

DYNAMICAL FRICTION

III. A MORE EXACT THEORY OF THE RATE OF ESCAPE
OF STARS FROM CLUSTERS

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Received March 26, 1943

ABSTRACT

A more exact estimate of the rate of escape of stars from clusters is made than in an earlier paper by properly allowing for the dependence of the coefficient of dynamical friction on the velocity. It is found that the probability that a star will have acquired the necessary velocity of escape (assumed to be equal to twice the root mean square velocity of the stars in the system) in a time τ (measured in units of the time of relaxation of the system) is given by

$$Q(\tau) = (1 - e^{-0.0075\tau}) .$$

On this basis, half-lives for galactic clusters of the order of 3×10^9 years are provided for, and it is further concluded that dynamical friction provides the principal mechanism for the continued existence of galactic clusters like the Pleiades for times of the order of 3×10^9 years.

1. *Introduction.*—In the two earlier papers of this series on “Dynamical Friction”¹ we have shown how stars must experience dynamical friction during their motion and how in the rate of escape of stars from clusters we can look for direct evidence for the operation of this force. However, in estimating this rate of escape of stars from clusters in II we assumed (for the sake of simplicity) that the coefficient of dynamical friction, η , and the diffusion coefficient, q (in the *velocity space*), were both constants. On the other hand, an explicit evaluation of the coefficient of dynamical friction on the two-body approximation for stellar encounters gave

$$\eta = 8\pi m^2 G^2 \left(\log_e \left[\frac{D_0 |\mathbf{u}|^2}{2Gm} \right] \right) \frac{1}{|\mathbf{u}|^3} \int_0^{|\mathbf{u}|} N(v) dv . \quad (1)$$

According to this formula,

$$\eta \rightarrow \eta_0 = \text{constant as } |\mathbf{u}| \rightarrow 0 \quad (2)$$

and

$$\eta \rightarrow \text{constant } |\mathbf{u}|^{-3} \text{ as } |\mathbf{u}| \rightarrow \infty . \quad (3)$$

In view particularly of (3) it does not appear entirely satisfactory that we ignore the dependence of η on $|\mathbf{u}|$. It is therefore a matter of some importance that we make proper allowance for the variation of η with $|\mathbf{u}|$ according to equation (1) in estimating the rate of escape of stars from clusters. This is the main purpose of this paper.

2. *The general theory of the rate of escape of stars from clusters allowing for the variation of η with $|\mathbf{u}|$.*—As in II, we shall suppose that, in order that a star may escape from a cluster, it is only necessary that it acquire a velocity greater than (or equal to) a certain critical velocity, v_∞ , which we may call the “velocity of escape.” On this assumption the probability that a star will have acquired the necessary velocity for escape during a certain time can be determined very simply in terms of the probability, $p(v_0, t) dt$, that a star having initially a velocity $|\mathbf{u}| = v_0$ at time $t = 0$ will acquire for the *first time* the velocity $|\mathbf{u}| = v_\infty$ between t and $t + dt$. And as we have already explained in

¹ *Ap. J.*, 97, 255 and 266, 1943. These two papers will be referred to as “I” and “II,” respectively.

II, § 2, this probability function $p(v_0, t)$ can be derived in turn from the spherically symmetric solution of the equation

$$\frac{\partial W}{\partial t} = \text{div}_{\mathbf{u}} (q \text{grad}_{\mathbf{u}} W + \eta W \mathbf{u}), \quad (4)$$

which satisfies the boundary conditions

$$W(|\mathbf{u}|, t) = 0 \quad \text{for } |\mathbf{u}| = v_{\infty} \quad \text{for all } t > 0 \quad (5)$$

and

$$W(|\mathbf{u}|, t) \rightarrow \frac{1}{4\pi v_0^2} \delta(|\mathbf{u}| - v_0) \quad \text{as } t \rightarrow 0, \quad (6)$$

where δ stands for the usual δ -function of Dirac.

