ERRORS OF TIME DETERMINATION

L. S. T. Symms and C. Joy A. Penny

(Communicated by The Astronomer Royal and The Astronomer Royal for Scotland)

(Received 1946 May 25)

Summary

The investigation refers to clock errors determined with a (non-reversible) transit circle of standard type, the collimation errors being measured against collimators and the level errors obtained by nadir observations. It is shown that there is a strong correlation between the clock errors and the measured collimation errors indicating that a good deal of the day-to-day scatter in clock error is due to accidental error in the measurement of collimation. Apart from this a good deal of the personal equation in clock error for an observer, that is, the (slowly changing) systematic error in his determinations of time, is found to arise from personal equation in his measures of collimation. The effect of errors of nadir setting in the measurement of level error is also discussed. For the instrument investigated (the Edinburgh Transit Circle) it is found that the probable error of a single night's determination of time (using the mean of ten stars) is ± 32 milliseconds when no corrections for personal equation are applied. By applying personal equations this figure is reduced to about ± 20 milliseconds, and of this that part of the probable error not due to errors in measuring collimation and level is ± 10 milliseconds or less. This figure applies to a hand-driven micrometer and includes contributions arising from erroneous star places.

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1. For the last five years observations of clock error have been made with the Transit Circle at the Royal Observatory, Edinburgh to provide, in combination with those obtained at the Abinger station of the Royal Observatory, Greenwich, the basis of the Greenwich Time Service.

The introduction of quartz clocks has, among other things, made it easier to derive the probable error of a single night's determination of clock error. The erratics of rate of quartz clocks are very small, the mean absolute scatter of daily rate being of the order of $\pm 0{\cdot}05$ milliseconds per day, and this means that over a considerable period of time the plot of clock error against date is of the form $E = a + bt + ct^2$ where E is the clock error and t is the time in days reckoned from some convenient zero date. (The term ct^2 arises from the frequency drift of the quartz oscillator—the frequency changes at a steady rate after the initial "ageing" has been completed.) Accordingly, given a set of measured clock errors extending over a period of months, a quadratic form can be fitted to them by a least square adjustment, and the probable error of a single night's determination of time then can be deduced from the residuals in the ordinary way. But previous to the introduction of quartz clocks the pendulum clocks in use were subject to erratics of relatively large amount—so large that it was necessary to credit the clock with a natural tendency to wander and so to fit a curve, often of a sinusoidal form, to the transit observations. This was mathematically equivalent to introducing a larger number of unknowns into the equations of condition used to determine the clock error as a function of date and the uncertainty in the p.e. of a night's determination of clock error was thus enhanced.

By referring the transit observations to quartz clocks it appears that the p.e. of a time determination (10 stars) at Edinburgh is of the order of ± 30 milliseconds. But the internal agreement of the transits of the individual stars used for a single night's determination indicates a probable error much smaller than this, and it has long been suspected that the measurement of instrumental corrections—collimation and level errors—contributes very heavily to the total p.e. The present paper is an attempt to determine these contributions on a numerical basis. The investigation arose from a remark by Professor Greaves that there appeared to be some evidence that a systematic difference in the measures of collimation error by the two observers (LS and JP) remaining at Edinburgh was responsible for an apparent large difference of personal equation between the clock errors obtained by them. This suspicion was confirmed, as will be seen, but the centre of interest shifted as the investigation proceeded; and it became apparent that not only were differences in personal equation in the measurement of collimation largely responsible for correspondingly large systematic differences in clock error between the various observers—as was only to be expected—but that there was a striking correlation between the residuals of the collimation measures from the means for the various observers and the corresponding residuals in clock error. In other words, quite apart from the systematic differences to which the term "personal equation" is usually applied, the day-to-day scatter in measurements of clock error is largely due to the day-to-day scatter in measurements of collimation with this instrument, i.e. the probable error of the clock errors, is largely due to the probable error of the measures of collimation and the probable error of measurement of the actual transits is relatively small. This is a striking result which, as far as we know, has not been demonstrated before on a numerical basis.

2. The transit circle is of standard type and incorporates a hand-driven impersonal micrometer. Its aperture is 8·5 inches and its focal length is 8 feet 11 inches. It is a reversible instrument, but over the period discussed it was not reversed. The collimation errors are determined against two collimators, and the level error by nadir settings with a Bohnenberger eyepiece in the normal way. A defect in the instrument is the employment of collimators with apertures less than the aperture of the transit instrument itself; the collimator apertures are 6·1 inches, their focal length being 6 feet 3 inches. The collimators are mounted in the same observing house as the main instrument; their mountings are satisfactorily stable and the seeing conditions for the collimator operations are very good. Care is taken to achieve steady and uniform illumination of the collimator wires. A single wire is mounted in the focal plane of each collimator and the north collimator is set on the south prior to setting the instrument on each. Over the period discussed a reversing prism was *not* employed in any of the measurements. The transit instrument is carefully focussed for the R.A.s, the travelling wire being in the focal plane of the objective.

No attempt was made to determine time fundamentally, i.e. to determine star places, and to relate them to the Sun. For each night's work some ten stars were observed and the transit times, after correction for collimation and level, were subtracted from the FK3 Right Ascensions as published in *Apparent*

Places of Fundamental Stars. The resulting differences were plotted against the tangent of the declination and a straight line fitted to the plotted points. The clock error was then deduced from the ordinate of the point of this line corresponding to abscissa $\tan\phi$, where ϕ is the latitude ($=55^\circ\ 55'\cdot5$ for Edinburgh). The azimuth error corresponds to the slope of the line, but the stars were chosen by the observer so that the mean $\tan\delta$ approximates closely to $\tan\phi (=1\cdot48)$. This ensures that the resulting clock error is practically independent of the azimuth error of the instrument. It is stressed that for each night's work the transits were reduced using the collimation and level errors as measured by the observer on duty on the night concerned. Corrections for adopted personal equations for two of the observers were applied to the clock errors: $+18$ milliseconds for LS and $+3$ milliseconds for B.

The transits were observed against a sidereal free pendulum clock (Shortt No. 4) and this clock was compared with the quartz clock A1 (a mean time clock) before, during and after each period of observation so that the result of each night's work was referred at once to the quartz clock and gave the error of the latter. Further, since the Rugby G.B.R. radio time-signals at 10^h U.T. were recorded against A1, the transits can be transferred to any other clock, situated at any station, against which the 10^h G.B.R. radio time-signals were also recorded, since daily comparisons between such a clock and A1 are given by the radio link. In the present investigation the transits have been used, in this way, to give the clock errors of the quartz clock IVC at the Post Office Radio Research Laboratories, Dollis Hill. The Dollis Hill clocks were more suitable than the Observatory quartz clock since, being battery controlled, they are not subject to dislocations due to mains' surges (the clock A1 at Edinburgh has been mains operated). Of the three clocks available at Dollis Hill, IVC was selected as being the clock which had put up the best performance over the period considered. Accordingly, all transits have been transferred so as to give, for each occasion, a clock error of IVC (a *positive* error denoting clock *slow*) and these clock errors have, in effect, been plotted against date. But it is desirable that over a lengthy period the plotted points should lie approximately on a straight line when a chart with a large scale of ordinates is used and that the first differences of the plotted points should be reasonably small. Accordingly, a quadratic expression in t (the term in t^2 representing the frequency drift of the oscillator) is removed from the figures before plotting, where t is the time in days from a chosen epoch. The resulting plot for IVC is shown in Fig. 1 (Plate 11). Each observer's transit points are joined by straight lines, not as an indication of the performance of the clock, but merely to assist the eye in separating the observations of the different observers. Other features of the figure will be described later.

