , & T_0 = 12{,}600^\circ , \\ \theta_0 = 0.30 , & T_0 = 16{,}800^\circ , \end{array}\right\} \log g = 2,4,6,8 .$$

ɪr log g = 8, only a few points along the spectrum were determined, since, in the re-ɪn considered, no trace of a line spectrum is left. The results of the computations : given in Table 3. From these data the number of visible Balmer lines can be found. ʒure 3 shows the dependence of this number on temperature and gravitation.

[1] E. Hopf, *Mathematical Problems of Radiative Equilibrium* (Cambridge: At the University Press, 4), p. 46.

[2] S. Chandrasekhar, *Ap. J.*, **100**, 76, 1944.

TABLE 3

H_0 AS A FUNCTION OF λ, θ_0, g

λ	θ_0	LOG $g=2$			LOG $g=4$			LOG $g=6$			LOG $g=8$		
	No H	0.50	0.40	0.30	0.50	0.40	0.30	0.50	0.40	0.30	0.50	0.40	0.30
3659.42	33	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.2
63.41	29	.20	.20	.20	....	....	....	....	....	....	....	....	....
73.76	23	.20	....	.20	....	....	....	....	....	....	....	....	....
75.25	....	.21	.20	.21	....	....	....	....	....	....	....	....	....
76.37	22	....	.19	....	....	....	....	....	....	....	....	....	....
77.74	....	....	....	....	.20	.20	....	....	....	....	....	....	....
79.36	21	.17	....	.20	....	....	....	....	....	....	....	....	....
81.10	....	....	....	.21	....	....	....	....	....	....	....	....	....
82.81	20	....	....	....	.20	....	....	....	....	....	....	....	....
84.92	....	.25	.23	.22	.20	.20	....	....	....	....	....	....	....
86.83	....	....	....	....	....	.20	....	....	....	....	....	....	....
89.14	....	.29	.25	.23	....	....	....	....	....	....	....	....	....
91.56	18	.12	.13	.16	....	....	....	....	.20	....	....	....	....
94.18	....	.35	.29	.25	....	....	....	....	....	....	....	....	....
3697.15	17	.10	.12	.14	.19	....	....	....	....	....	....	....	....
3700.62	....	.41	.28	....	.21	....	....	....	....	....	....	....	....
01.62	....	....	.28	....	....	....	....	....	....	....	....	....	....
03.12	....	.14	....	....	....	....	....	....	....	....	....	....	....
03.86	16	.08	.10	.12	.18	.19	.20	....	....	....	.19	.20	.2
04.60	....	.14	....	....	....	....	....	....	....	....	....	....	....
06.60	....	.40	.31	.27	....	....	....	....	....	....	....	....	....
07.60	....	.45	.35	.29	.23	.21	....	....	....	....	....	....	....
08.60	....	.44	.34	.28	....	....	....	....	....	....	....	....	....
11.35	....	.11	.12	.14	....	....	....	....	....	....	....	....	....
11.97	15	.06	.08	.11	.16	.18	.19	.19	.19	....	....	....	....
12.59	....	.11	.12	.14	....	....	....	....	....	....	....	....	....
15.34	....	.47	.35	.29	....	....	....	....	....	....	....	....	....
16.84	....	.50	.37	....	.25	.23	.21	.19	....	....	....	....	....
18.34	....	.46	.35	.29	....	....	....	....	....	....	....	....	....
21.34	....	.11	.11	.12	....	....	....	....	....	....	....	....	....
21.94	14	.05	.07	.09	....	.16	.18	....	....	....	....	....	....
22.54	....	.11	.11	.12	....	....	....	....	....	....	....	....	....
25.04	....	.44	.34	....	....	....	....	....	....	....	....	....	....
26.54	....	.53	.40	.32	....	....	....	....	....	....	....	....	....
28.04	....	.55	.41	.33	....	....	....	....	....	....	....	....	....
29.54	....	.53	.40	.32	....	....	....	....	....	....	....	....	....
33.97	....	.10	.10	.11	....	....	....	....	....	....	....	....	....
34.37	13	.04	.06	.07	.10	.13	.15	.19	.19	.20	....	....	....
34.95	....	.10	.10	.11	.11	....	.17	....	....	....	....	....	....
37.70	....	.46	.35	.29	....	....	....	....	....	....	....	....	....
40.20	....	.57	.43	.33	.31	.28	.25	....	....	....	....	....	....
41.70	....	.59	.44	.34	.33	.29	.25	.20	.19	....	....	....	....
43.20	....	.59	.43	.34	.32	.29	.25	....	....	....	....	....	....
44.70	....	.57	.43	.33	....	....	....	....	....	....	....	....	....
46.20	....	.50	.38	....	....	....	....	....	....	....	....	....	....
48.70	....	.24	.21	.20	.12	.14	.15	....	....	....	....	....	....
50.15	12	0.03	0.04	0.06	0.08	0.10	0.13	0.18	0.18	0.19	0.19	0.19	0.[illegible]

6. The number of visible Balmer lines is the quantity most readily obtained from bservations. Limitations to its dependability, however, are set by the possible rotation f the observed stars and by the resolving power of the spectrograph used.

