

## OVERTONE PULSATIONS FOR THE STANDARD MODEL

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## - ABSTRACT

The periods of the first four spherically symmetrical overtone pulsations are derived for the standard model.

Recent observations<sup>1</sup> on short-period variables indicate that overtone pulsations may actually occur in variable stars. To make possible a comparison between these observations and the pulsation theory, the periods of the fundamental mode and the first four overtones (all corresponding to radially symmetrical pulsations) have been computed for the standard model (which is likely to be a sufficiently good approximation for the short-period variables) for four different values of the ratio of the specific heats,  $\gamma$ .<sup>2</sup>

According to the pulsation theory,<sup>3</sup> the periods are determined by those values of

$$\omega^2 = \left(\frac{2\pi}{P}\right)^2 (\pi G \gamma \rho_{oc})^{-1} \quad (1)$$

(where  $P$  is the period,  $G$  the constant of gravitation, and  $\rho_{oc}$  the mean density in center), for which the pulsation equation

$$\frac{d^2 \xi_1}{d\xi_0^2} + \left(\frac{4}{\xi_0} + \frac{4}{u} \cdot \frac{du}{d\xi_0}\right) \frac{d\xi_1}{d\xi_0} + \left(\frac{\omega^2}{u} + a \frac{4}{\xi_0 u} \cdot \frac{du}{d\xi_0}\right) \xi_1 = 0 \quad (2)$$

(where  $\xi_1$  is the amplitude of displacement,  $\xi_0$  the mean distance from center,  $a = 3 - (4/\gamma)$ , and  $u$  is the Emden function<sup>4</sup> of polytropic index 3) has a solution nonsingular both at  $\xi_0 = 0$  and at  $\xi_0 = \xi_{0R}$ .

Close to the surface the standard model and the corresponding boundary condition for the pulsation are not good approximations. But the errors introduced into the periods by this inaccuracy should not be appreciable.

The characteristic values of  $\omega^2$  were determined in the following way. A trial value of  $\omega^2$  was chosen. For the neighborhood of  $\xi_0 = 0$  and of  $\xi_0 = \xi_{0R}$  equation (2) was solved by power series; the series developed at  $\xi_0 = 0$  (neglecting sixth and higher powers of  $\xi_0$ ) was used to compute  $\xi_1$  at  $\xi_0 = 0.4$  and at  $\xi_0 = 0.5$ ; the series developed at  $\xi_0 = \xi_{0R}$  (neglecting the fourth and higher powers of  $\xi_0 - \xi_{0R}$ ) was used to compute  $\xi_1$  at  $\xi_0 = 6.80$  and at  $\xi_0 = 6.75$ . Two numerical integrations of equation (2) were performed; the first integration, using  $\xi_1(0.4)$  and  $\xi_1(0.5)$  as starting-values and  $\Delta = +0.1$  as step-value, was led outward to  $\xi_0 = 5.1$ ; the second integration, using  $\xi_1(6.80)$  and  $\xi_1(6.75)$  as

<sup>1</sup> T. E. Sterne, *Proc. Nat. Acad. Sci.*, **26**, No. 9, 537, 1940; M. Schwarzschild, *Pub. A.A.S.*, **10**, No. 3, 117, 1941.

<sup>2</sup> The period of the second overtone for  $\gamma = 1.54$  had already been computed by Miss H. A. Kluyver, *Bull. Astr. Inst. Netherlands*, No. 276, 1936. Dr. Sterne had found certain models, rather different from the standard model, for which the overtone periods could be derived analytically, *M.N.*, **97**, 582, 1937.

<sup>3</sup> A. S. Eddington, *The Internal Constitution of the Stars*, pp. 188 and 192, 1930.

<sup>4</sup> British Association for the Advancement of Science, *Mathematical Tables*, **2**, 1932.

starting-values and  $\Delta = -0.05$  as step-value, was led inward to  $\xi_0 = 4.9$ . The integration formula employed was

$$\xi_1(\xi_0 + \Delta) = \xi_1(\xi_0) \times B(\xi_0) - \xi_1(\xi_0 - \Delta) \times A(\xi_0),$$

with

$$A(\xi_0) = 1 - \frac{2\Delta a}{2 + \Delta a} \quad \left( a = \frac{4}{\xi_0} + \frac{4}{u} \cdot \frac{du}{d\xi_0} \right)$$

and

$$B(\xi_0) = 2 - \frac{2\Delta a + 2\Delta^2 b}{2 + \Delta a} \quad \left( b = \frac{\omega^2}{u} + a \frac{4}{\xi_0 u} \cdot \frac{du}{d\xi_0} \right);$$

the integrations were performed with the help of the punched-card machines of the Thomas J. Watson Astronomical Computing Bureau.<sup>5</sup> The ratio  $R = \xi_1(5.1)/\xi_1(4.9)$