3. If on any particular occasion the measured collimation is in excess of its true value by an amount c then (since in determining the level error the reading for the nadir setting is combined with the measured reading for the line of no collimation) the level error will also exceed its true value by c (in addition to the effect of an error in the nadir reading) and the final effect of the error c on clock error is that the determined clock error is in excess of its correct value by $-2c\sec\phi$, in addition to the mean error of observation of the transits themselves and the errors originating from an erroneous nadir setting. Accordingly, the clock errors will be correlated with the collimation errors owing to the common errors c and the correlation coefficient will indicate the extent to which erroneous

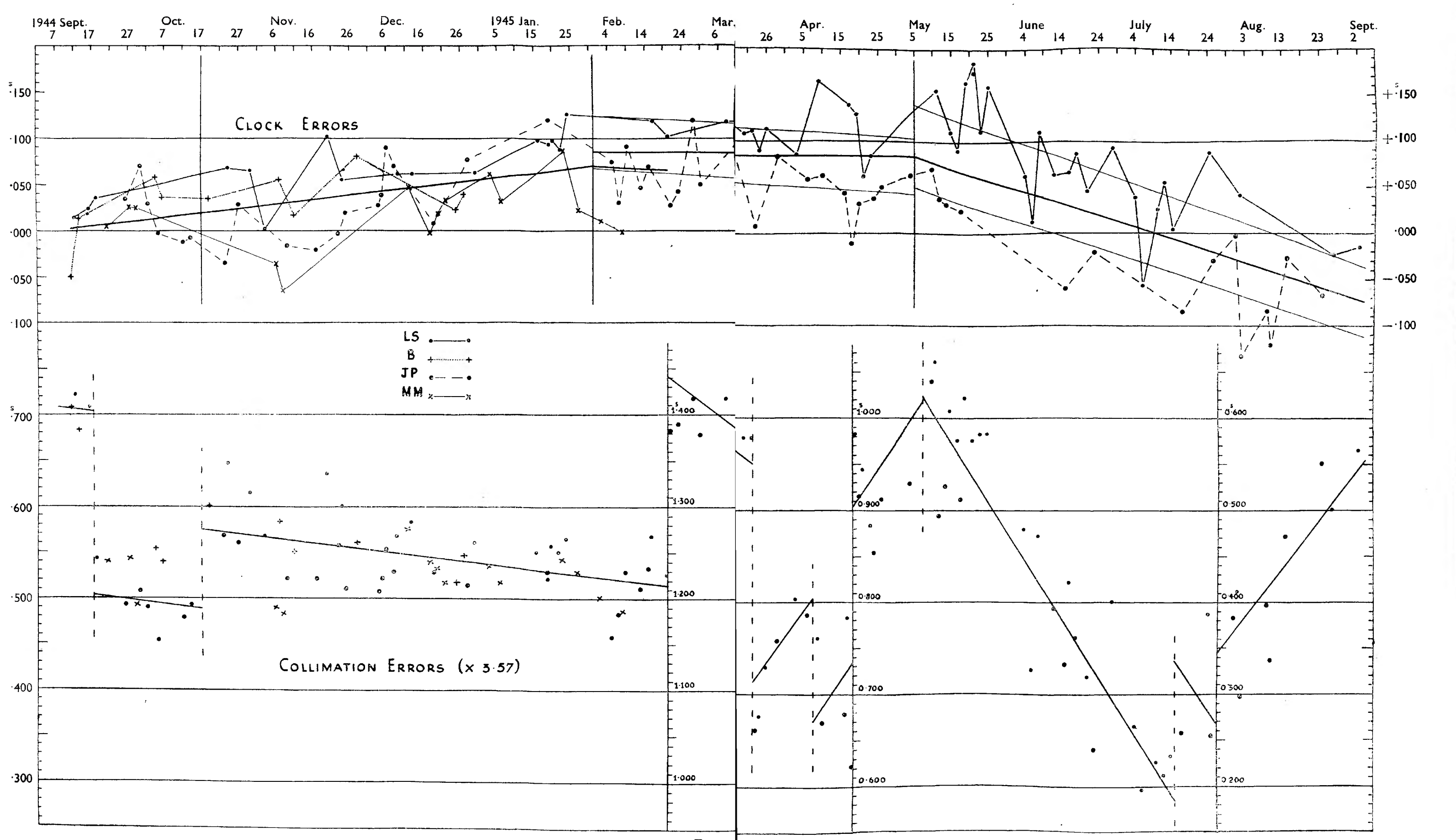

Fig. 1.

L. S. T. Symms and C. J. A. Penny, Errors of Time Determination.

measures of collimation contribute to the total probable error of time determination. As regards the effect of erroneous nadir settings on the clock errors if l be the excess of the measured level error over its true value owing to this cause, the resulting excess in clock error will be $-l \sec \phi$; thus there should be correlation between the clock errors and the nadir readings and the correlation coefficient should indicate the extent to which the errors l contribute to the total probable error of the clock errors.

The following general remarks are made at this point:—

(*a*) We shall be concerned with correlations between quantities which are themselves subject to secular change. Clock errors can clearly change progressively with date owing to clock rate and it is found that the collimation errors and nadir readings of the Edinburgh Transit Circle are also subject to secular drift. Now we are concerned with the calculations of correlation coefficients between the *accidental* parts of the quantities involved. Clearly then we must analyse, not the quantities themselves but their residuals from datum lines which take up the secular drifts. Thus instead of clock errors we must consider the residuals from a datum line drawn on the plot of clock error against date. Similarly for the collimation errors and nadir readings.

(*b*) We have already emphasized that we are investigating the correlation between the day-to-day variations of the clock errors and the instrumental errors. This means that we cannot use the same datum lines for all observers. Each observer is subject to personal equations in his transit measures and in his measures of the instrumental errors, and we are concerned with the accidental scatter of the measures about values representing the personal equations. Further, we have noticed that the personal equations are subject to slow secular changes. Accordingly, since we are dealing with the correlations between the accidental departures from the slowly changing values corresponding to the personal equations, we must aim at taking up the personal equations and their secular changes in the datum lines. It thus becomes necessary to adopt a different set of datum lines for each observer, and we shall be concerned with the evaluation of correlation coefficients between the residuals of the individual measures from these lines. The methods used for obtaining the datum lines will be described below; they incidentally give information regarding personal equations, but we shall not be primarily concerned with these.

