THE SHORT-PERIOD ERRATICS OF FREE PENDULUM AND QUARTZ CLOCKS

W. M. H. Greaves, F.R.S., and L. S. T. Symms

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Summary

The erratics of short period (derived from measures taken at intervals of one day) of clocks are discussed and three types of erratic are recognised. The first of these (type A) represents erratics of phase only, consisting of accidental fluctuations of phase, and also includes errors of chronographic recording and measurement. The third type (type C) represents erratics of rate only, consisting of accidental changes of rate.

The second or intermediate type of erratic (type B) may arise either from accidental fluctuations of rate about a mean value or from accidental changes of phase, the effects of these two kinds of erratic being indistinguishable. A method of analysis is given whereby a number is derived (the r-ratio) which indicates the degree of admixture of the three types of erratic.

It is found that for clocks of the standard free pendulum type all three types of erratic are present in appreciable amounts, but that for quartz clocks the erratics of types B and C are too small to be detected by the analysis and the erratics actually present are predominantly of type A. It is pointed out that type A erratics do not affect the long-period performance of a clock, but that the erratics of the other two types, especially those of type C, produce "wandering" of the clock by accumulation. It is accordingly concluded that in so far as the long-period performance of a clock arises from the integrated effects of the short-period erratics, good quartz clocks are very superior in long-period performance to the best clocks of the standard free pendulum type.

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1. The investigation set forth in the present paper was carried out early in 1942. As far as the portion dealing with Shortt clocks is concerned, the material was the daily intercomparisons of three such clocks carried out at one of the stations responsible for the maintenance of the Greenwich time-service. A discussion of the relative performance of two quartz-controlled clocks at the National Physical Laboratory was included in the original discussion, which has now been supplemented by data relating to other quartz-controlled clocks.

The investigation deals with short-period erratics only, but it should be mentioned that recent developments tend to show that the superior performance of the best quartz-controlled clocks extends to long-term performance.

2. The short-period erratics of clocks can be conveniently studied by tabulating the measured daily changes of rate—that is, the second differences of the relative clock errors given by a series of chronographic comparisons taken regularly at 24-hour intervals. For a perfect clock running at a uniform rate these second differences would be all zero, and it would thus be expected that information as regards clock erratics would be obtained by forming the mean absolute second differences, that is, the mean measured daily changes of rate irrespective of sign, over any prescribed period, provided of course that this period is long enough to ensure that the results are statistically significant. This

procedure assumes that the second differences are of an accidental character, and this appears to be so apart from any tendency of the clocks concerned to exhibit a "curvature term" involving a continuous systematically positive or negative change of rate. Such a tendency is an undoubted feature of present-day precision clocks, whether they be pendulum clocks or whether they incorporate quartz vibrators, but in practice the systematic change of rate is less than one millisecond per day per day, and its effect upon the mean absolute second differences is negligible.

But a little consideration will show that a knowledge of the mean absolute second differences does not by itself give a true picture of the performance of clocks from the point of view of short-period erratics. It will be shown below, for example, that the mean absolute second differences of relative clock error for two particular quartz clocks is of the same order of magnitude as for two good Shortt clocks, but that the real nature of the erratics is essentially different in the two cases. It is necessary to discuss the various contributory causes which produce the variations in the second differences, and three such contributions have to be distinguished, viz.:

(*a*) Accidental erratics in the clock errors themselves arising from random fluctuations, about a mean value, of the phase at which the seconds impulses are taken, or from errors of chronographic recording and measurement.

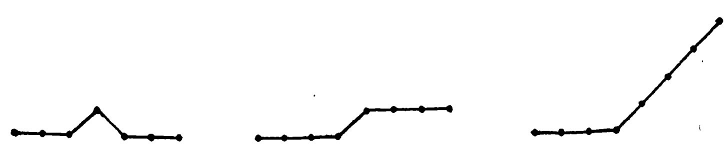
(*b*) Accidental errors affecting the first differences. These may arise from accidental fluctuations of rate about a mean value or from accidental changes of phase.

(*c*) Accidental errors affecting the second differences arising from accidental changes of rate.

We shall refer to these three types of erratic as types A, B and C respectively.

It is important to realise the difference between erratics of type B arising from random fluctuations of rate, and erratics of type C arising from random changes of rate. In the first case the daily rates will be distributed in a random fashion about a mean value, but in the case of type C erratics there is no tendency for the rate to recover following a change. There is a similar distinction between erratics of types A and B arising from fluctuations of phase and changes of phase.

The distinction between the three types can be visualised by considering the effect of abnormally large isolated cases on a plot of clock error against time. The diagram shows the effect upon such a plot of isolated instances of the three types.



On studying actual examples of intercomparisons of clocks in which the daily differences of clock error are plotted against the calendar date, occasional extreme cases of all three types can be recognised. It should be noted that type B can arise from random changes of phase or random fluctuations of rate. Known causes of type A erratics in free pendulum clocks are the hit-and-miss action of slave synchronisers, erratic variations in the actions of the seconds switches whereby the time is taken off the slaves, and erratic variations in the actions of the free pendulum remontoire switches which control the slave synchronisers. Errors of chronographic recording and measurement are also included in type A.

Type B erratics can arise from fluctuations in the arcs of swing of a free pendulum arising possibly from variations in the intensity of impulsing. The origin of erratics of type C is obscure, but they undoubtedly exist in pendulum clocks and, from the point of view of long-period performance, are the most serious defects in modern precision clocks, as their effect on clock error is cumulative. Type B erratics also tend to produce a cumulative effect on clock error, but they are less serious than type C as far as long-period performance is concerned.

From the foregoing it is clear that the tabulation of mean absolute second differences by themselves do not give a complete picture of performance, as they do not distinguish between the three types of erratic. It is possible, however, to get further information by tabulating an additional quantity as will now be shown.

3. Consider a series of chronographic comparisons between two clocks made at regular intervals (in the present investigation the intervals are 24 hours). Denote the differences of clock error thereby obtained by v_n.

