THE ASTRONOMICAL JOURNAL

FOUNDED BY B. A. GOULD

PUBLISHED BY THE AMERICAN ASTRONOMICAL SOCIETY

VOLUME 51 1944 August ~ No. 1150 *NUMBER 2*

THE SECULAR PERTURBATIONS OF THE SATELLITES OF MARS

BY EDGAR W. WOOLARD

Observations of the satellites of Mars at successive oppositions show that the orbits have large secular variations. The line of apsides of each satellite rotates in the orbital plane, while the position of the orbital plane in space varies in such a way that the pole of the orbit describes a small circle on the celestial sphere at a uniform rate in a retrograde direction around a fixed center. Discussions by H. Struve,[1] Burton,[2] and Sharpless[3] of the oppositions over the interval 1877–1941 have given the following values for the radius ρ of this circle, the rate K' of the polar motion per tropical year, the orbital eccentricity, tropical mean daily motion, and semi-major axis at unit distance for each satellite:

	Phobos	Deimos
ρ	$1^\circ.13$	$1^\circ.77$
K'	$159^\circ.1457 \pm 0^\circ.5044$	$6^\circ.54382 \pm 0^\circ.00070$
e	0.0210	0.0028
n	$1128^\circ.844133$	$285^\circ.161922$
a	$12''.895$	$32''.389$

The line of apsides of Phobos is observed to advance about $158^\circ.5$ per year. The principal cause of these variations is the disturbing force of the polar flattening of Mars, but the motion of each satellite is also appreciably disturbed by the sun; the satellites are so small and so close to Mars that possible mutual perturbations and perturbations by Jupiter may be neglected at present.

In previous investigations, two theoretical relations have been applied in discussing the motions of the satellites. One is the principle that the rate of advance of the apsides is equal to the rate of the retrograde circular motion of the orbital pole; this principle is an approximation, but in the case of the Martian satellites it proves to be valid within the accuracy to which the pericenters can be determined. The other relation is an expression for the ratio of the polar motion of Phobos to that of Deimos in terms of the sidereal mean motions:

$$\frac{K_P'}{K_D'} = \left(\frac{n_P}{n_D}\right)^{7/3}. \qquad (1)$$

This relation also is an approximation; and it is not satisfied by the observed values. The ratio of the observed polar motions is 24.32 $\pm$ 0.08, whereas the observed mean motions give $(n_P/n_D)^{7/3} = 24.79$, a discrepancy amounting to six times the probable error.

To investigate the source of this discrepancy, a more accurate expression for the ratio of the polar motions will first be derived. The disturbing function for the perturbations caused by the flattening of Mars is[4]

$$R_0 = k^2 m_0 (f - \tfrac{1}{2}\kappa + \tfrac{3}{2}f^2 + \cdots)\frac{b_0^2}{r^3}(\tfrac{1}{3} - \sin^2 d) + \cdots,$$

in which k^2 is the constant of gravitation, m_0 the mass of Mars, f the flattening, b_0 the equatorial radius, κ the ratio of the centrifugal force of rotation at the Martian equator to mean gravity on Mars, r the radius vector of the satellite, and d the angle between r and the equatorial plane of Mars. Within the parentheses, the further quantities are of the order of f^2 or smaller; and the next term of R_0 is of the order of the product of this first term by $f(b_0/r)^2$. The disturbing function for the action of the sun is[5]

$$R_1 = k^2 \frac{m_1}{r_1}\left\{[\tfrac{3}{2}\cos^2(r, r_1) - \tfrac{1}{2}]\left(\frac{r}{r_1}\right)^2 + \cdots\right\},$$

in which m_1 is the mass of the sun, and r_1 the distance of the sun from Mars; the next term has the factor $(r/r_1)^3$. These functions may readily be expressed in terms of the orbital longitudes and elements of the satellite and its primary. The periodic variations in the disturbing force that depend on the position of the satellite in its orbit do not lead to any progressive changes, and their small short-period effects may be disregarded; therefore the quantities that depend only on the radius vector and true anomaly of the satellite may be replaced by their mean values. The terms that depend on the orbital longitude of Mars may produce appreciable periodic perturbations, for which allowance is made in reducing the observations; omitting these terms, and neglecting terms in f^2 and in the higher powers of b_0/r, r/r_1 and e, we have the disturbing function for the secular perturbations produced by the flattening of Mars and the action of the sun:

$$R = k^2 m_0 \frac{(f - \frac{1}{2}\kappa) b_0^2}{a^3(1 - e^2)^{3/2}} \{\tfrac{1}{3} - \tfrac{1}{2}\sin^2\gamma\} + k^2 m_1 \frac{3a^2}{8a_1^3(1 - e_1^2)^{3/2}} \{(1 + \tfrac{3}{2}e^2)\cos^2 i + \tfrac{5}{2}e^2 \sin^2 i \cos 2(\pi - \Omega) - \tfrac{1}{2}e^2 - \tfrac{1}{3}\}, \quad (2)$$

where γ is the inclination of the satellite orbit to the Martian equator, i its inclination to the Martian orbit, π and Ω the longitudes of the pericenter and the node on the orbit of Mars, and a_1 and e_1 the semimajor axis and eccentricity of the Martian orbit. From this expression for R, the theoretical secular variations may be derived by means of the classical equations for the variations of the orbital elements[6] referred to any desired fundamental reference plane.

As the node and the inclination on the reference plane vary, the pole of the orbital plane traces a curve on the celestial sphere. Since the eccentricities are so small, and no differentiation with respect to e occurs in the equations for the variations of the node and inclination, the form of this curve may be found without appreciable error by neglecting e^2; and as a is constant to the order of approximation of (2), R then reduces to a function of only i and Ω. Adopting the plane of the orbit of Mars at a selected epoch as the reference plane, it follows that an integral of the equations for di/dt and $d\Omega/dt$ is[7]

$$R = C_1 \cos^2 i + C_2 \cos^2 \gamma = \text{const.}, \quad (3)$$

in which

$$C_1 = k^2 m_1 \frac{3a^2}{8a_1^3(1 - e_1^2)^{3/2}},$$

$$C_2 = k^2 m_0 (f - \tfrac{1}{2}\kappa)\frac{b_0^2}{2a^3}.$$

The relation (3) that must be satisfied by the angular distances of the pole of the satellite orbit from the poles of the Martian orbit and Martian equator represents the locus of the pole of the orbit on the celestial sphere. It follows from (3) by abstract geometry that this locus is the curve in which the sphere is intersected by an elliptic cylinder with an axis that passes through the center of the sphere and is perpendicular to the plane of the great circle through the pole of the Martian orbit and the pole of the equator of Mars.[7] The locus is therefore a non-plane oval curve, with its longer axis on the great circle through the poles of the Martian equator and orbit; furthermore, the center of this oval lies between these two poles, at a distance I_1' from the pole of the Martian equator that is given by

$$\tan 2I_1' = \frac{\frac{C_1}{C_2}\sin 2\gamma_1}{1 + \frac{C_1}{C_2}\cos 2\gamma_1}, \quad (4)$$

in which γ_1 is the inclination of the Martian equator to the Martian orbit. That is, the center of the locus is the pole of a fixed plane which is inclined at an angle I_1' to the plane of the Martian equator, and which passes through the intersection Ω_0 of the equatorial and orbital planes of Mars and lies between them. This fixed plane is the Laplacian plane, characteristic of satellite systems.

However, in the case of the satellites of Mars the eccentricity of the intersecting cylinder is so great, and the oval is so small, that the path of the pole of the satellite orbit differs inappreciably from a circle: The radius vector ρ of the pole in its path is the inclination of the orbital plane to the fixed Laplacian plane; and the angle Ω' which this radius vector makes with the longer axis of the oval, measured eastward through the south from the vertex nearest the pole of the Martian orbit, is equal to the longitude of the node of the satellite orbit on the fixed plane, reckoned eastward from Ω_0. In terms of ρ, Ω', the disturbing function (3) becomes

$$R = (C_1 + C_2)(\cos^2\rho\cos^2 B + \sin^2\rho\sin^2 B\cos^2\Omega'), \quad (5)$$

in which B is an auxiliary defined by

$$\tan^2 A = C_2/C_1, \quad \sin 2B = \sin \gamma_1 \sin 2A.$$

Adopting the Laplacian plane as the fundamental plane of reference, the equations for the variations of the inclination and node on this plane are

$$\frac{d\rho}{dt} = + \frac{C_1 + C_2}{na^2} \sin \rho \sin^2 B \sin 2\Omega',$$
$$\frac{d\Omega'}{dt} = - 2\frac{C_1 + C_2}{na^2} \cos \rho \cos^2 B \times \{1 - \tfrac{1}{2} \tan^2 B(1 + \cos 2\Omega')\}. \quad (6)$$

It is evident from the definition that A cannot be much less than 90°, and therefore B is a very small angle, while observation shows that ρ is always small; consequently the periodic variation of ρ with Ω' during the circuit of the pole around its path is negligible, while the rate of variation of Ω' is practically constant. The orbital plane keeps a practically constant inclination to the fixed plane, while the node regresses on this plane at a virtually uniform rate and the pole of the orbit revolves in a circle in a retrograde direction at the same rate.