Observational material was put at the writer's disposal through the kindness of Dr. C. D. Shane, director of the Lick Observatory. A large number of microdensitometer racings on a small scale of spectra made by C. S. Yü were available, as well as others, aken by the writer at the Lick Observatory and traced on a larger scale.

The parameters entering into a comparison of theory and observations are the temerature and surface gravity of a star. The latter depends on mass and radius. For two lasses of objects these quantities are known with fair accuracy. The first one consists f bright stars, as given, e.g., in Schlesinger's *Bright Star Catalogue*, which furnishes all he data required as far as dependable parallaxes are known. The second group is formed y eclipsing binaries. The data for these objects were taken from a publication by S. Gaoschkin.[13] As to a third source of observational material, we were fortunate in having ccess to the spectra of two white dwarfs.

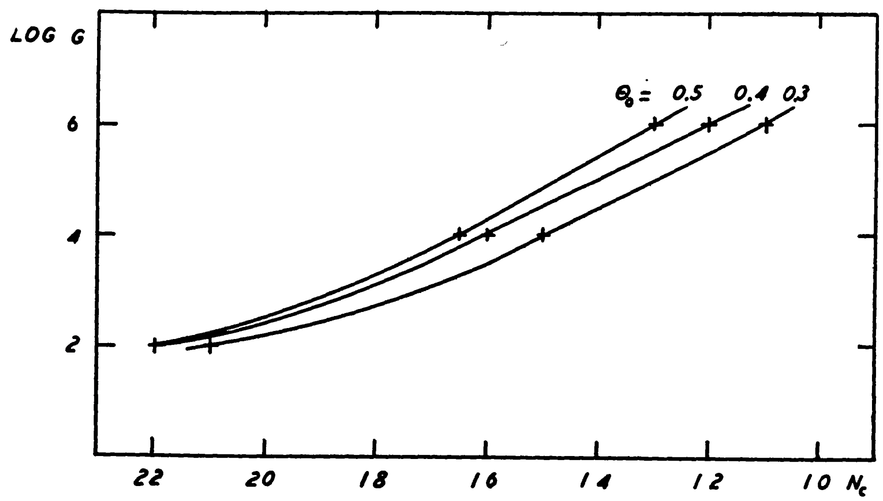

FIG. 3.—Last-resolved Balmer line N_c as a function of surface gravity G and surface temperature θ_0

The groups require a different treatment, and the single stars will be discussed first.

7. The mass of a star can be found from its absolute magnitude by the application of mass-luminosity relation. In the reductions the form given by Russell and Moore[14] is used:

$$\log \mathcal{M} = -0.105\,(M - 5.23)\,, \tag{13}$$

which $\mathcal{M}$ stands for the mass and M is the bolometric absolute magnitude. The correcns to be applied to visual magnitudes in order to reduce them to bolometric magnitudes by no means easily determined. Hertzsprung has computed these corrections, asning the stars to radiate like black bodies, but, as has been shown by Pannekoek[15] d Unsöld,[16] stellar radiation deviates from black-body radiation. From Pannekoek's rk Kuiper[17] derived corrections to be applied to Hertzsprung's numbers. Some un-

[13] *Proc. Amer. Phil. Soc.*, Vol. **82**, No. 3, 1940.

[14] *The Masses of the Stars* (Chicago: University of Chicago Press, 1940), p. 112.

[15] *M.N.*, **95**, 529, 1935.

[16] *Zs. f. Ap.*, **8**, 32, 1933; **8**, 225, 1934.

[17] *Ap. J.*, **88**, 429, 1938.

certainty may still exist as to these corrections, since they were derived from Pannekoek's theory, which, although it is qualitatively correct, still leaves discrepancies with observations. However, these corrections enter into log $\mathcal{M}$ only multiplied by 0.1, and hence they do not affect its value strongly.

The determination of the visual absolute magnitudes involves many more uncertainties, especially for the B stars. Spectroscopic determinations seemed to be the most reliable.

Many of the stars used here of classes B1–A7 have been classified by Morgan, Keenan and Kellman in their *Atlas of Stellar Spectra*, in which data on the absolute magnitudes of the various subclasses are also given. It should be kept in mind, though, that the authors themselves do not consider these determinations as final. Dr. Morgan, moreover, kindly allowed the writer to use some of his unpublished classifications and absolute magnitudes. The probable errors range from 0.4 to 1.0 mag.

For many of the stars of classes B1–B8 (on the Victoria classification) Stebbins, Huffer, and Whitford[18] give spectral classifications, together with a list of the mean absolute magnitudes for each spectral subclass.

Finally, absolute magnitudes for a small number of A stars were found in the Mount Wilson Catalogue.[19]

For the stars for which none of these sources give absolute magnitudes, the parallaxes as given by Schlesinger in the *Catalogue of Bright Stars*,[20] were used to derive the absolute magnitudes.