Using a slightly modified form of the ordinary central difference notation, denote the first differences (daily rates) of the v_ns by $\delta_1 v_{n+\frac{1}{2}}$ and the second differences (daily changes of rate) by $\delta_1{}^2 v_n$.

Then

$$\delta_1 v_{n+\frac{1}{2}} = v_{n+1} - v_n \tag{1}$$

and

$$\delta_1{}^2 v_n = \delta_1 v_{n+\frac{1}{2}} - \delta_1 v_{n-\frac{1}{2}}, \tag{2}$$

so that

$$\delta_1{}^2 v_n = v_{n+1} - 2v_n + v_{n-1}. \tag{3}$$

Following up a suggestion which was originally due to Professor E. T. Whittaker, we write

$$\delta_2{}^2 v_n = v_{n+2} - 2v_n + v_{n-2}. \tag{4}$$

The quantities $\delta_1{}^2 v_n$ and $\delta_2{}^2 v_n$ are particular cases of more general expressions $\delta_m{}^2 v_n$, to which reference will be made below. All these quantities are of the same dimensions as the second differences $\delta_1{}^2 v_n$, which in the present investigation are given in milliseconds per day per day.

From (1), (2) and (4) the expressions for $\delta_2{}^2 v_n$ in terms of the first and second differences are

$$\delta_2{}^2 v_n = \delta_1 v_{n+\frac{3}{2}} + \delta_1 v_{n+\frac{1}{2}} - \delta_1 v_{n-\frac{1}{2}} - \delta_1 v_{n-\frac{3}{2}} \tag{5}$$

and

$$\delta_2{}^2 v_n = \delta_1{}^2 v_{n+1} + 2\delta_1{}^2 v_n + \delta_1{}^2 v_{n-1}. \tag{6}$$

Apart from systematic effects produced by acceleration, which in practice amount to less than one millisecond per day per day, and which are ignored in the present analysis, the quantities $\delta_1{}^2 v_n$ and $\delta_2{}^2 v_n$ possess the property that their algebraic averages over a sufficiently long period of time are zero. They are, however, not necessarily purely accidental series of quantities, since each $\delta_1{}^2 v_n$ is in general correlated with some of the preceding and following values, and similarly for the $\delta_2{}^2 v_n$s. We proceed to examine the way in which the $\delta_1{}^2 v_n$s and the $\delta_2{}^2 v_n$s are built up from the erratics of the three types.

We first consider erratics of type A which arise from purely accidental variations in the clock errors v_n, such as would be produced by accidental variations in the phase at which the time is taken from a pendulum, or by accidental errors of chronographic recording and measurement. The variations being purely accidental they are thus independent, there being no correlation between the erratic on a particular day and the erratics on the preceding and following days. The erratics contribute to the quantities $\delta_1{}^2 v_n$ and $\delta_2{}^2 v_n$, but these contributions do not form purely accidental series since the values are now correlated with some of the preceding and following values. From (3) it follows that $\delta_1{}^2 v_n$ involves v_n negatively and v_{n-1} positively, whereas $\delta_1{}^2 v_{n-1}$ involves v_n positively and v_{n-1} negatively, and this implies that there is a negative correlation between $\delta_1{}^2 v_n$ and $\delta_1{}^2 v_{n-1}$. Thus, for example, the effect of a positive type A erratic in v_n is to produce a negative contribution to the corresponding $\delta_1{}^2 v_n$ and a positive contribution to the immediately preceding second difference $\delta_1{}^2 v_{n-1}$. A negative erratic in v_{n-1} has the same effect. The statistical result is that if we select all the large negative values of $\delta_1{}^2 v_n$ it will be found that the mean of the immediately preceding second differences $\delta_1^2 v_{n-1}$ is positive. Again, $\delta_1^2 v_n$ and $\delta_1^2 v_{n-2}$ both involve v_{n-1} positively so that there is a positive correlation between $\delta_1^2 v_n$ and $\delta_1^2 v_{n-2}$. Similarly there is a negative correlation between $\delta_1^2 v_n$ and $\delta_1^2 v_{n+1}$ and a positive correlation between $\delta_1^2 v_n$ and $\delta_1^2 v_{n+2}$. In the same way it follows from (4) that $\delta_2^2 v_n$ is correlated negatively with $\delta_2^2 v_{n-2}$ and $\delta_2^2 v_{n+2}$ and positively with $\delta_2^2 v_{n-4}$ and $\delta_2^2 v_{n+4}$.

We denote by $\pm\epsilon_a$ the mean absolute value (*i.e.* the mean value irrespective of sign) of the purely accidental variations of v_n which constitute the type A erratics. Then from (3) and (4) it follows that the resulting mean absolute values of $\delta_1^2 v_n$ and $\delta_2^2 v_n$ in a case in which type A erratics alone are operative are both $\pm\epsilon_a\sqrt{1^2+2^2+1^2} = \pm\sqrt{6}\epsilon_a$.

This application of the square rule of combining errors requires some justification since, as has been pointed out, the $\delta_1^2 v_n$ and $\delta_2^2 v_n$ are not series of independent accidental quantities. But if we divide the second differences into three sub-sequences, the first being $\delta_1^2 v_1$, $\delta_1^2 v_4$, $\delta_1^2 v_7$, etc., the second $\delta_1^2 v_2$, $\delta_1^2 v_5$, $\delta_1^2 v_8$, etc., and the third $\delta_1^2 v_3$, $\delta_1^2 v_6$, $\delta_1^2 v_9$, etc., then for each sub-sequence the values are independent accidental quantities and the mean absolute value for the sub-sequence will be $\pm\sqrt{6}\epsilon_a$. Accordingly the final mean absolute value for the whole sequence will also be $\pm\sqrt{6}\epsilon_a$. A similar argument applies to the $\delta_2^2 v_n$s.

It should also be noticed that we are tacitly assuming that the law of distribution of errors is such that mean absolute values can be combined by the square rule, as is the case for the Normal Error Law. From a strict theoretical point of view it would be more correct to use mean square values instead of mean absolute values. This would increase the labour of computation and it is exceedingly unlikely that any real advantage would be gained in the present application.