4. Fig. 1 (Plate 11) shows that over the whole period considered there were two small changes of rate of IVC, towards the end of January and towards the beginning of May. The transit observations are not accurate enough to define the dates where these changes occurred, but fortunately other data were available. The three Group IV clocks at Dollis Hill are compared in pairs, the oscillations of the crystals being made to produce beats by virtue of the slight departures of the actual frequencies from the nominal value of 100 kilocycles per second, and the beats are counted by electrically actuated counters. The daily totals of the counters give the relative errors of the three clocks in units of one-hundredth of a millisecond, and these totals have been supplied by the Post Office Radio Branch. From them it appears that the changes of rate took place around January 31 and May 5 but were apparently spread over a few days. These two dates have been adopted as the dates where the changes of rate occurred, and the changes of rate were evaluated from the clock errors of the observers LS and JP, means of the changes of rate given by these two observers being adopted.

The clock errors over the short period January 31 to February 20 were then corrected for the effect of the change of rate on January 31, and over the period October 19 to February 20 least square solutions of the form $y = a + bt$ were fitted to the clock errors for each observer so corrected. As will be seen in Section 5, it was necessary to restrict the investigation to the period October 19 to February 20 because of the large discontinuities in the collimation error. These least square solutions provided the datum lines (shown in Fig. 2) from which clock error residuals were reckoned.

5. The collimation errors appear in the transit reduction forms in equatorial seconds of time. They have been multiplied by $-2 \sec \phi = -3{\cdot}57$ and are plotted in Fig. 1 (Plate 11) beneath the clock errors. At this stage an examination was made to detect any correlation between the two graphs. Throughout the whole period there are some small compact groups of points, usually by the same observer, for which the day-to-day variations of the individual collimation errors are rather strikingly reproduced in the clock errors. There is also evidence that the systematic differences of the measures of collimation by different observers is also reproduced in the clock errors. For the last two-thirds of the whole period this relationship of the personal equations is less obvious, and on close examination it was realized that this is due to sudden discontinuities in the collimation error. On the assumption that the collimation error of the telescope suffered some nine abrupt changes and varied linearly between the changes, preliminary datum lines were drawn from which to measure residuals. These are shown as full lines in Fig. 1 (Plate 11). It still remained to adopt different datum lines for each observer.

It was decided to confine the arithmetical analysis to the period October 19 to Feburary 20, for which the collimation error did not appear to have undergone any abrupt changes. For each observer a datum line was determined by a least square fit of the collimation errors (multiplied by $-2 \sec \phi$) to an expression $a + bt$, where t is the time (in units of 100 days) from a chosen epoch. These datum lines are shown in the lower part of Fig. 2. The residuals of the individual collimation errors from these lines were formed and were available for correlation analysis against the corresponding clock error residuals.

6. In passing we may note that the values of a and b, obtained in the solutions for the clock error and collimation datum lines, give information about the (slowly changing) personal equations. They are collected in the following tables; the values given are referred to the mean of the four observers; consequently the separate b's do not coincide with the slopes shown in Fig. 2. The epoch, $t = 0$, corresponds to December 20, i. e. to the middle of the period.

Personal Equations. (Values of a.)

Observer	No. of obs.	Clock Error	Collimation
LS	15	+37 msecs.	+38 msecs.
B	6	− 3	− 3
JP	18	− 1	−15
MM	12	−32	−20

These give a correlation coefficient of $0{\cdot}92 \pm {\cdot}10$. The clock error residuals include the provisional personal equations of $+18$ for LS and $+3$ for B, previously mentioned. If these are removed and the uncorrected clock errors are used, the correlation is weakened to $0{\cdot}84 \pm {\cdot}19$.

The slopes of the lines, given by the values of b, with the mean removed, are:

Observer	Clock Error	Collimation
	(msecs. per 100 days)	
LS	+ 9	−24
B	−61	−44
JP	+26	+ 9
MM	+26	+59

There appears to be correlation between these which, expressed in the form of a correlation coefficient, gives $0{\cdot}76 \pm {\cdot}25$, indicating that even the different

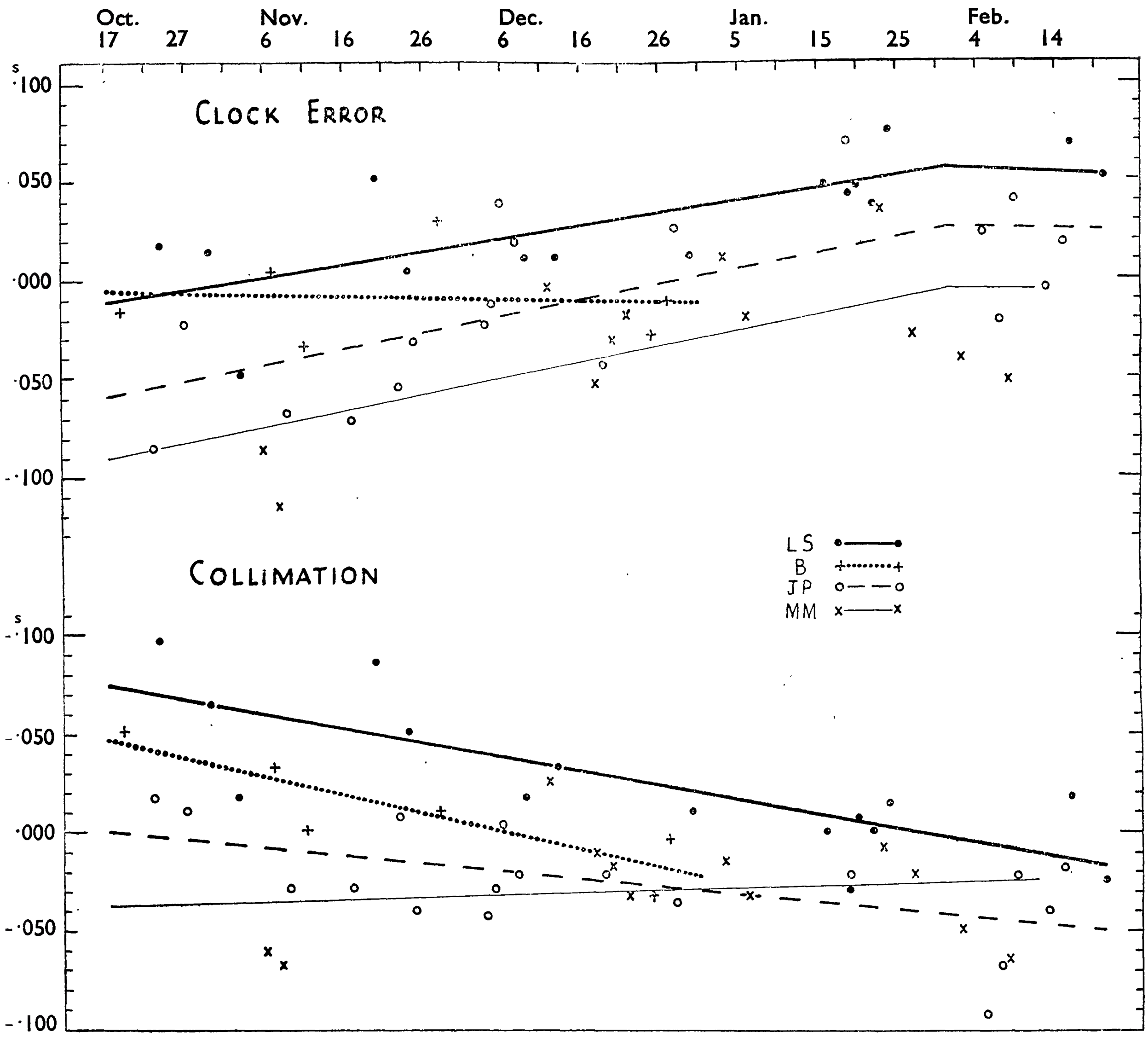

FIG. 2.—*Datum lines.*

slopes in the collimation datum lines are reflected in the clock error residuals, as they should be, when the collimation measures are erroneous.