For the case under discussion we have (I, eq. [36])

$$\eta = 8\pi N m^2 G^2 \left(\log_e \left[\frac{D_0 \overline{|\mathbf{u}|^2}}{2Gm} \right] \right) \frac{1}{|\mathbf{u}|^3} [\Phi(j|\mathbf{u}|) - j|\mathbf{u}|\Phi'(j|\mathbf{u}|)], \quad (7)$$

where Φ and Φ' denote, respectively, the error function and its derivative. Further, in equation (7), j is the parameter which occurs in the assumed Maxwellian distribution of velocities:

$$\frac{j^3}{\pi^{3/2}} e^{-j^2|\mathbf{u}|^2} d\mathbf{u}; \quad j = \left(\frac{3}{2\overline{|\mathbf{u}|^2}} \right)^{1/2}. \quad (8)$$

The formula (7) for η can be written more conveniently as

$$\eta = \eta_0 \nu(j|\mathbf{u}|), \quad (9)$$

where

$$\eta_0 = 8\pi N m^2 G^2 \left(\log_e \left[\frac{D_0 \overline{|\mathbf{u}|^2}}{2Gm} \right] \right) \left(\frac{3}{2\overline{|\mathbf{u}|^2}} \right)^{3/2} \frac{4}{3\pi^{1/2}} \quad (10)$$

and

$$\nu(\rho) = \frac{3\pi^{1/2}}{4} \rho^{-3} [\Phi(\rho) - \rho\Phi'(\rho)]. \quad (11)$$

With $\nu(\rho)$ defined in this manner,

$$\nu(\rho) \rightarrow 1 \quad \text{as } \rho \rightarrow 0 \quad (12)$$

and

$$\nu(\rho) \sim \frac{3\pi^{1/2}}{4} \rho^{-3} \quad \text{as } \rho \rightarrow \infty. \quad (13)$$

Again, since q and η are quite generally related according to

$$q = \frac{1}{3} \overline{|\mathbf{u}|^2} \eta, \quad (14)$$

we have

$$q = \frac{1}{3} \overline{|\mathbf{u}|^2} \eta_0 \nu(j|\mathbf{u}|). \quad (15)$$

The function $\nu(\rho)$ is tabulated in Table 1.

Returning to equation (4), we now introduce a change of the independent variables \mathbf{u} and t . Let

$$\tau = \eta_0 t; \quad \mathbf{u} = \left(\frac{2}{3} \overline{|\mathbf{u}|^2} \right)^{1/2} \boldsymbol{\rho}. \quad (16)$$

Equation (4) now takes the dimensionless form

$$\frac{\partial W}{\partial \tau} = \text{div}_{\mathbf{p}} \left[\frac{1}{2} \nu (|\mathbf{p}|) \text{grad}_{\mathbf{p}} W + \nu (|\mathbf{p}|) W \mathbf{p} \right]. \tag{17}$$

For a spherically symmetric solution $W(|\mathbf{p}|, \tau)$ equation (17) reduces to

$$\rho \frac{\partial w}{\partial \tau} = \frac{\partial}{\partial \rho} \left[\nu(\rho) \left\{ \frac{1}{2} \rho \frac{\partial w}{\partial \rho} + \left(\rho^2 - \frac{1}{2} \right) w \right\} \right], \tag{18}$$