The rate at which the line of apsides is dynamically rotated within the moving orbital plane by the disturbing forces is

$$\frac{\sqrt{1 - e^2}}{ena^2}\frac{\partial R}{\partial e} = \frac{2C_1\sqrt{1 - e^2}}{na^2}\{1 - \tfrac{3}{2}\sin^2 \gamma_1\} + \frac{2C_2}{na^2(1 - e^2)^2}\{1 - \tfrac{3}{2}(\sin^2 \rho + \sin^2 I_1')\}, \quad (7)$$

in which R is given by (2), and the terms that have been neglected are inappreciable; the variation of the longitude of the pericenter referred to the Laplacian plane is obtained by adding to (7) the small kinematic term $2 \sin^2 \tfrac{1}{2}\rho \dfrac{d\Omega'}{dt}$. Since the term in C_1 due to the action of the sun is small, this advance of the apsides is at very nearly the same rate as the regression (6) of the node on the fixed plane: an example of the general principle that to a first approximation the pericenter of a disturbed body advances at a rate equal to the recession of the node on the orbit of the disturbing body or, in perturbations from polar flattening, on the plane of the equatorial excess of mass. It is evident from a comparison of (7) and (6), without actually deriving expressions for the motions of the node on the orbit and equator of Mars, that this principle is valid for each of the disturbing influences separately. In general, however, this principle must be applied with caution, because the approximation that it gives is sometimes greatly in error.[8]

From (6), omitting the minute periodic terms, putting $k^2m_1 = n_1^2a_1^3$ in C_1 and

$$k^2m_0 = n^2a^3(1 - \tfrac{1}{2}\sigma),$$
$$\sigma = 2\left(\frac{b_0}{a}\right)^2(f - \tfrac{1}{2}\kappa) - \left(\frac{n_1}{n}\right)^2, \quad (8)$$

in C_2, we obtain for the solar component of the secular regression of the node on the fixed plane

$$K'_\odot = \frac{3n_1^2}{4n(1 - e_1^2)^{3/2}} \cos \rho \cos^2 B(1 - \tfrac{1}{2} \tan^2 B), \quad (9)$$

and for the component produced by the flattening of Mars

$$K_f' = (1 - \tfrac{1}{2}\sigma)(f - \tfrac{1}{2}\kappa) \times \left(\frac{b_0}{a}\right)^2 n \cos \rho \cos^2 B(1 - \tfrac{1}{2} \tan^2 B), \quad (10)$$

where

$$\cot^2 A = K'_\odot/K_f', \quad \sin 2B = \sin \gamma_1 \sin 2A, \quad (11)$$

and the factor in σ is the correction necessary to reduce the observed values n and a to the undisturbed values to which Kepler's Law applies.[9] Then, in place of (1),

$$\frac{K_P'}{K_D'} = \left(\frac{\sin A_D}{\sin A_P}\right)^2 \frac{[\cos \rho \cos^2 B(1 - \tfrac{1}{2} \tan^2 B)]_P}{[\cos \rho \cos^2 B(1 - \tfrac{1}{2} \tan^2 B)]_D} \times \frac{(1 - \tfrac{5}{6}\sigma)_P}{(1 - \tfrac{5}{6}\sigma)_D}\left(\frac{n_P}{n_D}\right)^{7/3}, \quad (12)$$

in which a has been eliminated from the last factor by means of (8).

To calculate the values of these theoretical expressions from the directly observed quantities, the auxiliary angles A and B must first be determined by successive approximation: The angle B and the component $K'_\odot$ are so small that a very good first approximation may be obtained by putting $B = 0$; then $K'_\odot$ is easily calculated by means of (9), and subtracting it from the observed K' gives K_f', whence a provisional value of A follows. The inclination I_1 of the Laplacian plane to the Martian orbit is known from the elements of Mars and the observed center and radius of the path of the orbital pole of the satellite, whence the simultaneous solution of the relation $I_1 = \gamma_1 - I_1'$ with (4) gives γ_1 and I_1'; in this way, Burton obtained $I_1' = 0^\circ.01$ for Phobos and $0^\circ.92$ for Deimos, together with $\gamma_1 = 25^\circ.20$ as the mean of determinations from the two satellites. From γ_1 and A, a provisional value of B is computed, and the calculation then successively repeated until the values remain unchanged. From the final K_f', the value of $(f - \frac{1}{2}\kappa)$ is found from (10); κ may be determined from the period of rotation of Mars and the quantities n, a, b_0, whence f follows. The flattening must lie between the Clairaut limits $\frac{5}{4}\kappa$ and $\frac{1}{2}\kappa$.