It should be pointed out that if the real personal equations in clock error are required the effect of these differing ordinates and slopes in the collimation residuals must be eliminated.

7. Returning to our main purpose, we now have two sets of residuals, viz. clock error residuals and collimation residuals, and the coefficient of correlation between them will indicate the extent to which the day-to-day accidental errors of measurement of collimation contribute to the total scatter in the observed clock errors.

Before giving the numerical results we shall investigate the significance of this correlation coefficient in some detail.

Let c = error of measurement of collimation error, a positive value of c meaning that the collimation error is measured too large.

Let C = real excess of collimation error over its mean value.

Let l = error of measurement of level error due to error of measurement of the nadir setting.

Let L = real excess of level error over its mean value.

These quantities are supposed to be expressed in equatorial seconds of time. The mean values referred to are, in each case, functions of some known parameter or parameters, such as the date and temperature.

In obtaining the level error the nadir is combined with the determination of the line of collimation and so the total error of measurement $= l+c$.

The total excess of the *measured* collimation error over its mean value is then

$$\chi = C + c,$$

and the total excess of the *measured* level error over its mean value is

$$\lambda = L + l + c.$$

The real clock error will be a function of date. The clock error actually obtained on any occasion will differ from the real clock error as a result of (i) the error of measurement of the transits themselves, (ii) the error of measurement of collimation and (iii) the error of measurement of level.

As regards (i), let e = that part of the excess of the clock error over its true value due to errors of measurements of the transits.

As regards (ii), the erroneous collimation c produces an erroneous excess $-c \sec \phi$ for the clock error.

As regards (iii), the erroneous level error, $l+c$, produces an erroneous excess $-(l+c) \sec \phi$ for the clock error.

Hence the total excess of the measured clock error over its true value is

$$\epsilon = e - (2c + l) \sec \phi.$$

Consider now a sample of n observations of clock, collimation and level errors. To begin with, let us consider the case in which n is very large and the errors ϵ, χ and λ can be identified with the clock error, collimation and level residuals.

We shall, as usual, denote the standard deviation (S.D.) of any error x by σ_x, so that

$$\sigma_x^2 = \frac{1}{n} \Sigma\, x^2.$$

We shall also denote by σ_{xy} the mean product (M.P.) of any two correlated errors x, y, so that

$$\sigma_{xy} = \frac{1}{n} \Sigma\, xy.$$

Let r_1 be the coefficient of correlation between the clock error residuals, ϵ, and the collimation residuals, $-2\chi \sec \phi$.

Hence by the definition of correlation coefficient,

$$r_1 = \frac{-\sigma_{\epsilon\chi}}{\sigma_\epsilon \sigma_\chi},$$

but

$$\sigma_{\epsilon\chi} = \frac{1}{n}\Sigma[e-(2c+l)\sec\phi][C+c]$$

$$= -2\frac{1}{n}\sec\phi \, . \, \Sigma \, c^2,$$

the other sums of products vanishing, assuming that there is no correlation between their components; i. e.

$$\sigma_{\epsilon\chi} = -2\sigma_c^2 \sec\phi, \tag{1}$$

so that

$$r_1 = \frac{2\sigma_c^2 \sec\phi}{\sigma_\epsilon \sigma_\chi},$$

where

$$\sigma_\epsilon^2 = \sigma_e^2 + (4\sigma_c^2 + \sigma_l^2)\sec^2\phi, \tag{2}$$

and

$$\sigma_\chi^2 = \sigma_C^2 + \sigma_c^2. \tag{3}$$

These equations show that a large value of r_1 (i. e. near to unity) will be obtained if σ_c is predominant over the other standard deviations σ_C, σ_e, σ_l.

Up to now we have, for simplicity, considered the number in the sample to be large. We can now drop this restriction provided that, in forming the standard deviations and mean products, we divide the sums by $n-m$ instead of n, where m is the number of unknowns that have been (similarly) introduced into each of the sets of data to define the datum lines. If we treat the residuals for all observers together we shall have $n=51$ and $m=8$ since, for each observer, a linear datum line involving two unknowns was taken out. For the results of an individual observer $m=2$. We ignore the circumstances that the datum lines for the nadir readings (considered later) contain a further unknown, the temperature coefficient, which was determined from all observations over the period considered; and that the datum lines in the clock errors involve a term in t^2, the coefficient of which was determined from a much more extended set of data. A rigorous attempt to take these into account would be very difficult, but the correct results should be intermediate between those obtained by taking $m=8$ and $m=9$, and to the order of accuracy with which we are concerned the error introduced is immaterial. We therefore compute the quantities using $m=8$.

The numerical results for the correlation between the clock error residuals, ϵ, and the collimation residuals, $-2\chi\sec\phi$, for the separate observers and for the mean of all four observers are:

Observer	No. of obs.	r_1 and probable error of r_1
LS	15	$0{\cdot}82 \pm 0{\cdot}07$
B	6	$0{\cdot}59 \pm 0{\cdot}27$
JP	18	$0{\cdot}29 \pm 0{\cdot}16$
MM	12	$0{\cdot}80 \pm 0{\cdot}09$
All Observers	51	$0{\cdot}59 \pm 0{\cdot}07$

The probable errors of the correlation coefficients have been calculated from the approximate formula,

$$\text{P.E. of } r = 0{\cdot}6745 \frac{1-r^2}{\sqrt{n-m}}\left[1 + \frac{11r^2}{4(n-m)}\right],$$

where n is the number of pairs of residuals and m is the number of unknowns introduced into the determinations of each of the datum lines from which the residuals are calculated.*

It is difficult to say dogmatically that the values of r_1 differ for the different observers; but some confirmation is forthcoming, for the two observers LS and JP, from a discussion of the period May 10 to July 14, over which the collimation does not seem to have suffered any abrupt change. During this period LS secured observations on 19 nights and JP on 6, and the values obtained for r_1 over this period are:

Observer	r_1 and probable error of r_1
LS	0.75 ± 0.08
JP	0.38 ± 0.32

a result which, for what it is worth, is in agreement with the values for those observers obtained above.