TABLE 1
THE FUNCTIONS $\nu(\rho)$ AND $-d \log \nu/d\rho$

ρ	$\nu(\rho)$	$-\frac{d \log \nu}{d\rho}$	ρ	$\nu(\rho)$	$-\frac{d \log \nu}{d\rho}$
0.00.....	1.00000	0.0000	1.45.....	0.33133	1.3062
0.05.....	0.99850	0.0600	1.50.....	.31026	1.3206
0.10.....	0.99402	0.1199	1.55.....	.29035	1.3323
0.15.....	0.98661	0.1795	1.60.....	.27157	1.3413
0.20.....	0.97634	0.2389	1.65.....	.25392	1.3477
0.25.....	0.96332	0.2979	1.70.....	.23734	1.3515
0.30.....	0.94770	0.3563	1.75.....	.22183	1.3528
0.35.....	0.92962	0.4141	1.80.....	.20732	1.3518
0.40.....	0.90927	0.4712	1.85.....	.19379	1.3486
0.45.....	0.88684	0.5274	1.90.....	.18117	1.3432
0.50.....	0.86257	0.5827	1.95.....	.16943	1.3358
0.55.....	0.83666	0.6369	2.00.....	.15852	1.3267
0.60.....	0.80936	0.6899	2.05.....	.14839	1.3159
0.65.....	0.78090	0.7417	2.10.....	.13898	1.3036
0.70.....	0.75152	0.7921	2.15.....	.13025	1.2901
0.75.....	0.72145	0.8409	2.20.....	.12216	1.2754
0.80.....	0.69093	0.8881	2.25.....	.11466	1.2597
0.85.....	0.66016	0.9336	2.30.....	.10770	1.2433
0.90.....	0.62936	0.9772	2.35.....	.10126	1.2262
0.95.....	0.59872	1.0188	2.40.....	.09528	1.2087
1.00.....	0.56842	1.0584	2.45.....	.08973	1.1907
1.05.....	0.53861	1.0958	2.50.....	.08458	1.1726
1.10.....	0.50944	1.1309	2.55.....	.07980	1.1544
1.15.....	0.48104	1.1636	2.60.....	.07536	1.1361
1.20.....	0.45350	1.1939	2.65.....	.07123	1.1179
1.25.....	0.42692	1.2216	2.70.....	.06739	1.0999
1.30.....	0.40137	1.2468	2.75.....	.06381	1.0820
1.35.....	0.37689	1.2693	2.80.....	.06048	1.0645
1.40.....	0.35354	1.2891	2.85.....	0.05737	1.0472

where we have written

$$\rho = |\mathbf{p}| \quad \text{and} \quad w = W \rho. \tag{19}$$

According to equations (5) and (6), we require a solution of equation (18) which satisfies the boundary conditions

$$w(\rho, \tau) = 0 \quad \text{for both} \quad \rho = 0 \quad \text{and} \quad \rho = \rho_{\infty} \quad \text{for all} \quad \tau > 0 \tag{20}$$

and

$$w(\rho, \tau) \rightarrow \frac{1}{4\pi\rho_0} \delta(\rho - \rho_0) \quad \text{as} \quad \tau \rightarrow 0. \tag{21}$$

Now, equation (18) is separable in the variables ρ and τ . Accordingly, we write

$$w = e^{-\lambda\tau}\phi(\rho), \quad (22)$$

where λ is, for the present, an unspecified constant; we then obtain for ϕ the differential equation

$$\frac{d}{d\rho} \left[\nu(\rho) \left\{ \frac{1}{2}\rho \frac{d\phi}{d\rho} + (\rho^2 - \frac{1}{2})\phi \right\} \right] + \lambda\rho\phi = 0. \quad (23)$$

If we now let

$$\phi = e^{-\rho^2/2}\psi, \quad (24)$$

equation (23) reduces to

$$\frac{d^2\psi}{d\rho^2} + \frac{d \log \nu}{d\rho} \frac{d\psi}{d\rho} + \left[2 \frac{\lambda}{\nu(\rho)} + 3 - \rho^2 - \frac{d \log \nu}{d\rho} \left(\frac{1}{\rho} - \rho \right) \right] \psi = 0. \quad (25)$$