TABLE 1  
CHARACTERISTIC VALUES OF  $\omega^2$

Mode	$a=0.0$	$a=0.2$	$a=0.4$	$a=0.6$	Max. Error
0.....	0.00000	0.05882	0.10391	0.13670	$\pm 0.00003$
1.....	.16643	.19139	.21998	.25090	.00010
2.....	.3392	.3648	.3920	.4209	.0004
3.....	.5600	.5863	.6136	.6420	.0020
4.....	0.8283	0.8554	0.8832	0.9117	$\pm 0.0060$

was computed from each of the two integrations, and the difference  $D = R_{\text{outw}} - R_{\text{inw}}$  was derived. The whole procedure described so far was repeated for another trial value of  $\omega^2$ . A first approximation for the characteristic value of  $\omega^2$  (for which  $D$  is zero) was obtained from the two available  $D$ -values by inter- or extrapolation, regarding  $D$  as a function of  $\omega^2$ . This approximation for  $\omega^2$  was used as a third trial value, for which  $D$  was again determined by the above procedure. Further successive approximations were obtained by using, every time, the previously derived  $D$ 's for the interpolation. The second approximation generally gave a sufficiently accurate value for the characteristic  $\omega^2$  (altogether 74 outward integrations, each of 46 steps, and as many inward integrations, each of 37 steps, had to be performed with the machines).

Table 1 gives the resulting characteristic  $\omega^2$ 's. The estimated maximal errors in the

<sup>5</sup> An improved integration method, found after the pulsation integrations had been performed, is described by L. Feinstein and M. Schwarzschild in *Rev. Sci. Instr.* (in press, 1941).

last column are based partly on comparisons with Miss Kluyver's integrations<sup>6</sup> and partly on special test integrations showing the errors in the final results produced by the errors in the starting-values and by the errors in the integration formula used.

Table 2 gives the periods of the overtones, in terms of the fundamental period, derived from the  $\omega^2$ 's with the help of equation (1).

TABLE 2  
OVERTONE PERIODS IN TERMS OF THE  
FUNDAMENTAL PERIOD

$a$ .....	0.2	0.4	0.6
$\gamma$ .....	1.43	1.54	1.67
$P_1/P_0$ .....	0.554	0.687	0.738
$P_2/P_0$ .....	.402	.515	.570
$P_3/P_0$ .....	.317	.412	.461
$P_4/P_0$ .....	0.262	0.343	0.387

TABLE 3  
RATIO OF SUCCESSIVE OVERTONE PERIODS

$a$ .....	0.0	0.2	0.4	0.6
$\gamma$ .....	1.33	1.43	1.54	1.67
$P_1/P_0$ .....	0.000	0.554	0.687	0.738
$P_2/P_1$ .....	.700	.724	.749	.772
$P_3/P_2$ .....	.778	.789	.799	.810
$P_4/P_3$ .....	0.822	0.828	0.833	0.839

Table 3 gives the ratios of the periods of successive overtones.

Table 4 gives the amplitudes ( $\xi_i$ ) of the displacements as functions of the mean distance from the center ( $\xi_0$ ), normalized so that  $\xi_1 = 1$  at  $\xi_0 = \xi_{0R}$ . The  $\xi_i$ 's given are taken from the integrations corresponding to the last trial values of  $\omega^2$ , which differ not more than one in the last place from the final values given in Table 1. The last place of the  $\xi_i$ 's in Table 4 is hardly significant; for the  $\xi_i$ 's near the surface in the higher overtones only the first two or three places are significant. For  $a = 0.4$  the  $\xi_i$ 's are shown also in Figure 1.

<sup>6</sup> *Bull. Astr. Inst. Netherlands*, No. 268, 1935, and No. 276, 1936.