The difference between these two extremes may be partly due to sampling error.† It is not explained by any large difference in the scatter of the correlated quantities for, as will be seen from a table of the collected results given later in Section 11, the probable errors of these quantities are very similar for these two observers. It may be due to an unsuspected correlation between the two components of a mean product which we have assumed to vanish in the expression for r_1. For instance, a bisection error ‡, variable from night to night, in all settings comprising the collimation and level measures (including the preliminary setting of the north collimator on to the south) would introduce a positive correlation between the collimation and level error residuals, and terms containing both c and l would not vanish. The expression for r_1 with these terms retained is:

$$r_1 = \frac{(2\sigma_c^2 + \sigma_{cl}) \sec\phi}{\sqrt{[\sigma_e^2 + (4\sigma_c^2 + \sigma_l^2)\sec^2\phi \pm 4\sigma_{cl}\sec^2\phi]}\sqrt{[\sigma_C^2 + \sigma_c^2]}}.$$

However, the strong correlation between the clock error residuals and the collimation residuals, for most observers, indicates that the errors of measurement of collimations are making a material contribution to the probable error of the time determinations. We can, in fact, obtain an upper limit to that part of the probable error of a time determination not due to erroneous measures of collimation by assuming that the whole of the collimation residuals arise from errors of measurement, i. e. assuming that there are no real fluctuations about the datum lines. On this assumption we take, for each observer, the datum line

* A proof for the case $m=1$ is given by Soper, *Biometrika*, IX, 1913.

† The standard error of sampling for σ_x^2 is $\frac{\sigma_x^2\sqrt{2}}{\sqrt{n-m}}$. For a mean product σ_{xy} it is $\frac{\sigma_{xy}}{\sqrt{n-m}} \cdot \frac{\sqrt{1+r^2}}{r}$. (Professor Greaves has proved this for $m=1$ and Dr Aitken has proved it for the general case.)

‡ In connection with this question of a possible bisection error, a considerable amount of work was done (after the period under discussion) using a reversing prism on all collimation and level settings. Only two observers (LS and JP) were available for this investigation, but fortunately, the personal equations for these two are very different. The results showed that the large difference in personal equation in the measurement of collimation is not merely a question of bisection error ; i. e. although the scatter in the measures was slightly reduced when a reversing prism was used, the difference between the observers' measures was, for some obscure reason, more than doubled.

values to represent the value of his collimation errors and we can thus subtract the collimation residuals from the clock error residuals (i. e. in effect recalculate the clock error with the required collimation error). The probable error of the resulting figures is then that part of the probable error of a determination of time not due to errors of measurement of collimations. The results are as follows:—

Observer	p.e. of clock errors (msecs.)	p.e. of clock errors corrected for collimation (msecs.)
LS	± 16	± 9
B	± 18	± 14
JP	± 22	± 23
MM	± 23	± 14
All observers	± 20	± 17

The first column refers to the probable errors of observations already corrected for (slowly changing) personal equations. The probable error of the original observations, including the effect of unknown personal equations, is ± 32 msecs. It should be stressed that the figures in the second column represent upper limits; they are, in fact, the values of $0{\cdot}6745\sigma_{\epsilon'}$, where $\epsilon' = e + (2C - l) \sec \phi$. They thus represent upper limits to the value of $0{\cdot}6745\sigma_e$ and incorporate the probable error of the real variations C of collimation about the steady drift, which we have assumed zero in deducing the figures. And, of course, they also include contributions due to errors of star places and clock erratics, the effects of these having been tacitly included in our σ_e.

8. The level errors have now to be considered. A straight determination of the correlation between clock and level errors is not the method used in practice. The level errors involve the collimation readings because they are obtained by differencing the nadir readings of the R.A. micrometer against the measured reading for the line of collimation. Consequently we have used the level errors with the collimations subtracted, that is the nadir readings. This subtraction is justified only if the removed collimations have approximated closely to a linear drift. For instance, if the collimation suffered an abrupt change the subtraction would introduce a discontinuity into the nadir readings. The figures, in terms of revolutions of the R.A. micrometer, were converted into seconds of time at the Edinburgh zenith by multiplying by $1{\cdot}306 \sec \phi = 2{\cdot}33$.

The nadir readings are subject to a variation with temperature. Accordingly, it was necessary to determine this temperature coefficient, in order to derive the nadir readings of level which would have been obtained if the temperature had remained steady.

Unfortunately, it has not been the practice to record the temperatures at the time of observation. The only records available at the Observatory were the routine meteorological readings taken at 21^{h} U.T. These have been supplemented by thermograph records kept by Mr J. Paton at the Physics Laboratory of the University of Edinburgh, and Mr Paton has kindly allowed Miss Angela Penny (sister of one of the authors) to extract the necessary information from the thermograph sheets. These figures have been used to correct the Observatory readings at 21^{h} U.T. to the mean times of observation of the transits. The temperatures so obtained are relatively unreliable and only a rough approximation to the temperature coefficient can be hoped for. Least square solutions were made for straight lines defining the temperatures and nadir readings separately when plotted against the date, and the residuals from each of these straight lines were then plotted against each other on a scatter diagram. The correlation of the residuals was not well defined but was considered sufficiently definite to give us confidence that the nadir readings would be smoothed somewhat by correction for temperature. An added difficulty was that the distribution of observers with temperature was not uniform, and as there was some indication of systematic difference in the levels as measured by the different observers, it was not enough merely to determine the slope of a line drawn through the points on the scatter diagram irrespective of observer. It was necessary to adopt different lines for the different observers but with a common slope. The adoption was made by a least square solution, and the temperature coefficient obtained was $+2{\cdot}8$ milliseconds, at the zenith, per degree Fahrenheit.

The coefficient was determined many years ago by Professor Sampson* who obtained $+8{\cdot}5$ equatorial milliseconds per degree Fahrenheit for the level error temperature coefficient. To compare this result with ours we must multiply it by $\sec\phi = 1{\cdot}785$ so as to convert to milliseconds at the Edinburgh zenith. This gives $+15{\cdot}2$, as against our $+2{\cdot}8$. It should be observed, however, that Sampson's result is for temperatures recorded by a thermometer with the bulb sunk some inches into one of the transit circle piers, whereas the temperatures we are using are referred to readings in a Stevenson screen. It is to be expected that Sampson's result would be more definite and would give a larger coefficient since temperature fluctuations of comparatively short period would be minimized on the pier as against the variations in the screen. Since we have only screen temperatures available we must naturally use a temperature coefficient applicable to them. The nadir readings for level (i. e. the level errors with the collimations subtracted and the resulting differences multiplied by $\sec\phi$) were accordingly corrected for the temperature coefficient $+2{\cdot}8$ and some smoothing resulted.

At this stage the corrected nadir readings were reversed in sign so that a positive correlation with clock errors was to be expected as a result of the common errors of measurement of the nadir settings. As in the case of collimations, it was necessary to adopt separate datum lines for the different observers to allow for any secular change in the personal equations in the measurement of level error. The lines were defined by least square fits of expressions $a+bt$ to the numbers, one such fit being made for each observer. Residuals from the datum lines were then deduced.