It is now seen that, in order that a solution of the foregoing equation may vanish both at $\rho = 0$ and at $\rho = \rho_\infty$, it is necessary that λ take one of an infinite enumerable set of discrete values

$$\lambda_1, \lambda_2, \dots, \lambda_n, \dots, \quad (26)$$

which may properly be called the "characteristic values" of the problem. Further, if

$$\psi_1, \psi_2, \dots, \psi_n, \dots \quad (27)$$

denote the solutions of equation (25) which satisfy the boundary conditions (20) at $\rho = 0$ and at $\rho = \rho_\infty$ and belong, respectively, to the values $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$, then it can be readily verified that these solutions form a complete set of orthogonal functions. Without loss of generality we can therefore suppose that these functions are all properly normalized. Consequently, in terms of the fundamental solutions

$$w_n = e^{-\lambda_n\tau} e^{-\rho^2/2} \psi_n(\rho) \quad (28)$$

which satisfy the boundary conditions (20) we can construct solutions which will satisfy any further arbitrary boundary condition for $\tau = 0$. Thus, the solution

$$w = \frac{1}{4\pi\rho_0} e^{-(\rho^2 - \rho_0^2)/2} \sum_{n=1}^{\infty} e^{-\lambda_n\tau} \psi_n(\rho) \psi_n(\rho_0) \quad (29)$$

clearly satisfies the boundary condition (21) for $\tau = 0$. Corresponding to the solution (29) for w , we have

$$W = \frac{1}{4\pi\rho\rho_0} e^{-(\rho^2 - \rho_0^2)/2} \sum_{n=1}^{\infty} e^{-\lambda_n\tau} \psi_n(\rho) \psi_n(\rho_0). \quad (30)$$

Using the foregoing solution for W , we can write down the probability function $p(\rho_0, \tau)$. For, since

$$p(\rho_0, \tau) = -2\pi\rho_\infty^2 \nu(\rho_\infty) \left(\frac{\partial W}{\partial \rho} \right)_{\rho=\rho_\infty}, \quad (31)$$

we have

$$p(\rho_0, \tau) = \frac{\rho_\infty}{2\rho_0} \nu(\rho_\infty) e^{-(\rho_\infty^2 - \rho_0^2)/2} \sum_{n=1}^{\infty} e^{-\lambda_n\tau} \left(-\frac{d\psi_n}{d\rho} \right)_{\rho=\rho_\infty} \psi_n(\rho_0). \quad (32)$$

To obtain the total probability $Q(\rho_0, \tau)$ that a star having initially a velocity corresponding to ρ_0 will have acquired during the time τ a velocity corresponding to ρ_∞ , we have simply to integrate equation (32) from 0 to τ . Thus,

$$Q(\rho_0, \tau) = \int_0^\tau p(v_0, \tau) d\tau; \quad (33)$$

or, using equation (32), we have

$$Q(\rho_0, \tau) = \frac{\rho_\infty}{2\rho_0} \nu(\rho_\infty) e^{-(\rho_\infty^2 - \rho_0^2)/2} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (1 - e^{-\lambda_n \tau}) \left(-\frac{d\psi_n}{d\rho} \right)_{\rho=\rho_\infty} \psi_n(\rho_0). \quad (34)$$

Finally, to obtain the expectation, $Q(\tau)$, that an "average" star will have acquired the necessary velocity for escape during a time τ , we must average the foregoing expression over all ρ_0 . The final result can therefore be expressed in the form

$$Q(\tau) = \sum_{n=1}^{\infty} Q_n(\tau), \quad (35)$$

where

$$Q_n(\tau) = A_n (1 - e^{-\lambda_n \tau}) \quad (36)$$

and

$$A_n = \frac{1}{2\lambda_n} \rho_\infty \nu(\rho_\infty) e^{-\rho_\infty^2/2} \left(-\frac{d\psi_n}{d\rho} \right)_{\rho=\rho_\infty} \left[\frac{e^{\rho_0^2/2}}{\rho_0} \psi_n(\rho_0) \right]. \quad (37)$$

3. *Numerical results.*—Now, since in a star cluster the root mean square velocity of escape is twice the root mean square velocity of the stars in the system,² it is clear that the values of ρ_∞ which come under discussion are in the general neighborhood of

$$\rho_\infty = \sqrt{6} \sim 2.45. \quad (38)$$

As we shall see presently, for these values of ρ_∞ , $Q(\tau)$ can be represented with ample accuracy by the first term on the right-hand side of equation (35). Accordingly, it would be sufficient to specify the lowest characteristic value of λ (for a given ρ_∞) and the normalized characteristic function ψ_1 belonging to it. For this purpose the following procedure appears suitable:

First we assign a value for λ and look for a solution $\Psi(\rho)$ of equation (25) whose behavior near the origin can be described by a series expansion of the form

$$\Psi = \rho + a_3 \rho^3 + a_5 \rho^5 + \dots \quad (39)$$

For any prescribed value of λ the coefficients a_3 , a_5 , etc., can be successively determined from the differential equation (25) for Ψ . Thus a_3 and a_5 are found to be

$$\left. \begin{aligned} a_3 &= -\frac{1}{6} (3 + 2\lambda), \\ a_5 &= \frac{1}{24} [2.2 - 1.2\lambda + \frac{1}{6} (3 + 2\lambda) (0.6 + 2\lambda)]. \end{aligned} \right\} \quad (40)$$

The higher coefficients can be similarly found, but the explicit formulae in terms of λ have no particular interest. However, it is clear that, starting a solution near the origin with a series expansion of the form (39), we can continue it for larger values of ρ by

² Cf., e.g., S. Chandrasekhar, *Principles of Stellar Dynamics*, pp. 206–207, University of Chicago Press 1942.

standard numerical methods until we reach the first zero $\rho_\infty(\lambda)$ of Ψ . Conversely, for the value of ρ_∞ thus determined, the solution Ψ satisfies the necessary boundary conditions at the origin and at $\rho = \rho_\infty$. The initially assigned value of λ is therefore the lowest characteristic value of λ for this value of ρ_∞ . If we now let α denote the normalizing factor for the solution Ψ determined in this fashion, we can express A_1 (cf. eq. [37]) alternatively in the form

$$A_1 = \frac{\alpha^2}{2\lambda} \rho_\infty \nu(\rho_\infty) e^{-\rho_\infty^2/2} \left(-\frac{d\Psi}{d\rho} \right)_{\rho=\rho_\infty} \left[\frac{e^{\rho_0^2/2}}{\rho_0} \Psi(\rho_0) \right]. \quad (41)$$

Now, it is found that, for the values of ρ_∞ in the neighborhood of 2.45, λ is very small and A_1 is very close to unity. Thus, for $\lambda = 0.0075$, a numerical integration of equation (25) gave

$$\rho_\infty = 2.4518; \quad A_1 = 0.9966 \quad (\lambda_1 = 0.0075). \quad (42)$$

Accordingly, for this case, equation (35) takes the explicit form

$$Q(\tau) = 0.9966 (1 - e^{-0.0075\tau}) + \sum_{n=2}^{\infty} A_n (1 - e^{-\lambda_n\tau}) \quad (\rho_\infty = 2.4518). \quad (43)$$

Since $Q(\tau)$ must, by definition, approach unity as $\tau \rightarrow \infty$, it is clear that

$$\sum_{n=2}^{\infty} A_n = 0.0034 \quad (\rho_\infty = 2.4518). \quad (44)$$

Again, since λ_2 must be in the neighborhood of 2 (cf. II, p. 270) and the higher characteristic values still larger, it is evident that, for $\tau > 5$, sufficient accuracy will be provided by

$$Q(\tau) = 1 - e^{-0.0075\tau} \quad (\tau \gtrsim 5). \quad (45)$$

The situation for other values of ρ_∞ is quite similar, as is apparent from Table 2, where the results for a few values of λ are collected together.