It follows easily from formulae (13) and (14) that a change in M amounting to $\Delta M = 1.0$ will result in a change of the computed surface gravity of $\Delta \log g = 0.3$. Hence, in spite of the existing uncertainties in M, the order of magnitude of log g, as computed, is believed to be essentially correct.

The radius of a star is related to its absolute magnitude and effective temperature by a formula, given by Russell, Dugan, and Stewart[21] as

$$\log R = \frac{5900}{T_e} - 0.2 M_v - 0.02 + x \,, \qquad (14)$$

where R is expressed in solar radii. This formula is based on a solar temperature of 6000° and a visual absolute magnitude of 4.83. If we take the values given by Kuiper,[17] viz., 5713° and 4.73, the constant -0.02 is changed into -0.09. The term x has the following values:

$T=$	8000°	10,000°	12,000°	15,000°	20,000°
$x=$	-0.01	-0.01	-0.02	-0.04	-0.06

The radius thus computed is that of the layer of the effective temperature of the star, or essentially that of the opaque disk. It is an open question how far the layers of smaller opacity extend and hence by how much log R should be increased. Radiation pressure or dynamical ejection might support the stellar atmosphere much higher than the negligibly small thickness of the photosphere found from computations, which do not take these effects into account.

Another difficult point in the discussion is that of the stellar temperatures. As mentioned before, the material for checking theory with observations consisted partly of a number of microdensitometer tracings made by Yü at the Lick Observatory. Yü has used the tracings to determine the color temperatures of these stars.[22] However, his de

[18] *Ap. J.*, 91, 20, 1940.

[19] *Mt. W. Contr.*, No. 511; *Ap. J.*, 81, 187, 1935.

[20] New Haven: Yale University Observatory, 1940.

[21] *Astronomy*, 2 (Boston: Ginn & Co., 1938), 738.

[22] *Lick Obs. Bull.*, No. 375, 1926.

terminations seem to be affected by systematic errors,[23] and, besides, the reduction of color temperatures to effective temperatures is an uncertain procedure. Therefore, in Table 4 all temperatures are based on Kuiper's scale and the Henry Draper classification, since this classification underlies Kuiper's determinations.

For the relation between effective temperature T_e and surface temperature T_0, the approximation $T_e = 1.20\ T_0$, or $T_0 = \frac{5}{6} T_e$ was used; T_0 was rounded off to the nearest 50°.

In Table 4 the stars used are listed, together with the quantities essential in the computation of log g. Column 7 gives the computed number of observable Balmer lines N_c. Columns 8, 9, and 10 give the observed numbers N_{o1} on Yü's tracings, N_{o2} on the larger scale tracings, and the logarithm of the effective surface gravity, log g_e, which follows from the observed number of lines. Both g_c and g_e are expressed in solar units. In a few cases, observed line numbers were available from other sources, as indicated in the column "Remarks." Log g_e was always derived from the largest observed number indicated and was entered only if $N_c < N_o$.

As can be seen from the table, the agreement between observations and theory is not very satisfactory, especially if we realize that the N_{o1} are lower limits, owing to the small scale of the tracings. One might have expected an effect in a direction opposite to the one shown, viz., that the number observed would be smaller than the number predicted, because of limited resolving power of the spectrograph, etc. However, the number observed is, in general, larger than what follows from the computations. The case of β Orionis presents a striking discrepancy, and ζ Tauri, which is known to be a shell star, is an even stronger example, with $N_c = 14$, and $N_o = 32$, as given by Miss Losh.[24] In terms of Stark effect, this means that the field is weaker than would follow from the theory based on temperature and surface gravity. It does not seem probable that the temperatures are much lower than the ones assumed, and certainly not to the extent required to bring observations and theory into agreement. Some other possibilities will be discussed later.

8. The double stars will be treated briefly, since observations are available for only a few systems.

As is well known, it is possible to derive, for well-observed systems of eclipsing binaries, the relative dimensions of the components. If a mass-luminosity relation is assumed and if the spectrum of at least one of the components is known, it is possible to determine the absolute dimensions and the masses. Some modifications were made in the treatment of the extensive material compiled by Gaposchkin.[13] The temperature scale was changed slightly so as to agree with Kuiper's scale, used in the previous section. Furthermore, the mass-luminosity relation (13) was substituted for Gaposchkin's, and the ratio of the masses of two stars with known difference in brightness was computed from that relation. Table 5 gives the data for the five stars for which the lines could be counted. As before, the number of lines observed is, in almost every case, considerably larger than the number computed.