These nadir residuals were not analysed for correlation with the original clock error residuals but with the clock error residuals from whch the collimation residuals had been subtracted, i. e. with the clock error residuals obtained on the assumption that the collimation drifted linearly with time.

We shall, as before, investigate the significance of this correlation coefficient before giving the numerical results.

To obtain the residuals of clock error corrected for collimation, subtract the quantities $-2\chi\sec\phi$ from the quantities ϵ, thereby forming the quantities

$$\epsilon' = \epsilon + 2\chi\sec\phi = e + (2C - l)\sec\phi.$$

To obtain the nadir residuals, subtract the quantities χ from the quantities λ,

* *M.N.*, **75**, 69, 1914.

forming the quantities

$$\lambda' = \lambda - \chi = L + l - C.$$

The coefficient r_2 is defined as the coefficient of correlation between ϵ' and $-\lambda' \sec\phi$, so that

$$r_2 = -\frac{\sigma_{\epsilon'\lambda'}}{\sigma_{\epsilon'}\sigma_{\lambda'}},$$

where $$\sigma_{\epsilon'\lambda'} = -(2\sigma_C^2 + \sigma_l^2)\sec\phi, \tag{4}$$

assuming that the other sums of products vanish as before,

also $$\sigma_{\epsilon'}^2 = \sigma_e^2 + (4\sigma_C^2 + \sigma_l^2)\sec^2\phi, \tag{5}$$

and $$\sigma_{\lambda'}^2 = \sigma_C^2 + \sigma_L^2 + \sigma_l^2. \tag{6}$$

The numerical values of the coefficient of correlation between the nadir and corrected clock error residuals are:

Observer	r_2 and p.e. of r_2
LS	$0{\cdot}07 \pm 0{\cdot}19$
B	$0{\cdot}78 \pm 0{\cdot}19$
JP	$0{\cdot}14 \pm 0{\cdot}17$
MM	$-0{\cdot}13 \pm 0{\cdot}21$
All observers	$0{\cdot}10 \pm 0{\cdot}10$

For three observers these figures do not provide definite evidence of correlation. For the observer B the correlation coefficient 0·78 suggests strong correlation, but there are only six observations by this observer over the period considered. However, the probable error of ±·19 suggests that the coefficient obtained cannot be entirely due to sampling error and an examination of the six pairs of residuals involved strengthens our belief that there is a pronounced correlation for B. These residuals are as follows:—

Residuals, in units of one millisecond

Of clock error after correction for collimation.	−17	+6	−4	+34	−3	−16
Of the nadir readings.	−10	+20	−16	+18	−3	−9

Quite apart from the actual residuals and correlation coefficients there is reason to believe that the correlation should be stronger in the case of observer B. For any observer there must necessarily be some correlation since the quantity σ_l cannot be zero but the correlation will be small if σ_l and σ_C are much smaller than one of the other standard deviations involved in (5) and (6). Now one of these is σ_L, the standard deviation of the real variations of the level error about the datum taken as the "mean value". But this datum involves the temperature, and we have already referred to the unsatisfactory nature of the temperatures used. It is, we think, significant that the observations of the three observers LS, JP and MM were made over a temperature range twice as large as that for the observations of B. The suggestion is that for the three observers σ_L is enhanced and the enhanced value reduces the correlation coefficient whereas for B σ_L is smaller. This suggestion receives support from the table of collected results given below in Section 11; from this table it appears that the probable error of the nadir measures, corrected for temperature, is, in fact, nearly twice as large for the three observers as it is for B.

It should be mentioned that a possible additional complication would be a variation of *collimation* with temperature. This would introduce a correlation between C and L and such a correlation would give rise to additional terms in r_2 which would be enhanced for the three observers LS, JP and MM as compared with observer B on account of the smaller temperature range for B's observations. But without knowledge of the sign of the possible correlation between C and L it is impossible to say whether its effect would be to decrease or increase the correlation coefficients.

Since the correlation coefficients r_2 for the three observers LS, JP and MM are small, we cannot obtain reduced upper limits to that part of the probable error of a time determination not due to erroneous measures of instrumental errors in a way similar to our treatment of collimation. For observer B we can obtain such a reduced upper limit by differencing the two columns of residuals quoted above, thereby obtaining the quantities $\epsilon'' = \epsilon' + \lambda' \sec\phi = e + (L + C)\sec\phi$. For this observer the probable error of a time determination corrected for erroneous collimation measures is $0{\cdot}6745\ \sigma_{\epsilon'} = \pm 14$ msecs. (see Section 7); it is reduced, when corrected for error of measurement of the nadir settings to $0{\cdot}6745\ \sigma_{\epsilon''} = \pm 9$ msecs. These figures are, from what has been said, upper limits to $0{\cdot}6745\ \sigma_e$, but it must be remembered that they are subject to sampling error which is necessarily enhanced for a sample of only six observations.

9. By forming the products of the collimation and nadir residuals we obtain a further mean product,

$$\sigma_{\chi\lambda'} = -\sigma_C^2. \tag{7}$$

This supplies a value of σ_C, the standard deviation of the real excess of the collimation over the adopted linear drift, assuming, as before, that the mean products of the components vanish.

This completes a set of seven equations involving all the unknown quantities we have been discussing.

In the following table are collected the numerical values corresponding to equations (1) to (7) for the mean of the four observers:—

M.P. of C.E. × Colln. (msecs.)

$= \sqrt{-2\sigma_{\epsilon\chi}\sec\phi} \quad = 2\sigma_c \sec\phi \qquad \times 0{\cdot}6745 = \pm 14$ (i)

S.D. of C.E.

$= \sigma_\epsilon \quad = \sqrt{\sigma_e^2 + (4\sigma_c^2 + \sigma_l^2)\sec^2\phi} \qquad ,, \quad = \pm 20$ (ii)

S.D. of Collimation

$= 2\sigma_\chi \sec\phi \quad = 2\sqrt{\sigma_C^2 + \sigma_c^2}\sec\phi \qquad ,, \quad = \pm 16$ (iii)

M.P. of (C.E. − Colln.) × Nadir

$= \sqrt{-\sigma_{\epsilon'\lambda'}\sec\phi} \quad = \sqrt{2\sigma_C^2 + \sigma_l^2}\sec\phi \qquad ,, \quad = \pm\ 6$ (iv)

S.D. of (C.E. − Collimation)

$= \sigma_{\epsilon'} \quad = \sqrt{\sigma_e^2 + (4\sigma_C^2 + \sigma_l^2)\sec^2\phi} \qquad ,, \quad = \pm 17$ (v)

S.D. of Nadir

$= \sigma_{\lambda'}\sec\phi \quad = \sqrt{\sigma_C^2 + \sigma_L^2 + \sigma_l^2}\sec\phi \qquad ,, \quad = \pm 21$ (vi)

M.P. of Colln. × Nadir

$= \sqrt{-2\sigma_{\chi\lambda'}\sec^2\phi} \quad = \sqrt{2}\,.\,\sigma_C \sec\phi \qquad ,, \quad = \pm 11$ (vii) where C.E. stands for clock error. The values are multiplied by 0·6745 to put them on the same footing as probable errors, used throughout the rest of the paper.