Next consider erratics of type B such as may arise from accidental fluctuations of clock rates about mean values or from accidental changes of phase. In this case the first differences are subject to purely accidental and independent variations. We denote the mean absolute value of these variations by $\pm\epsilon_b$. It then follows from (2) and (5) that in a case in which type B erratics are solely operative the mean absolute value of $\delta_1^2 v_n$ is $\pm\epsilon_b\sqrt{1^2+1^2} = \pm\sqrt{2}\epsilon_b$, and the mean absolute value of $\delta_2^2 v_n$ is $\pm\epsilon_b\sqrt{1^2+1^2+1^2+1^2} = \pm 2\epsilon_b$, the argument being justified by a division of the $\delta_1^2 v_n$ and $\delta_2^2 v_n$ into sub-sequences as before. There is again correlation between the neighbouring values of both $\delta_1^2 v_n$ and $\delta_2^2 v_n$.

Thirdly we consider type C erratics which arise from accidental changes of rate. When these are solely operative the second differences $\delta_1^2 v_n$ are purely accidental and independent quantities and we denote their mean absolute value by $\pm\epsilon_c$. From (6) it follows that the resulting mean absolute value of $\delta_2^2 v_n$ is $\pm\epsilon_c\sqrt{1^2+2^2+1^2} = \pm\sqrt{6}\epsilon_c$, the application of the square rule being again justified by the division of the $\delta_2^2 v_n$s into sub-sequences. The neighbouring values of $\delta_2^2 v_n$ are correlated.

In general all three types of erratic will operate simultaneously, and the individual $\delta_1^2 v_n$s and $\delta_2^2 v_n$s will be the sum of the resulting three components. Accordingly if we denote the mean absolute values of $\delta_1^2 v_n$ and $\delta_2^2 v_n$ by ϵ_1 and ϵ_2, we shall have in the general case

$$\epsilon_1^2 = 6\epsilon_a^2 + 2\epsilon_b^2 + \epsilon_c^2 \tag{7}$$

and

$$\epsilon_2^2 = 6\epsilon_a^2 + 4\epsilon_b^2 + 6\epsilon_c^2, \tag{8}$$

where ϵ_a, ϵ_b and ϵ_c are as have been defined, that is, they are respectively the mean absolute values of those components of v_n, $\delta_1 v_{n+\frac{1}{2}}$ and $\delta_1^2 v_n$ which are purely accidental and independent.*

* We have assumed that the erratics of the same type are mutually uncorrelated. It will be noticed that this implies no correlation between $\delta_1^2 v_n$ and $\delta_1^2 v_{n\pm p}$ for values of p greater than 2. This point has been tested for intercomparisons between three Shortt clocks using the same data as is analysed in section 4. For values of p from $p=3$ to $p=10$ the mean correlation coefficients obtained from the three sets of comparisons are all numerically less than ± 0.10 and appear to be accidental.

We now write

$$\epsilon_2/\epsilon_1 = r. \tag{9}$$

The quantity r is a measure of the degree of admixture of the three types of erratic. If erratics of type A are solely operative $r=1$, if erratics of type B are solely operative $r=\sqrt{2}=1\cdot41$, and if erratics of type C are solely operative $r=\sqrt{6}=2\cdot45$. Accordingly a value of r near to unity will indicate that erratics of type A are predominant, and a value near to 2·45 will indicate that we are mainly concerned with erratics of type C. A value of r greater than 1·41 indicates that erratics of type C are certainly present.

It is necessary to notice the effect of erratics of types B and C on the relative clock errors v_n. If the only erratics present were of type A, a plot of relative clock error against calendar date would consist of a number of points scattered in a random manner about a straight line. Thus for comparisons at 24-hour intervals we should have $v_n = \alpha + n\beta + \epsilon_{a,n}$, where α is the initial value of the relative clock error apart from the erratic for day 0, β is the daily rate, and $\epsilon_{a,n}$ is the type A erratic for day n. But if type B erratics are also present, the erratic of this type in $\delta_1 v_{n+\frac{1}{2}}$ being denoted by $\epsilon_{b,n+\frac{1}{2}}$, then

$$v_n = \alpha + n\beta + \sum_{p=0}^{p=n-1} \epsilon_{b,p+\frac{1}{2}} + \epsilon_{a,n},$$

β being now the average value about which the daily rate fluctuates in an accidental fashion. The summation of the type B erratics in the expression for v_n produces a "wandering" in relative clock error, the smoothed plot of this against date being no longer a straight line.

If now type C erratics are superposed, the erratic of this type in $\delta_1^2 v_n$ being denoted by $\epsilon_{c,n}$, we have

$$v_n = \alpha + \sum_{p=0}^{p=n-1} \left[\left\{ \beta + \sum_{q=0}^{q=p} \epsilon_{c,q} \right\} + \epsilon_{b,p+\frac{1}{2}} \right] + \epsilon_{a,n}. \tag{10}$$

An equivalent form of (10) is

$$v_n = \alpha + n\beta + \sum_{p=0}^{p=n-1} [(n-p)\epsilon_{c,p} + \epsilon_{b,p+\frac{1}{2}}] + \epsilon_{a,n}. \tag{10'}$$

The expression (10) represents mathematically the physical conception of the three types of erratic*, namely that the actual rate on any day consists of accidental fluctuations (represented by the terms $\epsilon_{b,p+\frac{1}{2}}$) about a mean value $\beta + \sum_{q=0}^{q=p} \epsilon_{c,q}$ which itself is subject to the accidental changes which constitute the type C erratics.† In addition the phase at which the time is taken from the clock is subject to accidental fluctuations $\epsilon_{a,n}$ about a mean value. Just as we must distinguish between accidental fluctuations of rate about a mean value (type B erratics) and accidental changes of rate (type C erratics), so we must similarly distinguish between accidental fluctuations of phase about a mean value (type A erratics) and accidental changes of phase, the distinguishing feature of the latter (and of the accidental changes of rate) being that the several daily changes are independent, there being no correlation between the change on a given day and those

* But (10) is not a complete representation of clock performance, although it is a good approximation over a limited period of time. The curvature terms have been ignored, and in a long term discussion of performance they would have to be taken into account.