AMPLITUDES OF THE DISPLACEMENTS,  $\xi_i$

$\xi_0$	FUNDAMENTAL			FIRST OVERTONE				SECOND OVERTONE			
	$a=0.2$	$a=0.4$	$a=0.6$	$a=0.0$	$a=0.2$	$a=0.4$	$a=0.6$	$a=0.0$	$a=0.2$	$a=0.4$	$a=0.6$
0.00	+0.31767	+0.11046	+0.04457	-0.03382	-0.02848	-0.02291	-0.01691	+0.01142	+0.01004	+0.00869	+0.00733
0.40	.31873	.11122	.04504	-.03373	-.02851	-.02302	-.01706	+0.01135	+0.01003	+0.00871	+0.00737
0.50	.31933	.11166	.04532	-.03368	-.02853	-.02308	-.01714	+0.01132	+0.01002	+0.00872	+0.00740
0.60	.32007	.11210	.04505	-.03361	-.02855	-.02316	-.01724	+0.01127	+0.01000	+0.00873	+0.00742
0.70	.32094	.11283	.04605	-.03353	-.02857	-.02325	-.01736	+0.01121	+0.00999	+0.00875	+0.00746
0.80	.32196	.11357	.04652	-.03343	-.02859	-.02336	-.01751	+0.01115	+0.00996	+0.00876	+0.00750
0.90	.32312	.11442	.04706	-.03331	-.02862	-.02348	-.01767	+0.01107	+0.00994	+0.00877	+0.00754
1.00	.32443	.11538	.04767	-.03317	-.02864	-.02361	-.01785	+0.01097	+0.00990	+0.00878	+0.00759
1.10	.32589	.11646	.04836	-.03301	-.02866	-.02375	-.01805	+0.01086	+0.00986	+0.00879	+0.00764
1.20	.32750	.11766	.04913	-.03283	-.02868	-.02391	-.01827	+0.01074	+0.00981	+0.00880	+0.00769
1.30	.32928	.11898	.04999	-.03261	-.02869	-.02407	-.01851	+0.01060	+0.00974	+0.00880	+0.00774
1.40	.33122	.12044	.05094	-.03237	-.02869	-.02425	-.01877	+0.01043	+0.00967	+0.00880	+0.00779
1.50	.33333	.12203	.05198	-.03209	-.02868	-.02443	-.01906	+0.01024	+0.00957	+0.00879	+0.00784
1.60	.33562	.12377	.05313	-.03176	-.02865	-.02462	-.01936	+0.01003	+0.00947	+0.00877	+0.00789
1.70	.33808	.12566	.05439	-.03140	-.02861	-.02481	-.01968	+0.00979	+0.00934	+0.00874	+0.00794
1.80	.34074	.12771	.05577	-.03098	-.02854	-.02500	-.02002	+0.00951	+0.00918	+0.00869	+0.00798
1.90	.34359	.12993	.05727	-.03050	-.02845	-.02519	-.02038	+0.00920	+0.00901	+0.00862	+0.00800
2.00	.34664	.13233	.05890	-.02995	-.02832	-.02538	-.02075	+0.00885	+0.00880	+0.00854	+0.00802
2.10	.34990	.13492	.06068	-.02934	-.02815	-.02556	-.02114	+0.00846	+0.00855	+0.00843	+0.00802
2.20	.35336	.13770	.06261	-.02864	-.02794	-.02573	-.02154	+0.00801	+0.00827	+0.00829	+0.00801
2.30	.35700	.14070	.06471	-.02785	-.02767	-.02588	-.02195	+0.00752	+0.00794	+0.00812	+0.00797
2.40	.36098	.14392	.06699	-.02695	-.02734	-.02600	-.02236	+0.00696	+0.00756	+0.00790	+0.00790
2.50	.36514	.14737	.06946	-.02594	-.02694	-.02609	-.02278	+0.00635	+0.00713	+0.00765	+0.00786
2.60	.36950	.15107	.07213	-.02479	-.02645	-.02614	-.02319	+0.00566	+0.00664	+0.00734	+0.00766
2.70	.37423	.15504	.07503	-.02350	-.02586	-.02614	-.02360	+0.00490	+0.00608	+0.00697	+0.00748
2.80	.37917	.15928	.07817	-.02205	-.02516	-.02607	-.02399	+0.00406	+0.00544	+0.00654	+0.00724
2.90	.38438	.16382	.08156	-.02042	-.02433	-.02594	-.02435	+0.00313	+0.00473	+0.00604	+0.00695
3.00	.38989	.16867	.08523	-.01858	-.02335	-.02571	-.02467	+0.00211	+0.00392	+0.00546	+0.00658