9. Dr. W. J. Luyten kindly put at the writer's disposal the spectra of two white dwarfs, taken at the McDonald Observatory. The stars are Co $-32°5613$ and o^2 Eri B;[25] both are A-type stars. The spectrum of o^2 Eri B shows the usual features of a white dwarf, namely, extremely broad hydrogen lines; the last visible member of the Balmer series is H 9. The lines in Co $- 32°5613$ are far less broadened, but here also the continuum begins between H 9 and H 10. Luyten's value for the absolute magnitude of this star, $M = 10.1$, and equation (13) yield a value for the mass of 0.3 solar masses. It is known, however, that the white dwarfs are less luminous than the main-sequence stars

[23] A. Arnulf, D. Barbier, and D. Chalonge, *J. d. Obs.*, **19**, 149, 1936.

[24] *Pub. Obs. U. Michigan* **4**, 1, 1932.

[25] A description of these stars is given by W. J. Luyten, *Ap. J.*, **102**, 382, 1945.

1949ApJ...109..452V

TABLE 4

Star (1)	Spectrum (2)	Authority* (3)	M_v (4)	θ_0 (5)	log g_c (6)	N_c (7)	N_{o1} (8)	N_{o2} (9)	log g_e (10)	Remarks (11)
o Per.....	B1 IV	M	−4.7	0.26	−0.5	14–15	15	....	−0.8	
ζ Per.....	B1 I	M	−5.5	.26	−1.0	15–16	16	....	....	
ε Per.....	B0.5 III	MKK	−4	.26	−0.6	14–15	14–15	....	....	
η Ori.....	B1 V	M	−3.8	.26	−0.5	14–15	14–15	....	−0.6	
β CMa...	B1 II–III	MKK	−4	.26	−0.5	14–15	15	....	−1.4	Marshal
β Cep....	B1 IV	MKK	−2.5	.26	−0.1	14	16	....	−1.7	Marshai
ν Eri.....	B2s	SHW	−3.0	.30	−0.4	15	15–16	....	−0.6	
γ Ori.....	B2 IV	MKK	−2.5	.30	−0.2	14	17–18	....	−1.2	
χ^2 Ori....	B2 I	M	−7.2	.30	−1.7	18	20	....	−2.0	
α Vir.....	B1 III–IV	MKK	−3	.30	−0.4	15	13–14	....	....	
ζ Cas....	B2.5 IV	MKK	−3	.33	−0.5	15	14–15	....	−1.2	Struve‡
ε Cas.....	B3 III	MKK	−3.5	.33	−0.7	16	15	....	....	
π^4 Ori....	B2s	SHW	−3.0	.33	−0.4	15	18	....	−1.6	
η Aur....	B3 V	MKK	−1.0	.33	+0.1	14	16	....	−0.8	
ζ Tau....	B3e	SHW	−2.2	.33	−0.3	14–15	18	....	<−3.0	Losh§
η UMa...	B3 V	MKK	−1.0	.33	+0.1	14	16–17	....	−0.8	
δ Per.....	B5 III	MKK	−0.5	.39	0.0	15	15	....	....	
17 Tau...	B5neα	SHW	−1.6	.39	−0.3	15–16	16–17	....	−0.7	
20 Tau...	B9s	SHW	−0.4	.39	+0.1	15	15–16	....	−0.2	
η Tau....	B5p	HD	−1.5	.39	−0.2	15–16	16–17	....	−0.7	
τ Ori.....	B8	SHW	−0.8	.39	0.0	15	16	....	−0.6	
γ CMa...	B5	HD	−0.5	.39	+0.1	15	15–16	....	−0.2	
κ Dra....	B5e	SHW	−1.6	.39	−0.3	15–16	16–17	....	−0.7	
67 Oph...	B5 I–II	M	−5.5	.39	−1.4	18	15?	....	−2.2	Struve‡
ζ Dra....	B5	HD	+0.3	.39	+0.1	15	19:	....	−2.0	Struve‡
τ Her....	B5 IV	MKK	−1.0	.39	−0.1	15	16–17	....	−0.7	Struve‡
ξ Tau....	B8nn	SHW	−0.8	.50	−0.3	16	13	16–17	−0.5	
β Ori.....	B8 Ia	MKK	−6.5	.50	−2.0	20	19–20	26	−3:	Marshal
β Tau....	B8 III	MKK	−2.0	.50	−0.7	17	15–16	....	....	
α Leo....	B8 V	MKK	0.0	.50	−0.1	16	14–15	16	....	
β Lib.....	B8 V	MKK	0.0	.50	−0.1	16	15	....	....	
α Del.....	B8	HD	+0.1	.50	0.0	16	13–14	....	....	
α And....	A0p	HD	+0.2	.57	−0.2	16–17	15–16	19–20	−1.4	
ν Tau....	A0	HD	+0.4	.57	−0.1	16–17	15–16	....	....	
μ Lep....	A0	HD	+0.7	.57	0.0	16	18	19–20	−1.4	
β Aur....	A2 IV	MKK	+0.5	.57	−0.3	17	17	....	....	
θ Aur....	A0p	MKK	−1?	.57	−0.6?	17–18	16	15–16	....	
γ Gem...	A1 V	MKK	+1	.57	0.0	16	18–19	....	−1.0	
α CMa...	A1 V	MKK	+1	.57	0.0	16	16–17	18–19	−1.0	
c Hya....	A0	HD	+0.6	.57	−0.1	16	14	....	....	
θ UMa...	A0	HD	+0.2	.57	−0.2	17	16–17	17	....	
κ Hya....	A0	HD	+0.2	.57	−0.2	17	16	15–16	....	
η Leo.....	A0p	HD	(+0.7)	.57	−0.1	16	18–19	....	−1.0	
β UMa...	A1 V	MKK	+1	.57	0.0	16	16	19	−1.4	
γ UMa...	A0 V	W	+0.5	.57	−0.1	16	18	....	−1.0	
ε UMa...	A0	HD	0.0	.57	−0.3	17	16–17	18–19	−1.0	
α CVn....	A0p	MKK	−1?	.57	−0.6	17	15–16	....	....	
α Dra....	A0p	HD	+0.5	.57	−0.1	16	15–16	19–20	−1.4	
α CrB....	A0 V	MKK	+0.5	.57	−0.1	16	15–16	18–19	−1.0	
μ Ser.....	A0	HD	+0.4	.57	−0.1	16	15	....	....	
λ Oph....	A0	HD	+1.1	.57	−0.1	16	15	....	....	
ε Her.....	A0	HD	+0.8	0.57	0.0	16	15	....	....	