TABLE 2
THE RATE OF ESCAPE OF STARS FROM CLUSTERS INCLUDING DYNAMICAL FRICTION
AND ALLOWING FOR ITS DEPENDENCE ON $|u|$

λ	ρ_∞	$\nu(\rho_\infty)$	$-\Psi'(\rho_\infty)$	α^2	$\left[\frac{e^{\rho_0^2/2}}{\rho_0} \Psi(\rho_0) \right]$	$Q_1(\tau)$
0.0025	2.6642	0.07011	0.4077	2.3083	0.9891	1.0000 $(1 - e^{-0.0025\tau})$
.0050	2.5320	.08148	.5183	2.3458	.9813	0.9978 $(1 - e^{-0.0050\tau})$
.0075	2.4518	.08954	.5932	2.3787	.9748	0.9966 $(1 - e^{-0.0075\tau})$
.0100	2.3936	.09601	.6503	2.4089	.9689	0.9941 $(1 - e^{-0.0100\tau})$
0.0125	2.3476	0.10156	0.6969	2.4373	0.9634	0.9921 $(1 - e^{-0.0125\tau})$

4. *The half-life of a cluster.*—From our results of § 3 it follows that for $\tau \gtrsim 5$ we can write

$$Q(\tau) = 1 - e^{-\lambda_1\tau} \quad (\tau = \eta_0 t) \quad (46)$$

for the values of ρ_∞ which come under discussion. Since $Q(\tau)$ gives the expectation that an average star will have escaped during a time τ (in units of η_0^{-1}), we can properly regard $1/\lambda_1\eta_0$ as a measure of the half-life of the cluster. Thus,

$$\text{Half-life of the cluster} = (\lambda_1\eta_0)^{-1}, \quad (47)$$

where η_0 is defined in equation (10). The precise value of λ_1 will depend, of course, on circumstances; but it is clear that greatest interest attaches to a value of $\rho_\infty \simeq 2.45$. For this value of ρ_∞ we have found that $\lambda_1 \simeq 0.0075$, so that the half-life of the cluster may be defined by

$$\text{Half-life of the cluster} = 133\eta_0^{-1}. \quad (48)$$

For the Pleiades, η_0^{-1} is of the order of 2×10^7 years, so that its half-life is of the order of 3×10^9 years. In judging this value it should be remembered that, when dynamical friction is ignored, a half-life for the Pleiades of the order of only 5×10^7 years is predicted, while our own earlier calculations in II, in which we ignored the dependence of the coefficient of dynamical friction on $|\mathbf{u}|$, gave half-lives which are about seven to eight times shorter than those indicated by our present calculations. More explicitly, we have found that (cf. II, eqs. [28] and [62])

$$\left. \begin{aligned} Q(\tau) &\simeq 1.3(1 - e^{-0.82\tau}) && \text{(dynamical friction ignored),} \\ Q(\tau) &= (1 - e^{-0.059\tau}) && \text{(dynamical friction included, but the dependence of } \eta \text{ on } |\mathbf{u}| \text{ ignored),} \\ Q(\tau) &= (1 - e^{-0.0075\tau}) && \text{(dynamical friction included and the dependence of } \eta \text{ on } |\mathbf{u}| \text{ allowed for).} \end{aligned} \right\} \quad (49)$$

There can thus be hardly any doubt that dynamical friction provides the principal mechanism for the continued existence of the galactic clusters like the Pleiades for times of the order of 3×10^9 years. But, even with dynamical friction properly allowed for, it will be hard to account for such clusters' half-lives of the order of 10^{10} years. This, in turn, provides another strong argument in favor of the "short-time scale."

The results of Table 2 allow us also to infer something about the relative rates of escape of stars of different masses: for stars with masses appreciably different from the average value, ρ_∞ may be expected to change according to³

$$\rho_\infty(m) = \left(6 \frac{m}{\bar{m}}\right)^{1/2}. \quad (50)$$

From Table 2 we now see that even a 10 per cent increase of ρ_∞ prolongs the half-lives by a factor of the order 3, while a similar decrease in ρ_∞ shortens the half-life by a factor of the order 2. The general conclusion to be drawn from this is simply that a cluster loses its less massive members rather more rapidly than the average ones, while the more massive members continue to remain, on the average, for longer times. We hope to return to these questions in greater detail on a later occasion.

In conclusion, I wish to record my indebtedness to Mrs. T. Belland, who undertook most of the numerical work involved in the preparation of this paper, and in particular for the care with which she performed the necessary numerical integrations.

³ *Ibid.*, pp. 209–213.