3.10	.39569	.17386	.08920	-.01652	-.02221	-.02537	-.02494	+0.00100	+0.00302	+0.00478	+0.00613
3.20	.40181	.17940	.09349	-.01420	-.02087	-.02491	-.02515	+0.00022	+0.00202	+0.00401	+0.00559
3.30	.40824	.18531	.09812	-.01161	-.01931	-.02429	-.02528	+0.00154	+0.00091	+0.00313	+0.00496
3.40	.41502	.19162	.10313	-.00871	-.01751	-.02351	-.02530	+0.00297	+0.00032	+0.00213	+0.00421
3.50	.42214	.19835	.10855	-.00546	-.01543	-.02252	-.02520	+0.00450	+0.00166	+0.00101	+0.00334
3.60	.42961	.20552	.11439	-.00183	-.01304	-.02129	-.02495	+0.00614	+0.00313	+0.00024	+0.00234
3.70	.43747	.21317	.12071	+0.0222	-.01029	-.01979	-.02451	+0.00788	+0.00472	+0.00163	+0.00119
3.80	.44571	.22132	.12753	+0.0673	-.00714	-.01798	-.02384	+0.00971	+0.00644	+0.00316	+0.00111
3.90	.45435	.23000	.13489	+0.1176	-.00355	-.01580	-.02291	+0.01162	+0.00827	+0.00484	+0.00157
4.00	.46341	.23925	.14283	+0.1737	+0.00055	-.01321	-.02167	+0.01359	+0.01020	+0.00666	+0.00320
4.10	.47290	.24909	.15141	+0.2360	+0.00521	-.01014	-.02005	+0.01561	+0.01223	+0.00861	+0.00500
4.20	.48284	.25958	.16067	+0.3052	+0.01050	-.00653	-.01801	+0.01763	+0.01433	+0.01069	+0.00696
4.30	.49325	.27073	.17065	+0.3821	+0.01650	-.00231	-.01545	+0.01961	+0.01646	+0.01287	+0.00908
4.40	.50413	.28261	.18143	+0.4674	+0.2329	+0.00263	-.01230	+0.02150	+0.01858	+0.01512	+0.01134
4.50	.51552	.29524	.19306	+0.5620	+0.39095	+0.00836	-.00846	+0.02323	+0.02064	+0.01739	+0.01371
4.60	.52743	.30868	.20560	+0.6669	+0.39600	+0.01500	-.00382	+0.02471	+0.02256	+0.01903	+0.01614
4.70	.53987	.32298	.21913	+0.7829	+0.49334	+0.02267	+0.00174	+0.02584	+0.02425	+0.02176	+0.01857
4.80	.55287	.33818	.23371	+0.9113	+0.60030	+0.3151	+0.00337	+0.02649	+0.02558	+0.02368	+0.02093
4.90	.56645	.35435	.24944	+1.0533	+0.7261	+0.4166	+0.01023	+0.02649	+0.02642	+0.02525	+0.02308
5.00	.58063	.37154	.26641	+1.2101	+0.8643	+0.5330	+0.02550	+0.02566	+0.02658	+0.02631	+0.02490
5.05	.58795	.38054	.27538	+1.2946	+0.9397	+0.5972	+0.03972	+0.02485	+0.02634	+0.02657	+0.02561
5.10	.59543	.38982	.28469	+1.3833	+1.0193	+0.6660	+0.03638	+0.02375	+0.02584	+0.02663	+0.02617
5.15	.60308	.39938	.29436	+1.4765	+1.1037	+0.7394	+0.04250	+0.02231	+0.02504	+0.02645	+0.02654
5.20	.61090	.40924	.30440	+1.5744	+1.1928	+0.8178	+0.04911	+0.02049	+0.02392	+0.02598	+0.02667
5.25	.61889	.41940	.31483	+1.6772	+1.2872	+0.9014	+0.05625	+0.01825	+0.02242	+0.02510	+0.02654
5.30	.62705	.42987	.32564	+1.7851	+1.3860	+0.9907	+0.06395	+0.01554	+0.02049	+0.02402	+0.02609
5.35	.63538	.44066	.33688	+1.8984	+1.4924	+1.0859	+0.07226	+0.01231	+0.01809	+0.02243	+0.02527
5.40	.64389	.45170	.34853	+2.0173	+1.6038	+1.1874	+0.08121	+0.00851	+0.01516	+0.02036	+0.02403
5.45	.65257	.46326	.36064	+2.1421	+1.7215	+1.2955	+0.09085	+0.00407	+0.01163	+0.01774	+0.02231