* The authorities quoted in the third column for classification and absolute magnitudes are: M: W. W. Morgan, unpublishe MKK: Morgan, Keenan, and Kellman, *An Atlas of Stellar Spectra*. SHW: Stebbins, Huffer, and Whitford, *Ap. J.*, **91,** 20, 19 W: Miss M. E. Walther of the Harvard Observatory, who kindly made some tentative classifications on the MKK syste HD: *Henry Draper Catalogue* and Schlesinger's *Bright Star Catalogue*.

† *Pub. Obs. U. Michigan*, **5,** 137, 1934.

‡ *Ap. J.*, **91,** 365, 1940.

§ *Pub. Obs. U. Michigan*, **4,** 1, 1932.

TABLE 4—*Continued*

Star (1)	Spectrum (2)	Authority* (3)	M_v (4)	θ_0 (5)	log g_c (6)	N_c (7)	N_{o1} (8)	N_{o2} (9)	log g_e (10)	Remarks (11)
γ Oph....	A0	HD	0.0	0.57	−0.3	17	13	....	....	
Lyr....	A0 V	MKK	+0.5	.57	−0.1	16	16–17	18–19	−1.0	
γ Lyr....	B9 V	MKK	+0.5	.57	−0.1	16	17:	....	....	
Lac.....	A0 V	W	+0.5	.57	−0.1	16	14–15	17–18	−0.6	
Gem...	A2 I	W	−6.5	.62	−2.3	21	17	19	....	
Gem...	A2	HD	+1.4	.62	0.0	16–17	15	18	−0.8	
Leo....	A4s	MtW	+1.0	.62	−0.1	16–17	17–18	....	−0.6	
UMa...	A2 IV	MKK	+0.5	.62	−0.3	17	18	19	−1.4	
Leo....	A3 V	MKK	+2.0	.62	+0.2	16	16–17	18–19	−0.8	
UMa...	A3 V	MKK	+2.0	.62	+0.2	16	13	17–18	−0.6	
$_1$ UMa...	A2 V	MKK	+1.3	.62	−0.1	17	17	....	....	
Vir.....	A3 V	MKK	+2.0	.62	+0.2	16	16	....	....	
γ UMi...	A3 II–III	MKK	−1.0	.62	−0.8	18	14	18	....	
Ser.....	A2	MtW	+1.8	.62	+0.2	16	15	....	....	
Cyg....	A2 I	M	−7.2	.62	−2.6	22	18–19	19–20	<−3:	Wright‖
Eri.....	A3	HD	+1.5	.65	0.0	16–17	15	15–16	....	
Leo.....	A3	MtW	+2.1	.65	+0.1	16–17	16	19?	−1.2:	
Tri.....	A5 III	MKK	0.0	.71	−0.6	18:	15–16	19	−1.2:	
Aql.....	A7 V	MKK	+2.5	.71	+0.1	16–17:	15	16–17	......	
Cep....	A7 V	MKK	+2.5	0.71	+0.1	16–17:	15	18	−0.8:	

‖ *Lick Obs. Bull.*, **10**, 100, 1921.

›f the same mass. In Figure 1 of Kuiper's article on the empirical mass-luminosity rela-
.ion[26] and in Figure 1 of Russell and Moore's book[14] it seems indicated that the mass of
white dwarf corresponds to a bolometric absolute magnitude of about 3.5 mag. brighter
han that of the dwarf. Therefore, a mass of 0.75⊙ was adopted for this star. With a
emperature of 10,000°, formula (14) then yields log $g = 3.0$ in solar units, or 7.4 abso-
ıte. For o^2 Eri B, Kuiper[26] gives log $g = 7.6$.