Theoretically a solution of best fit could be obtained giving all the probable errors involved, and this would be feasible provided the number of observations was big enough. But for an effective number of 43 observations the sampling errors are not small enough to justify any attempt at a rigorous solution. Apart from this the table indicates the approximate magnitude of the various quantities, some of which have already been quoted.

Four of the equations can be used to obtain a rough measure of the quantity $0{\cdot}6745\,\sigma_e$.

From (i) and (ii) we get $0{\cdot}6745\sqrt{\sigma_e^2+\sigma_l^2\sec^2\phi}=\pm 15$.

From (v) and (vii) we get $0{\cdot}6745\sqrt{\sigma_e^2+\sigma_l^2\sec^2\phi}=\pm 7$.

The disagreement illustrates the effects of sampling error and of the assumption that there is no correlation between the various components of the collimation and level measures. It can be taken that $0{\cdot}6745\sqrt{\sigma_e^2+\sigma_l^2\sec^2\phi}=\pm 15$ is an upper limit. Therefore, $0{\cdot}6745\,\sigma_e$, the probable error of that part of the scatter in clock error not due to the effect of erroneous measures of collimation and level, is less than ± 15 msecs. It would require much more extended data to arrive at a more definite conclusion by these methods.

10. Approaching the problem from a totally independent direction, values for $0{\cdot}6745\sigma_e$ can be obtained from the internal agreement of the results from individual stars for the same night's work.

On a given night 10 stars are usually employed and the probable error of the resulting clock error can be deduced from the residuals of the individual stars from the line of best fit in the plot against $\tan\delta$, of $\alpha - T$ (corrected for collimation and level), where α denotes R.A. and T the observed time of transit of a star. The probable errors obtained in this way are practically independent of the effects of personal equation and of erroneous measures of the instrumental errors. The values for each observer, and for the mean of all four observers are :

LS	$\pm$ 7 msecs.
B	$\pm$ 9
JP	$\pm$ 8
MM	$\pm$10
All Observers	$\pm$ 9

For three observers these values are similar to, though rather smaller than, the values obtained by correcting the clock errors for erroneous measures of collimation and level. The values of $0{\cdot}6745\,\sigma_e$, derived in this way, may be expected to be lower limits to the true values, because errors, such as would be caused for example by a lateral refraction, which are common to the observations during a night, are eliminated. If, therefore, the larger values, derived in Section 7, are accepted as most nearly representing the true values of the probable errors, they are seen to be considerably smaller than the total probable error of a measure of clock error (± 32 msecs.).

11. The results for the individual observers so far obtained are given in the following table (referred to in Sections 7 and 8):—

Probable Errors (msecs.)

Observer	Collimation	Nadir	Clock Error: Uncorrected (All Observers)	Clock Error: Corrected for personal equations	Clock Error: Corrected for collimation	Clock Error: Corrected for level	Clock Error: From internal agreement
LS	±15	±22		±16	± 9		± 7
B	±11	±12	±32	±18	±14	± 9	± 9
JP	±15	±22		±22	±23		± 8
MM	±19	±20		±23	±14		±10

12. We have been forced to confine the analysis to the short period October 19 to February 20 for which the collimation error did not seem to be subject to appreciable discontinuities, and we have seen that the effect of erroneous measures of collimation betrays itself by a strong correlation between collimation and clock errors. Although detailed analysis is not feasible for the remainder of the whole period covered by the data, it is easy to show by a graphical method that the correlation persists.

The heavy continuous line in Fig. 1 (Plate 11) over the period October 19 to February 20 represents a datum line for the mean of all four observers. The heavy continuous line extending from February 1 to September 3 (over the greater part of this period only two observers, LS and JP, shared in the transit work) represents a datum line for the mean of the two observers LS and JP. The thin lines represent least square solutions for these two observers for the period February 1 to May 5 and May 6 to September 3, and the heavy continuous line is the mean of these, but adjusted so that the two portions intersect at May 5, one of the dates on which a change of rate of IVC took place.

The residuals of the clock errors for the different observers from the heavy continuous lines for the period October 19 to Feburary 20 and February 20 to September 3, are plotted in Fig. 3 (Plate 12), and this figure also exhibits a plot of the collimation residuals reckoned from the datum lines for collimation shown in Fig. 1 (Plate 11). On comparing the two graphs it is at once apparent that the fluctuations in the collimation graph are reproduced in the graph of clock error residuals, that is that there is a strong correlation between clock error and collimation over the whole period.

The graphs also appear to show that there is a correspondence between the general trends of collimation and clock errors over the rest of the whole period for the two observers LS and JP, that is that the varying difference of personal equation in collimation is reflected in the varying difference of personal equation in clock error. But it must be pointed out that we have really preserved the feature by the method used in the adoptions of the collimation datum lines shown in Fig. 1 (Plate 11). This procedure will, however, not produce the detailed day-to-day correlation which is a striking feature of the graphs and which indicates that the erroneous measures of collimation contribute materially to the total scatter in clock error.