† The initial mean daily rate on day 0 is $\beta + \epsilon_{c,0}$. We may put $\epsilon_{c,0} = 0$, thereby defining β as the initial mean daily rate.

on the preceding and following days. But accidental changes of phase are indistinguishable in their effect from accidental fluctuations of rate, both of these erratics being represented by the accidental errors $\epsilon_{b,\, n+\frac{1}{2}}$ in the first differences. Accordingly they are both grouped together as erratics of type B. Erratics of type A also include the effect of accidental errors of chronographic recording and measurement.

We have so far been considering relative clock errors, obtained by chronographic comparisons of two clocks, but the same conceptions apply to the absolute errors of a single clock. These absolute errors can only be determined by transit observations, and if erratics of type A alone were present, a linear expression $\alpha + n\beta$ could be fitted to the transit observations over a considerable period of time, this expression being then capable of use for the extrapolation of the clock error over a correspondingly considerable period. In practice the type A erratics are of an order of magnitude smaller than the accidental errors of transit observations and so to a large extent they become immaterial. Under these circumstances the possibility of an extended linear extrapolation would provide a useful means of maintaining an accurate time-service over a prolonged period of cloudy weather when few transits are available.

But if erratics of types B and C are present in any appreciable degree, then, as a result of "wandering," the error accumulated by using a linear extrapolation of transit observations can rapidly exceed the limits of tolerance demanded from a first-class service, and this is especially true when appreciable erratics of type C are involved, there being a double summation of these erratics in the expression (10) for the clock error. It is thus a matter of importance to base the maintenance of a time-service on clocks for which the erratics of types B and C are as small as possible. Type C erratics represent the most objectionable defect in a precision clock.

The main object of any analysis of the kind indicated in this section is accordingly to compare the incidence of erratics of types B and C, and especially of type C, for different types of clock, the aim being to base the maintenance of a time-service on clocks for which the "wandering" produced by these types of erratic is least. In the succeeding sections the method of analysis will be applied to free pendulum and quartz clocks, and it will be seen that the latter are definitely much superior to free pendulum clocks, since for the quartz clocks the r-ratios are indistinguishable from unity whereas they are certainly greater than unity for clocks of the free pendulum type.

A practical point to which attention should be directed is that it will be found convenient to compute the quantities $\delta_2{}^2 v_n$ from formula (6) and not from the equivalent form (4). This arises from the circumstance that the second differences $\delta_1{}^2 v_n$ (which have to be formed in any case so as to obtain the mean absolute value ϵ_1) are comparatively small numbers in an actual case, and consequently the calculation of the $\delta_2{}^2 v_n$s from formula (6) is convenient and comparatively rapid.

[Added 1943 June.] It has been remarked to us by Dr. H. R. Hulme that a more sensitive discrimination between the three types of erratic would be achieved by using, instead of $\delta_2{}^2 v_n$, one of the higher members of the expressions

$$\delta_m{}^2 v_n = v_{n+m} - 2v_n + v_{n-m}. \tag{11}$$

If we denote the mean absolute value of $\delta_m{}^2 v_n$ by ϵ_m, then it can be verified that the general formula of which (8) is a particular case is

$$\epsilon_m{}^2 = 6\epsilon_a{}^2 + 2m\epsilon_b{}^2 + \tfrac{1}{3}m(2m^2+1)\epsilon_c{}^2. \tag{12}$$

If we now write

$$\epsilon_m/\epsilon_1 = r_m, \tag{13}$$

then if erratics of type A are solely operative $r_m = 1$, if erratics of type B are solely operative $r_m = \sqrt{m}$, and if erratics of type C are solely operative $r_m = \sqrt{\tfrac{1}{3}m(2m^2+1)}$.

Clearly by taking higher values of m the values of r_m corresponding to the three types of erratic become more widely separated. Also the calculation of three of the quantities ϵ_m would permit of the complete derivation of the separate mean absolute erratics ϵ_a, ϵ_b and ϵ_c*, although this step would possibly only be justified for more extensive and more homogeneous sets of data than those subsequently analysed in this paper.

The tabulation of any series of the general expressions $\delta_m{}^2 v_n$ for $m > 2$ would increase the labour of computation which, as has been said, is comparatively rapid and convenient for $m=2$. It has not been felt to be worth while to undertake the additional computations in the present instance. The investigation was primarily undertaken with the object of comparing the performances, from the point of view of short-period erratics, of free pendulum and quartz clocks, and, as will be seen, the simple analysis based on the use of the $\delta_2{}^2 v_n$s leads at once to the conclusion that quartz clocks are affected by erratics of types B and C to a very much smaller extent than is the case for pendulum clocks. This means that quartz clocks are much less affected by "wandering" than are pendulum clocks, and it seems likely that horology has now reached a stage at which for precision purposes existing types of pendulum clock will be replaced by clocks involving the use of vibrating quartz crystals. It is very improbable whether, at the present stage, justification could be found for the spending of undue time on a more elaborate investigation of the erratics of existing pendulum clocks.

But as soon as more extensive series of intercomparisons of quartz clocks become available, it would certainly be very worth while to embark on a more elaborate investigation of their erratics, and it is possible that improvements may be introduced into free pendulum clocks which would justify a careful discussion of the results obtained with the improved type. For the present, therefore, we have not attempted to extend the original discussion, and we confine ourselves to drawing attention to Dr. Hulme's remark which indicates how future analysis can be elaborated when the data justifies elaboration. In such an elaborated analysis it would probably be necessary to take into account the so-called curvature or acceleration terms which produce small systematically positive or negative values of the second differences. It may even be desirable to consider the possibility of erratic changes in the curvature terms, thereby bringing into consideration a fourth type of erratic which would produce purely accidental variations in the third differences.