In the course of the computations of H_0, the net stream at the stellar surface, it was
›und that a one-to-one correspondence exists between this quantity and the value of q^2
ǝq. [12]) at optical depth 0.05. It therefore seems reasonable to assume that a line will be
ǝparately visible if, for the F_0 prevailing at $\tau = 0.05$, the absorption coefficient of that
ne has not yet attained the value of the continuous band of absorption. It was found
ıat log $F_0 = 2$ is only about 0.20 higher than the values of log F_0 in these two dwarfs.
able 6 gives log α for a range of wave lengths for log $F_0 = 2$ and 1.5. The former cor-
sponds to log $g = 8$, the latter to log $g = 6.6$, which is probably far below the dynami-
.l value of log g in the stars considered. It is seen from the table that, for the higher
ılue of log g, H 10 should not be visible, whereas, at log $g = 6.6$, H 10 will not yet
ıve merged into the continuum. Hence the conclusion seems justified that, in the two
ıite dwarfs considered, the observations are in accordance with the theoretical expec-
tions.

10. In this section some effects will be investigated which may be the cause of the
screpancies found.

a) Collisional broadening, which was neglected altogether, would tend to decrease the
mputed numbers still further. It was found, however, that, on the basis of the theory
Weisskopf,[27] the collisional broadening is negligibly small in comparison with the
ırk broadening applied.

[26] *Ap. J.*, **95**, 472, 1938.

[27] *Phys. Zs.*, **34**, 1, 1933.

b) The discrepancy between observation and theory is in the sense that the theoretica field strength seems to be too large or that the Stark broadening has been exaggerated If the half-width of the Stark pattern is taken equal to the distance of the third-last com ponent from the center, then the factor $\{n(n-1)+2\}$ is changed into $\{n(n-3)+2\}$ This is equivalent to taking, for the field strength at which the lines begin to merge about 1.15 its present value; this adds 0.06 to log F_0. As can be seen from Figures 1 and 2 this will hardly ever increase N_c by one unit, and hence this correction will not remov the difficulty.

Inglis and Teller[28] have published an important investigation on the depressioı of series limits. They conclude that if n is the upper quantum number of the las line separately visible, the electrons do not contribute to the field at temperature $T > 10^5/n$. Unsöld[29] reached a similar conclusion, for $T > (4.6 \times 10^5)/n$. If we adop the first limit, this would mean that the value of P used is too large by a factor of 2. Cor

TABLE 5

DOUBLE STARS

Star	Spectrum	N_c	N_{o2}	Star	Spectrum	N_c	N_{o2}
AR Aur........	B9	17–18	19	β Lyr.........	B5–B8p	19	18
YZ Cas.........	A3	18–19	21–22	β Per.........	B8	16–17	18–19
α CrB..........	A0	16–17	18–19				

TABLE 6

LOG a FOR LOG F_0 = 2 AND 1.50

λ	IDENTIFICATION	LOG a	
		log F_0=2	log F_0=1.50
3970.08...........	$H7$	0.44	....
60..............	....	.36	....
50..............	....	.14	....
40..............	....	.89−1	....
30..............	....	.75−1	....
20..............	....	.80−1	....
10..............	....	.96−1	....
00..............	....	.12	....
3889.05...........	$H8$	.19	0.64
80..............	....	.17	.20
71..............	....	.10	.52−1
62..............	....	.045	.30−1
53..............	....	.03	.56−1
44..............	....	.06	.08
35.39...........	$H9$	.09	.40
26..............	....	.09	.12
17..............	....	.09	.79−1
08..............	....	.09	.98−1
3797.90...........	$H10$	0.08	0.19

[28] *Ap. J.*, **90,** 439, 1939.

[29] *Vierteljahrsschrift der Astronomischen Gesellschaft,* Vol. **78,** No. 4, 1943.

recting log P by 0.30 comes to the same as correcting log g by 0.60, since P is very nearly proportional to $g^{1/2}$. This reduction would remove many, though not all, discrepancies observed, as can be seen from Table 4.

c) Another way of interpreting the effect found is that the electrical field actually present is weaker because the electron pressure is lower than the one assumed. On the assumption that the ionization corresponds to the case of thermodynamical equilibrium, this would mean that the mean density of the absorbing material is lower, or that the pressure gradient is smaller, than what would follow from the dynamical gravitation GM/R^2. A reduction by a factor as large as 50 seems to be indicated by the values of log g_e in Table 4, and for ζ Draconis and some others even by more than 100. Similar effects were found by Pannekoek and Reesinck[30] and by Schalén.[31] The latter, using Verweij's computations, finds for α Cygni a surface gravity of log $g = 1.2$, whereas the dynamical value is 2.5. Pannekoek and Reesinck find log $g = 0.4$, -0.1, and -0.2 for the cepheids δ Cephei, α Ursae Minoris, and ζ Geminorum; this would lead to masses for these stars smaller than the sun's mass. Pannekoek then concludes that a strong radiation pressure must be present, which might eventually surpass the gravitational forces and result in an outward flow of gas with very small pressure gradient.