The results obtained in this way from the two satellites differ very little; and the component K_f', from which $f - \frac{1}{2}\kappa$ is evaluated, is such a large proportion of the total motion of the orbital pole, that in view of the much greater probable error of the observed K_P' we may adopt the value derived from Deimos as the best obtainable, which is $f - \frac{1}{2}\kappa = 0.002920$. Then with $\kappa = 1/218$, we have $f = 0.005214 = 1/191.8$. The corresponding theoretical values given by (9), (10), (11) are:

	Phobos	Deimos
A	88°49′.1	78°24′.4
B	30′.2	4°49′.5
$K'_\odot$	0°.0675	0°.26432
K'_f	158°.4841	6°.27950
K'	158°.5516	6°.54382

To these theoretical values of K' should be added small corrections for the effects of neglecting terms in f^2, e^2, and higher powers of b_0/r and r/r_1; the order of magnitude of these corrections is estimated at about $+0^\circ.2$ for Phobos, and $+0^\circ.0009$ for Deimos.

These theoretical rates of motion agree satisfactorily with the observed values. Their ratio, excluding the estimated corrections that should be added, is 24.23 as compared to the ratio of the observed values 24.32 ± 0.08, and the agreement can be brought within the probable error by a reasonable allowance for the theoretical corrections; but this ratio of the *two theoretical motions* must be distinguished from the *theoretical ratio* (12) of the motions, the value of which is found to be

$$(0.960)(1.0109)(0.99946)(n_P/n_D)^{7/3} = (0.97)(24.79) = 24.04.$$

The factor that multiplies the previous approximation reduces the disagreement with observation from $+0.47$ to -0.28, but a significant discrepancy is still left because Kepler's Law imposes on the mean distances of the two satellites a relation

$$\frac{a_D}{a_P} = \frac{(1 - \frac{1}{6}\sigma)_P}{(1 - \frac{1}{6}\sigma)_D}\left(\frac{n_P}{n_D}\right)^{2/3}, \qquad (13)$$

which was used in forming the last factor in (12) but which is not satisfied by the observed values of a_D, a_P that are used in computing $K'_\odot$, K_f'; the observed value of a_D/a_P is 2.51175, whereas the relation (13), with the observed mean motions, gives 2.50216. Theoretically, from the two conditions that (*a*) the ratio of the semimajor axes must satisfy the relation imposed by Kepler's Law, and (*b*) the motions of both the satellites must lead to the same value for the flattening of Mars, it would be possible to derive corrections to the observed mean distances; the corrected values would both give the same value for the mass of Mars. In practice, however, no significant results are obtained in this way. Corrections could also be determined from a sufficiently accurate value for the mass of Mars obtained independently of the satellites.

I am indebted to Mr. B. P. Sharpless for drawing my attention to this problem; and to the Superintendent of the U. S. Naval Observatory for the use of unpublished data.

REFERENCES

1. *Sitz.-Ber. Preuss. Akad. Wiss.* 1911, Stück 48, 1056.
2. *A. J.* **39**, 155, 1929.
3. Unpublished.
4. See Appell, *Traité de Mécanique Rationnelle*, **4**, Fasc. II, 2me éd. (by Véronnet), p. 269, 1937.
5. See, e.g., Brown, *An Introductory Treatise on the Lunar Theory*, p. 79, 1896.
6. See, e.g., Moulton, *An Introduction to Celestial Mechanics*, 2nd ed., p. 401, 1914; Brown and Shook, *Planetary Theory*, p. 135, 1933.
7. Tisserand, *Traité de Mécanique Céleste*, **4**, 93, 1889.
8. Kourganoff, *Bull. Astr.* (2), **10**, 109, 1937.
9. Tisserand, *ibid.* 17; Laplace, *Traité de Mécanique Céleste*, Livre VIII, ch. 2, art. 3, 1805.

Washington, D. C.,
1944 *April.*