4. *Discussion of Intercomparisons of Shortt Clocks.*—We proceed to apply this method of analysis to the intercomparisons between three Shortt clocks over the period 1941 August to December. The clocks are denoted by the letters X, Y and Z. Throughout this period these clocks have been intercompared twice a day within the hours before 10^h and 18^h U.T. The comparisons are reduced to 10^h and 18^h by application of the approximate daily rates. In the case of clock Z two slaves are controlled by the same free pendulum, one being a special slave, used to transmit the rhythmic signals radiated from Rugby, and called the "transmitter." The procedure has been to compare X and Y with the transmitter, these being the clock comparisons which are used to determine the error of the transmitter prior to setting it for the transmission of the radio signals. The error of the transmitter on each occasion is rectified by accelerating or retarding the whole Z ensemble, consisting of free pendulum, standard slave and transmitter, by the required amount, using magnetic correctors for this purpose, the total alteration per minute, when a prescribed current is passed through the correctors, having been measured. The times of passage of current through the correctors are recorded, and

* A more complete analysis would consist of plotting the quantities ϵ_m as functions of m for a number of values of m. This would, in principle, be an extension of the method introduced by Eddington and Plakidis (*M.N.*, 90, 65) and applied by them to a discussion of the irregularities of long-period variables.

in tabulating the data these records have been used to reduce comparisons involving Z to what they would have been if the correctors had never been used and the error of this clock never adjusted. In addition, regular comparisons have been made between the standard slave and transmitter of Z. As these are comparisons between two slaves controlled by the same free pendulum, they would be expected to show only erratics arising from phase variations, and this provides a check on the method of analysis.

The 10^h and 18^h series of clock comparisons have been analysed separately, and the results, meaned over the five months considered, are given in Table I, which gives the

TABLE I

Unit one millisecond/day²

	10^h Series			18^h Series		
	$\delta_1^2 v_n$	$\delta_2^2 v_n$	r	$\delta_1^2 v_n$	$\delta_2^2 v_n$	r
X/Z*t*	± 7·8	±12·0	1·54	± 8·8	±11·8	1·34
Y/X	9·3	16·9	1·82	10·6	15·1	1·42
Y/Z*t*	10·0	16·9	1·69	9·5	15·7	1·65
Z*s*/Z*t*	4·3	4·9	1·14	4·9	5·6	1·14

mean absolute values of $\delta_1^2 v_n$ and $\delta_2^2 v_n$ (these mean absolute values are the ϵ_1 and ϵ_2 of section 3) taken over the whole five months for the various intercomparisons. The values of r are also given.

The letters t and s attached to Z indicate whether the time is taken from the transmitter or from the ordinary slave.

It will be observed that for the comparisons involving Shortt clocks as a whole (free pendulums and slaves) the values of r range from 1·3 to 1·8, whereas for the comparisons Z*s*/Z*t*, in which two slaves under control of the same free pendulum are concerned, the value of r is much smaller. In this last case only phase variations enter, and there is no question of type C erratics being involved. The value of r found (1·14) is near to unity, indicating that the type A erratics (involving fluctuations of phase) are predominant. If the excess (0·14) over unity is real, it would imply that type B erratics (involving accidental changes of phase) are also present. The presence of erratics of type B would be expected to affect the difference between the two slaves by accumulation, thereby leading to a wandering of the difference apart from the day-to-day fluctuation due to erratics of type A. That this wandering actually does take place is shown by Table II, which gives for the 10^h and 18^h series separately the values of v_n for the comparisons between the two slaves (*i.e.* the phase difference between them) meaned over 10-day intervals. The data for the first week in 1942 January are included so as to complete the last 10-day group. An examination of the figures in this table shows that the wandering of phase between the two slaves is exhibited by each series independently. It will be noted that both series show a minimum at the interval September 30–October 9, a maximum towards the end of October, and a rise for the last 10-day interval.

The large difference in phase between the standard slave of clock Z and the transmitter was due to the circumstance that an attempt had been made to phase forward the transmitter by an amount approximately equal to the transmission time of the signals by land line to the radio station. This meant that the operation of the synchronising spring commenced when the pendulum of the transmitter was not in its central position,

TABLE II

Slave slow on Transmitter
10-day means
Unit one millisecond

	10^h	18^h		10^h	18^h
1941 Aug. 1–Aug. 10	102	101	1941 Oct. 20–Oct. 29	102	100
,, 11– ,, 20	100	99	,, 30–Nov. 8	98	97
,, 21– ,, 30	98	100	Nov. 9– ,, 18	95	95
,, 31–Sept. 9	96	96	,, 19– ,, 28	91	94
Sept. 10– ,, 19	96	95	,, 29–Dec. 8	88	90
,, 20– ,, 29	96	97	Dec. 9– ,, 18	85	87
,, 30–Oct. 9	93	94	,, 19– ,, 28	85	87
Oct. 10– ,, 19	99	100	,, 29–Jan. 7	90	91

and it was subsequently felt that this was undesirable. The phasing was accordingly adjusted in 1942 April so that the synchronising arm engaged the spring when the pendulum was central. Although this adjustment has seemed to decrease the comparatively small number of occasions when difficulty was met in keeping the transmitter under its proper control by the master, analysis of the results obtained since the re-adjustment does not show any significant improvement. Over the four months 1942 September to December inclusive the average $\delta_1{}^2v_n$ for Zs/Zt was $\pm 4{\cdot}4$ and the average $\delta_2{}^2v_n$ $\pm 5{\cdot}5$, giving $r = 1{\cdot}25$. The phase differences obtained by 10-day means ranged from 68 to 49 milliseconds,* this range of 19 milliseconds being comparable with, and actually slightly greater than, the range of the values in Table II.