In this connection the work of Elvey and Struve[32] and of Shajn[33] should be mentioned. Elvey and Struve conclude from the shape of the curve of growth in three supergiants that considerable turbulence must be present. Shajn finds a confirmation of outward streaming gases in supergiants. From a solution for galactic rotation of supergiants he finds a K-term of -5 km/sec, which he interprets as arising from an outward motion of the absorbing gases.

Finally, we may note here the increasing number of stars which have been found to possess extended atmospheres, and we recall the unsolved problem of the support of the solar corona.

In many investigations it has been assumed that the effect of decreased effective gravity can be attributed to the action of radiation pressure. A critical study shows, however, that, in general, the effect of radiation pressure is quite negligible. At first sight, one would be inclined to compute the net outward flux H, integrated over all frequencies, multiply it by the mean coefficient of mass absorption $\bar{k}$, and put the impulse communicated to the absorbing mass equal to $H\bar{k}/c$. This is essentially Eddington's procedure.[34] However, in doing so, one overlooks the important fact that the spectral region through which most of the outward flux passes is quite different from the spectral region of highest opacity. Hence the radiation passes mostly unobstructed, and no radiation pressure ensues. One must therefore consider both the absorption coefficient and the flux of radiation in their dependence on frequency.

The following treatment is due to Dr. Menzel, to whom the writer is indebted for permission to quote it here. Let

N_n = Number of H atoms per cubic centimeter in the nth quantum state;

N_i = Number of H ions per cubic centimeter ;

$N = \Sigma N_n$;

$a_{\nu n}$ = Atomic coefficient of continuous absorption for frequency ν of an atom in the nth quantum state ;

[30] *B.A.N.*, **3**, 47, 1925; **8**, 175, 1937.

[31] *Upsala U. Arsskrift*, No. 8, 1936.

[32] *Ap. J.*, **79**, 409, 1934.

[33] *Bull. Abastumani Obs.*, **7**, 83, 1943.

[34] *The Internal Constitution of the Stars* (Cambridge: At the University Press, 1926), §§ 233, 247, 248.

$a_{\nu n} \neq 0$ only to the violet of the nth series limit ;

$\nu_n =$ Frequency of the nth series limit ;

$a_\nu =$ Absorption coefficient per cubic centimeter for frequency ν ;

$F_\nu =$ Flux of radiation at frequency ν .

For F_ν the following expression is assumed:

$$F_\nu = \frac{2\pi h\nu^3}{c^2} \frac{1}{e^{h\nu/\kappa T} - 1} W ,$$

in which W is the dilution factor. The following relations exist:

$$\frac{N_n}{N_1} = \frac{g_n}{g_1} e^{-h(\nu_1 - \nu_n)/\kappa T} = n^2 e^{-h(\nu_1 - \nu_n)/\kappa T} ,$$

$$a_{\nu n} = \frac{\pi e^2}{m c} \cdot \frac{2^4}{3\pi \sqrt{3}} \frac{R^3}{\nu^3} \cdot \frac{1}{R n^5} = \frac{1}{n^5} a_{\nu 1} = \text{Const.} \frac{1}{n^5} \frac{1}{\nu^3} ,$$

$$a_\nu = N_1 a_{\nu 1} + N_2 a_{\nu 2} + \ldots .$$

The impulse on the material in 1 cm^3 resulting from absorption of radiation in the frequency interval ν to $\nu + d\nu$ is

$$\frac{F_\nu a_\nu d_\nu}{c} = \frac{a_{\nu 1}}{c} N_1 e^{-h\nu_1/\kappa T} \left(e^{h\nu_1/\kappa T} + \frac{1}{2^3} e^{h\nu_2/\kappa T} + \frac{1}{3^3} e^{h\nu_3/\kappa T} + \ldots \right) \times W \frac{2\pi h}{c^2} \frac{1}{e^{h\nu/\kappa T} - 1} d\nu .$$

Integrating this over all frequencies, we obtain

$$\text{Const. } N_1 e^{-h\nu_1/\kappa T} \left[\int_{\nu_1}^{\infty} \frac{e^{h\nu_1/\kappa T}}{e^{h\nu/\kappa T} - 1} d\nu + \frac{1}{2^3} \int_{\nu_2}^{\infty} \frac{e^{h\nu_2/\kappa T}}{e^{h\nu/\kappa T} - 1} d\nu + \ldots \right]$$

$$\sim \text{const. } N_1 e^{-h\nu_1/\kappa T} T \left(1 + \frac{1}{2^3} + \frac{1}{3^3} \ldots \right) ,$$

in which the constant is equal to 9.12×10^{-18}. Attention may be called here to the fact that the contributions from the continua other than the Lyman continuum are, in fact, negligible and independent of temperature. With increasing temperature the population of the higher levels increases, but the frequency of maximum intensity moves to the violet, and these two factors just balance. So much for the impulse arising from the absorption of radiation.