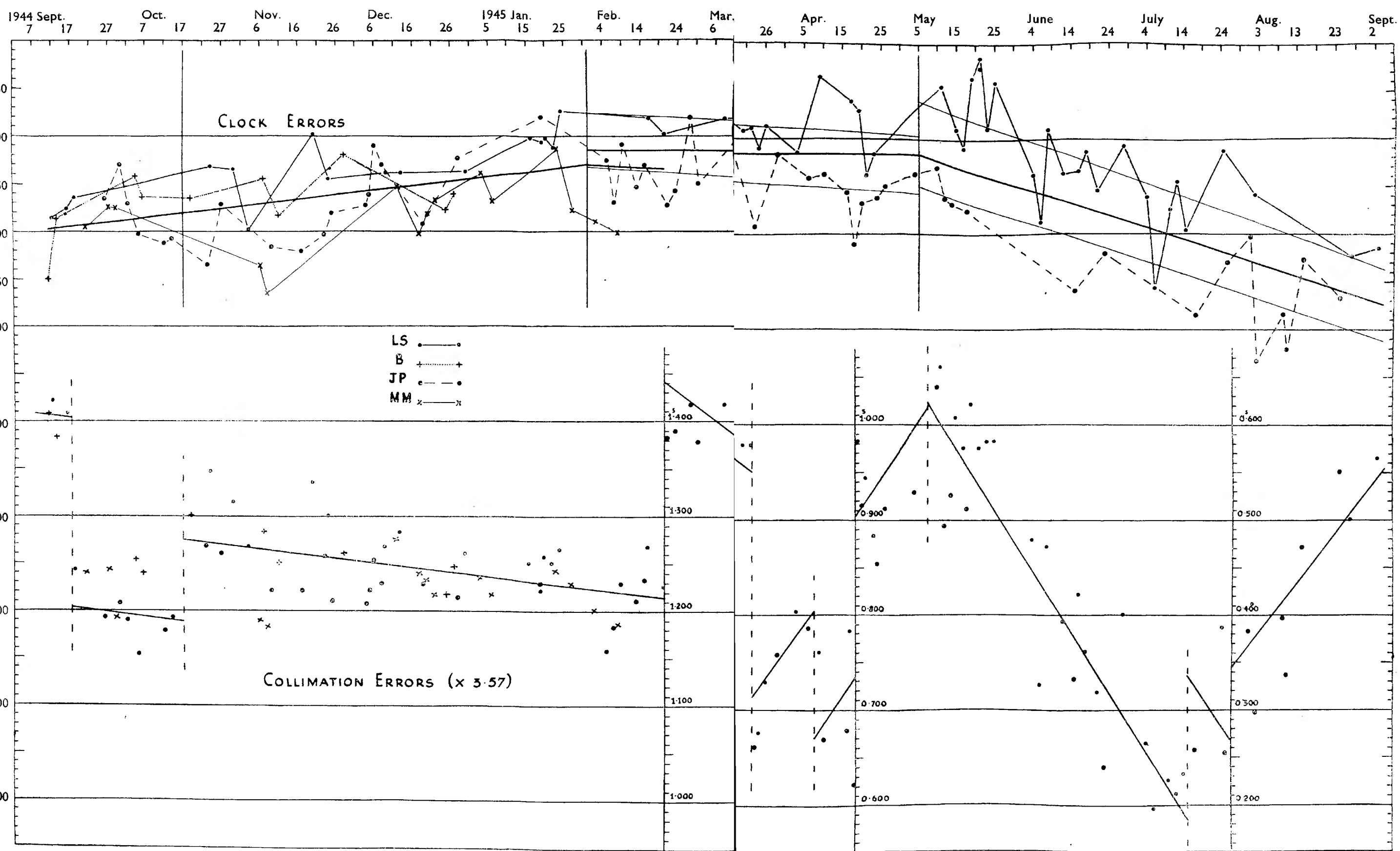

FIG. 1.

L. S. T. Symms and C. J. A. Penny, *Errors of Time Determination.*

At the bottom of Fig. 3 (Plate 12) there appears a plot of the clock error residuals with the collimation residuals subtracted. In the case of the observer B the level residuals have also been subtracted. It will be seen that the ranges of the fluctuations shown in the clock residual graph are materially decreased.

13. It has long been suspected that the major part of the probable error of a determination of time with a transit circle of classical pattern is due to the effect of erroneously determined instrumental constants and that the contribution made by the actual errors of observing the transits of the stars is relatively small. All the numerical evidence in this paper goes to confirm this suspicion. It is very well brought out in the case of the two observers with long observing experience:

Probable errors of clock errors (msecs.)

Observer	Uncorrected (from whole series)	Corrected for personal equations	Corrected for collimation	Corrected for level	From internal agreement
LS	±32	±16	± 9	...	±7
B		±18	±14	±9	±9

It should be noted that these probable errors are applicable to a hand-driven impersonal micrometer and include the effects of errors in the individual star places.

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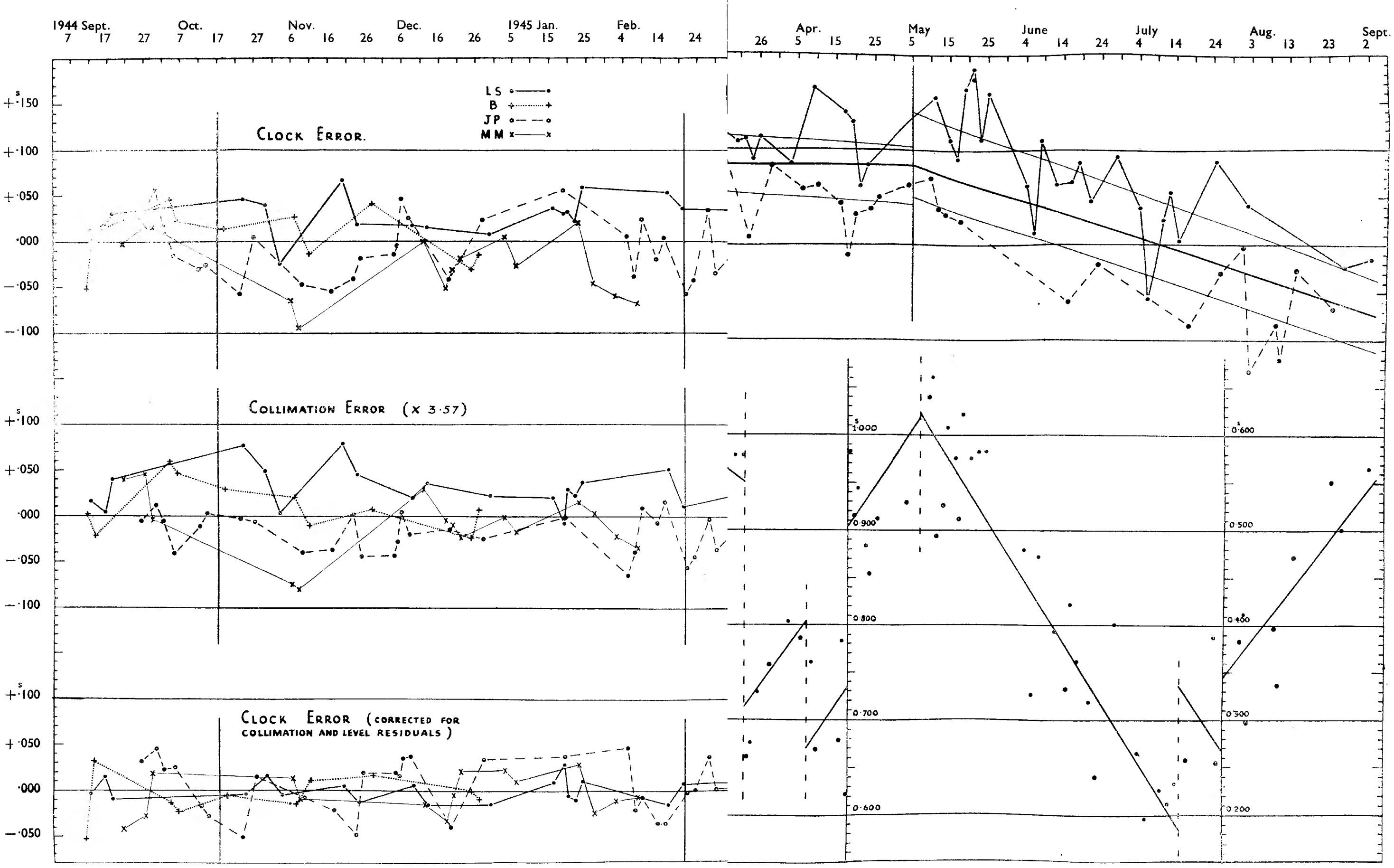

Fig.

L. S. T. Symms and C. J. A. Penny, Errors of Time Determination.