The results for the clock intercomparisons, which have been summarised in Table I, are given in a different form in Table III. In this table the mean absolute values of

TABLE III

Unit one millisecond/day²

	X/Zt			Y/X			Y/Zt			Zs/Zt		
	$\delta_1{}^2v_n$	$\delta_2{}^2v_n$	r	$\delta_1{}^2v_n$	$\delta_2{}^2v_n$	r	$\delta_1{}^2v_n$	$\delta_2{}^2v_n$	r	$\delta_1{}^2v_n$	$\delta_2{}^2v_n$	r
1941 Aug.	±11·9	±16·0	1·34	±10·4	±15·8	1·52	±12·4	±20·6	1·66	±5·8	±6·4	1·10
Sept.	6·2	9·8	1·58	8·4	15·8	1·88	8·4	15·4	1·83	3·8	4·9	1·29
Oct.	6·6	9·0	1·36	11·9	20·1	1·69	13·0	22·1	1·70	4·5	5·0	1·11
Nov.	10·1	13·5	1·34	11·4	17·4	1·53	8·2	12·0	1·46	4·1	5·1	1·24
Dec.	6·7	11·2	1·67	7·5	10·8	1·44	6·6	11·6	1·76	4·8	4·8	1·00
Means	± 8·3	±11·9	1·43	± 9·9	±16·0	1·62	± 9·7	±16·3	1·68	±4·6	±5·2	1·13

$\delta_1{}^2v_n$ and $\delta_2{}^2v_n$ for the 10^h and 18^h series have been meaned and the results are grouped according to calendar months. It will be seen that for the comparisons involving free pendulums as well as slaves there is a distinct variation from month to month in the mean absolute values of $\delta_1{}^2v_n$ and $\delta_2{}^2v_n$ and in the ratios r. It seems that the erratics of the three types are variable in amount from time to time, and that consequently the

* It should be noticed that the centring of the synchronising units of two slaves under the control of the same free pendulum does not necessarily produce a recorded phase difference of zero, since the recorded phase difference includes the differences of adjustment and operation of the two seconds switch mechanisms.

mean values in the last row of the table must be taken as representing average performance over the five months considered. At the same time it appears from Table I, in which the average values for the 10^h and 18^h series are given separately, that the agreement between the two series is good and that consequently the final means have a statistical significance.

But for the comparisons between the two slaves Zs and Zt, which are both controlled by the same master, the accordance of the figures from month to month is good, indicating that we are here dealing with erratics which are more homogeneous with respect to time. In particular the monthly values of the r-ratios are more accordant for the comparisons between these two slaves. This is probably due to the circumstance that the effect of erratics of one type (type A) is here predominant.

The final mean absolute values of $\delta_1^2 v_n$ and $\delta_2^2 v_n$ for the comparisons between Y and X and between Y and Z (transmitter) are practically the same. This means that the performance of X and Z are the same, and accordingly we can get the mean absolute values of $\delta_1^2 v_n$ and $\delta_2^2 v_n$ for either by dividing the values for the comparisons between them by $\sqrt{2}$. We thus get for either of these clocks $\delta_1^2 v_n = \pm 8 \cdot 3/\sqrt{2} = \pm 5 \cdot 9$ and $\delta_2^2 v_n = \pm 11 \cdot 9/\sqrt{2} = \pm 8 \cdot 4$, giving $r = 1 \cdot 42$. And for the comparison of either with Y we have $\delta_1^2 v_n = \pm 9 \cdot 8$ and $\delta_2^2 v_n = \pm 16 \cdot 2$, so that for Y alone we have

$$\delta_1^2 v_n = \pm\sqrt{(9 \cdot 8)^2 - (5 \cdot 9)^2} = \pm 7 \cdot 8 \quad \text{and} \quad \delta_2^2 v_n = \sqrt{(16 \cdot 2)^2 - (8 \cdot 4)^2} = \pm 13 \cdot 9,$$

giving $r = 1 \cdot 78$.

We thus get the following values for the three clocks:—

Table IV

	$\delta_1^2 v_n$	$\delta_2^2 v_n$	r
X, Zt	$\pm 5 \cdot 9$	$\pm 8 \cdot 4$	$1 \cdot 4$
Y	$\pm 7 \cdot 8$	$\pm 13 \cdot 9$	$1 \cdot 8$

The high value of the r-ratio for clock Y suggests that the inferior performance of this clock, as indicated by the value $\pm 7 \cdot 8$ for $\delta_1^2 v_n$, is due, in part at least, to the increased incidence of erratics of type C. Now type C erratics must be purely properties of free pendulums, and it accordingly would be of interest if we could separate the effects of the slaves and obtain results appertaining to the free pendulums alone. We can do this by assuming that the performances of all slaves are the same, in which case the mean absolute values of $\delta_1^2 v_n$ and $\delta_2^2 v_n$ for any slave are obtained by dividing the results for the comparisons Zs/Zt by $\sqrt{2}$.

This gives for a slave:

$$\delta_1^2 v_n = \pm 4 \cdot 6/\sqrt{2} = \pm 3 \cdot 3,$$
$$\delta_2^2 v_n = \pm 5 \cdot 2/\sqrt{2} = \pm 3 \cdot 7.$$

We can now deduce results appertaining to the free pendulums alone. Thus for the free pendulums of X and Z we get $\delta_1^2 v_n = \pm\sqrt{(5 \cdot 9)^2 - (3 \cdot 3)^2} = \pm 4 \cdot 9$. In this way we get the results given in Table V.

Table V

Results for Free Pendulums

	$\delta_1^2 v_n$	$\delta_2^2 v_n$	r
X, Z	$\pm 4 \cdot 9$	$\pm 7 \cdot 5$	$1 \cdot 5$
Y	$\pm 7 \cdot 1$	$\pm 13 \cdot 4$	$1 \cdot 9$

Taking Table V as a correct representation of the free pendulum performances, the question arises whether any erratics of type A are still present. Such erratics could only enter into the performance of a free pendulum as a result of a variable action of the switch which provides the synchronising currents controlling the slaves. If the time elapsing between the instant at which the gravity arm leaves the impulse wheel mounted on the free pendulum and the instant at which the switch arm makes contact (thereby sending out the synchronising signal and at the same time operating the free pendulum remontoire) be variable, type A erratics will be introduced.