The impulse communicated per second to the hydrogen atoms and ions in 1 cm^3 by the force of gravitation is equal to $(N + N_i) m_H g$, in which m_H is the mass of one hydrogen atom.

From the two expressions given and from the condition that the layer be semitransparent, Menzel computes the ratio of radiation pressure to gas pressure. The result found is that, for temperatures of 10,000°–15,000° and $\log g = 4$, this ratio is 0.001 to 0.01 and hence entirely negligible. Even in a completely opaque atmosphere the number of atoms that can be supported by radiation pressure is of the order of $10^{-10} \times$ the number present.

It must therefore be concluded that radiation pressure is not the explanation for the discrepancy found.

d) If a Milne-Eddington model of the atmosphere is assumed, in which s/k is constant through the atmosphere, then the line depth R (1 − residual intensity) in a line profile is given by

$$R = \frac{1}{2}\frac{s}{k} - \frac{3}{8}\left(\frac{s}{k}\right)^2 \ldots \qquad \text{for} \quad \frac{s}{k} \leqslant 1 \,,$$

$$R = 1 - \left(\frac{k}{s}\right)^{1/2} + \frac{1}{2}\left(\frac{k}{s}\right)^{3/2} \ldots \qquad \text{for} \quad \frac{s}{k} \geqslant 1 \,,$$

according to Pannekoek and van Albada.[35] It follows that the contrast between the line centers ($s/k > 1$) and line wings ($s/k < 1$) will be increased if s/k decreases. Therefore, the discrepancy found may have been caused by the use of too small a value for k. Moreover, the larger k is, the smaller is the pressure that corresponds to a certain value of the optical depth, and this again tends to make the line spectrum extend farther into the violet. The computations discussed here were carried through at a time when the war had cut off the writer nearly completely from contact with important developments abroad. For this reason the absorption by negative hydrogen has not been considered in the computations. It is a fortunate circumstance, however, that this source of absorption appears to be negligible at the temperatures considered originally. It may come in, however, at the lower temperatures for which line numbers were extrapolated. The discrepancies are by no means limited to these lower temperatures, and, therefore, the cause must be sought elsewhere.

Two effects that may influence the continuous absorption will be discussed here: electron scattering and admixture of other elements besides hydrogen. The scattering per electron is equal to

$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m c^2}\right)^2 .$$

If the atmosphere consists of hydrogen only, of which a fraction x is ionized, then there will be x electrons for every hydrogen atom. It follows that the electron scattering, σ, per gram of hydrogen will be equal to

$$\sigma = \frac{x}{m_H} \times \frac{8\pi}{3}\left(\frac{e^2}{m c^2}\right)^2 = 0.42x \,.$$

On the other hand, the mass-absorption coefficient per gram of hydrogen is given by $k = \kappa x P_e$. It follows that

$$\frac{\sigma}{k} = \frac{0.42}{\kappa P_e} . \tag{15}$$

Electron scattering is predominant over atomic continuous absorption if $\sigma/k \geqslant 1$; this will be the case for $\log \kappa P_e \leqslant -0.38$. If for κ the values given by Pannekoek[10] are used, the resulting values of $\log P_e$ at which σ becomes larger than k are found to be about 0.2 smaller than those given by Greenstein.[36] Since the details of neither computation are available, it is not clear what causes the difference. The order of magnitude, however, is the same in both cases.

In the computations of P_e as a function of optical depth the electron scattering was neglected. It was found there that $\log \kappa P_e$ fulfils condition (15) only for $\log g = 2$ and only for the outer layers, as deep down as $\tau = 0.10$. The effect of an additional factor 3 in k is found to come approximately to the same as reducing $\log g$ by 0.8 at most. Although this correction is appreciable, it does not explain all the discrepancies in the

[35] *Pub. Astr. Inst. U. Amsterdam*, No. 6, Part 2, 1946.

[36] *Ap. J.*, **95**, 299, 1942.

supergiants. Moreover, corrections for electron scattering are negligible in the main sequence stars. Hence electron scattering does not solve our problem either.

The influence of the admixture of other elements besides hydrogen has been treated by Greenstein.[36] He finds that the mean opacity per gram is much smaller for *He* than for *H* at almost all temperatures; only at $\theta = 0.3$ are they comparable. Hence the influence of *He* is very small, unless it greatly exceeds *H* by mass. For the elements *C*, *N*, *O*, and *Si*, Greenstein finds negligible contributions to the general opacity.

11. The general conclusion of this discussion must therefore be that the discrepancy found to exist between the theory, as developed here, and observation cannot be explained satisfactorily. Observational evidence deviates from the theoretical expectations in the sense that the stars considered seem to have effective gravities as much as a hundred times smaller than the dynamical surface gravities GM/R^2. The results of the present investigation seem to indicate that the theory which assumes the stellar atmospheres to be in a state of equilibrium, in which the force of gravitation is balanced by the gradients of gas and radiation pressures, is inadequate. Extensive study of shell stars together with fundamental theoretical investigations, may lead to a correct interpretation of the observed phenomena.

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