We proceed to deduce values of ϵ_a, ϵ_b and ϵ_c for the free pendulums from the data in Table V on two hypotheses which probably represent extremes. In the first place let us assume that the free pendulum switch is perfect and introduces no erratics of type A. The mean absolute values of $\delta_1^2 v_n$ and $\delta_2^2 v_n$ given in Table V can then be used to deduce the separate values ϵ_b and ϵ_c for the erratics of types B and C by solving equations (7) and (8) of section 3 with ϵ_a put equal to zero. This gives the following results for the free pendulums:—

	Free Pendulums X and Z	Free Pendulum Y	
ϵ_a	0	0	
ϵ_b	$\pm 3{\cdot}3$	$\pm 3{\cdot}9$	msecs./day
ϵ_c	$\pm 1{\cdot}4$	$\pm 4{\cdot}4$	msecs./day^2

Secondly, let us assume that the free pendulum switch introduces the same type A erratics as are introduced by the slaves. We have found that for a single slave $\delta_1^2 v_n = \pm 3{\cdot}3$ and $\delta_2^2 v_n = \pm 3{\cdot}7$. Solving with $\epsilon_c = 0$, this gives $\epsilon_a = \pm 1{\cdot}2$ and $\epsilon_b = \pm 1{\cdot}2$. We thus assume $\epsilon_a = \pm 1{\cdot}2$ for free pendulums and again solve from the data in Table V. There results:

	Free Pendulums X and Z	Free Pendulum Y	
ϵ_a	$\pm 1{\cdot}2$	$\pm 1{\cdot}2$	msecs.
ϵ_b	$\pm 2{\cdot}4$	$\pm 3{\cdot}2$	msecs./day
ϵ_c	$\pm 2{\cdot}1$	$\pm 4{\cdot}7$	msecs./day^2

Since the slaves cannot introduce erratics of type C, it follows that a value of ϵ_c found for a free pendulum will apply to the clock as a whole. It will be noticed that in both solutions the type C erratics (as indicated by the value of ϵ_c) are enhanced for clock Y, as is indeed shown by the high values of r for this clock in Tables IV and V. These erratics would be expected to influence the long-period performance of the clock owing to the cumulative nature of their effect, and it is in fact true that the long-term performance of Y, as shown by plots of its error derived from transit observations, is inferior to that of either of the other two clocks.

It is well known that when the errors of free pendulum clocks, as determined by transit observations, are plotted over an interval of several months, the fact emerges that these clocks are liable to occasional large changes of rate which may exceed 10 milliseconds per day. The question arises as to whether these occasional large changes of rate can be regarded as extreme cases of the normal type C erratic or whether they are to be classed separately as occasional catastrophic changes in the clock. There seems to be no reason for regarding the occasional large changes of rate as anything more than extreme cases of erratics of type C. If we assume in the first place that for a good clock erratics of this type contribute a mean absolute daily change of rate amounting to $\pm 1{\cdot}4$ msecs./day^2 (this being the value of ϵ_c found above for clocks X and Z on the first limiting assumption that the free pendulum itself does not contribute to type A erratics), then the probable error of the daily change of rate is $\pm 1{\cdot}4 \times {\cdot}8453 = \pm 1{\cdot}2$. The probable error of the accumulated change of rate in say five days is $\sqrt{5}$ times this or $\pm 2{\cdot}7$, and assuming a normal frequency distribution, this means that the probability that a change of rate amounting to more than 10 milliseconds per day will develop in any given five-day interval is 1/80, leading to an expectation of about one such change in a year. The accidental errors of transit determinations of clock error being what they are, a change of rate of this kind, developing in five days, would, if transit observations were taken as the sole criterion of performance, be interpreted as an instantaneous change of rate.

If, however, we take $\epsilon_c = \pm 2{\cdot}1$ msecs./day^2, this being the value found for clocks X and Z on the second limiting assumption that the free pendulums contribute equally with the slaves to the total type A erratics, a similar calculation gives an expectation of about six or seven changes of rate per year exceeding 10 msecs./day over a five-day interval.

As regards the actual performances of clocks X and Z over the period 1941 August to December, an examination of transit plots and plots of clock comparisons showed that the change of rate accumulated in five days by these two clocks exceeded 10 msecs./day on one and four occasions respectively in the five-month period. This, for what it is worth, seems to favour the second hypothesis, namely that the free pendulums contribute equally with the slaves to type A erratics. But evidence of this kind should not be stressed, as the excess of the number of actual changes of rate over the expectation calculated on the basis of $\epsilon_c = \pm 1{\cdot}4$ may be a result of sampling error and of the well-known tendency in experimental data for extreme errors to be more frequent than is predicted by the normal Law of Errors. It should be noticed that when the erratics of type C are small the calculated expectation is very sensitive to the exact value of ϵ_c used.

For clock Y the values of ϵ_c of $\pm 4{\cdot}4$ and $\pm 4{\cdot}7$ found above lead respectively to expectations over five months of thirteen and fourteen changes of rate exceeding 10 msecs./day and developing in five days. The actual number of such changes for 1941 August to December found by the examination of transit plots and clock comparisons was ten.

Owing to the various processes to which the mean values of $\delta_1{}^2 v_n$ and $\delta_2{}^2 v_n$ given in Table III have been subjected, stress must not be laid on the exact numerical values obtained above, and the results must be regarded as indicative of order of magnitude only. The conclusion seems to emerge that for good Shortt clocks, of which X and Z may be regarded as examples, the average daily changes of rate represented by type C erratics are of the order of magnitude of one or two milliseconds per day per day. For a clock such as Y the type C erratics are more pronounced. It has, unfortunately, not so far proved possible to associate the occurrence of enhanced erratics of this type with any particular kind of mechanical defect.

It is perhaps necessary to point out that the method of analysis used above, although possessing the advantage of discriminating between the three types of erratics discussed, cannot be expected to lead in itself to a complete picture of the performance of the clocks. One well-known feature of Shortts which is ignored by the analysis is the systematic acceleration often found—that is, a systematic tendency to a change of rate of the same sign persisting for periods of months at a time. Such "curvature terms" seem to be always less than one millisecond per day per day, and it has accordingly been possible to ignore them in the present investigation. Their origin is unknown, and one of the puzzling features about them is their comparatively sudden appearance, their persistence over a period of months, and then their disappearance or comparatively sudden change of value. Like the erratics of type C, curvature terms are probably associated with some particular mechanical features in the clocks, but in neither case has the origin of the disturbance been traced.

5. *Results for Quartz Clocks.*—It is of interest to compare the results obtained for Shortt clocks with corresponding results using the same method of analysis for quartz clocks, these consisting of phonic motors controlled *via* frequency dividers by vibrating quartz crystals. There are two such clocks (Q2 and Q6) at the National Physical Laboratory, and the time of reception of the Rugby 10^h signal is recorded daily on each of them. These measures, which are forwarded to Greenwich among other recipients,

have been used for the derivation of intercomparisons between the two clocks. But this set of data is not as satisfactory for statistical purposes as the Shortt clock intercomparisons discussed in the preceding section. In the first case only one series (at 10^h U.T.) is available, whereas for the Shortt clocks there was the 18^h series as well. There have been occasions when the 10^h signal from Rugby was not radiated, leading to gaps in the data. Further, Q6 stopped on 1941 September 17 and Q2 on October 28.

The unsatisfactory nature of the data makes it all the more remarkable that the intercomparisons between the two quartz clocks give very consistent results, as is shown by Table VI.

TABLE VI

Intercomparisons of Quartz Clocks

Unit one millisecond/day²

Q2/Q6

	$\delta_1{}^2v_n$	$\delta_2{}^2v_n$	r
1941 Aug. 14–Sept. 17	±10·4	± 9·6	0·92
Sept. 19–Oct. 27	8·1	8·3	1·02
Oct. 29–Nov. 30	8·2	9·1	1·11
Dec. 1–Dec. 31	10·0	10·3	1·03
Means	± 9·2	± 9·3	1·01

The outstanding feature of the results is that the value of r is very close to unity. This means that the erratics must be predominantly of type A, and on comparison with Table III it appears that this predominance is actually more pronounced for the intercomparison of the two quartz clocks than for the case of two Shortt slaves (Zs and Zt) under the control of the same free pendulum.

The mean absolute value of $\delta_1{}^2v_n$ is $\pm 9{\cdot}2$ for the quartz clock intercomparisons. Assuming that this is equally distributed between the two clocks, we get for each $\delta_1{}^2v_n = \pm 9{\cdot}2/\sqrt{2} = \pm 6{\cdot}5$. Assuming that the whole of this is due to erratics of type A, we get for the mean absolute variation in phase combined with the mean absolute error of chronographic recording and measurement the value $\epsilon_a = \pm 6{\cdot}5/\sqrt{6} = \pm 2{\cdot}7$ milliseconds. Genuine phase variations could conceivably arise from erratic action of the commutator switches by which the seconds signals are taken from the phonic motors, and it is possible that errors of chronographic recording and measurement contribute to the total type A erratics.

In so far as long-term performance is the result of the accumulation of short-period erratics of type C and, to a lesser extent, of type B, it would appear that for quartz clocks it should be very good. From the data given in Table VI it does not seem likely that the value of r for the intercomparison between the two quartz clocks exceeds 1·05. A value of r less than 1·05 combined with a mean absolute $\delta_1{}^2v_n$ equal to $\pm 9{\cdot}2$ means that $\epsilon_c < 1{\cdot}4$ msecs./day², and so for a single quartz clock the mean absolute value of the type C erratics would be less than $\pm 1{\cdot}4/\sqrt{2} = \pm 1{\cdot}0$ milliseconds per day per day. This represents a performance comparable to, and probably better than, the very best pendulum clocks.

When the long-term performance of quartz clocks is assessed by comparison with transit observations it transpires that, like clocks of the free pendulum type, they possess curvature terms indicating a progressive continuous change of rate. This feature is especially prominent for Q6. It also appears that sudden large changes of rate can take place. In the cases of the two quartz clocks here considered there were no large sudden changes of rate during the five months covered by the investigation, but Q6 changed rate on 1942 January 20–21 and Q2 on 1942 February 23–24. In the case of Q6 the clock suddenly commenced to run faster by about 30 msecs./day, and the change in Q2 was about 20 msecs./day in the opposite sense. These changes of rate are far too large to be regarded as extreme cases of normal type C erratics (which, as we have seen, must be very small for quartz clocks) and must be regarded as comparatively rare catastrophic changes in the periods of vibration of the crystals.

[Added 1943 April]

Some further information as to the performance of quartz clocks is given by three clocks at the Post Office Radio Branch Laboratories. These clocks are designated IV A, IV B and IV C, and the times of reception of the Rugby 10^h signals on them have been forwarded to Greenwich. Analysis of recent data gives the figures in Table VII.

Table VII

Intercomparison of P.O. Quartz Clocks 1942 *December* 11–1943 *March* 30

Unit one millisecond/day^2

	$\delta_1^2 v_n$	$\delta_2^2 v_n$	r
IV A/IV B	±7·1	±7·1	1·00
IV B/IV C	5·7	5·6	0·98
IV C/IV A	5·2	5·3	1·02

Again the values of r are close to unity, indicating that the erratics are predominantly of type A. It will be noticed that the mean absolute values of $\delta_1^2 v_n$ and $\delta_2^2 v_n$ are smaller than was the case in Table VI, indicating that for the P.O. clocks the erratics are smaller. It will also be noticed that the values for IV A/IV B are larger than for the comparison of either of these clocks with IV C, indicating that the erratics of IV C are smaller. As far as the erratics of type A are concerned, it appears that IV C is at least comparable to the best pendulum clocks and probably better than any of them. These erratics of type A do not of course affect the long-period performance of the clocks concerned by accumulation. But when we come to erratics of types B and C the great superiority of these quartz clocks over clocks of the pendulum type is very clear.

We are indebted to the Director of the National Physical Laboratory and to the Head of the Post Office Radio Department for their kind permission to utilise the results of quartz clock intercomparisons in the above discussion.