5.50	.66144	.47508	.37320	+2.2731	+1.8450	+1.4107	+0.10122	+0.00108	+0.00743	+0.01450	+0.02002
5.55	.67050	.48727	.38624	+2.4105	+1.9773	+1.5333	+0.11238	+0.00701	+0.00250	+0.01058	+0.01710
5.60	.67975	.49983	.39978	+2.5546	+2.1160	+1.6639	+0.12437	+0.01380	+0.00327	+0.00587	+0.01346
5.65	.68920	.51277	.41382	+2.7058	+2.2625	+1.8029	+0.13724	+0.02153	+0.00994	+0.00209	+0.00899
5.70	.69885	.52611	.42840	+2.8643	+2.4171	+1.9507	+0.15107	+0.03030	+0.01703	+0.00227	+0.00360
5.75	.70871	.53986	.44354	+3.0305	+2.5802	+2.1070	+0.16591	+0.04022	+0.02644	+0.00139	+0.00283
5.80	.71877	.55403	.45924	+3.2049	+2.7523	+2.2750	+0.18182	+0.05139	+0.03649	+0.02278	+0.01045
5.85	.72904	.56863	.47554	+3.3876	+2.9338	+2.4526	+0.19882	+0.06392	+0.04791	+0.03299	+0.01979
5.90	.73952	.58367	.49245	+3.5792	+3.1253	+2.6413	+0.21716	+0.07797	+0.06084	+0.04470	+0.02979
5.95	.75022	.59917	.51001	+3.7799	+3.3272	+2.8418	+0.23674	+0.09365	+0.07542	+0.05808	+0.04186
6.00	.76114	.61514	.52822	+3.9904	+3.5400	+3.0546	+0.25770	+0.11113	+0.09184	+0.07331	+0.05578
6.05	.77229	.63159	.54712	+4.2109	+3.7644	+3.2806	+0.28013	+0.13057	+0.11026	+0.09057	+0.07176
6.10	.78367	.64854	.56674	+4.4419	+4.0009	+3.5204	+0.30414	+0.15215	+0.13088	+0.11010	+0.09004
6.15	.79528	.66601	.58709	+4.6839	+4.2500	+3.7749	+0.32980	+0.17666	+0.15392	+0.13211	+0.11087
6.20	.80713	.68402	.60821	+4.9375	+4.5126	+4.0449	+0.35724	+0.20252	+0.17601	+0.15688	+0.13454
6.25	.81922	.70258	.63012	+5.2030	+4.7892	+4.3313	+0.38657	+0.23174	+0.20820	+0.18467	+0.16136
6.30	.83156	.72170	.65286	+5.4812	+5.0805	+4.6349	+0.41789	+0.26397	+0.23996	+0.21579	+0.19167
6.35	.84416	.74139	.67645	+5.7724	+5.3873	+4.9508	+0.45134	+0.29947	+0.27519	+0.25058	+0.22583
6.40	.85702	.76168	.70093	+6.0774	+5.7103	+5.2979	+0.48706	+0.33853	+0.31421	+0.28941	+0.26427
6.45	.87015	.78259	.72632	+6.3967	+6.0504	+5.6594	+0.52517	+0.38145	+0.35737	+0.33265	+0.30743
6.50	.88354	.80413	.75207	+6.7309	+6.4084	+6.0424	+0.56584	+0.42857	+0.40504	+0.38075	+0.35580
6.55	.89720	.82633	.78000	+7.0808	+6.7852	+6.4481	+0.60922	+0.48024	+0.45765	+0.43418	+0.40991
6.60	.91114	.84921	.80836	+7.4470	+7.1817	+6.8777	+0.65488	+0.53685			

TABLE 4—Continued

$\xi_0$	THIRD OVERTONE				FOURTH OVERTONE			
	$\alpha=0.0$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.0$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$
0.00	-0.00544	-0.00491	-0.00438	-0.00387	+0.00308	+0.00282	+0.00256	+0.00232
0.40	-0.00539	-0.00488	-0.00438	-0.00388	+0.00304	+0.00279	+0.00254	+0.00231
0.50	-0.00536	-0.00487	-0.00437	-0.00389	+0.00301	+0.00277	+0.00254	+0.00231
0.60	-0.00533	-0.00485	-0.00437	-0.00389	+0.00298	+0.00275	+0.00252	+0.00230
0.70	-0.00528	-0.00482	-0.00436	-0.00390	+0.00295	+0.00273	+0.00251	+0.00230
0.80	-0.00523	-0.00479	-0.00435	-0.00390	+0.00290	+0.00270	+0.00249	+0.00229
0.90	-0.00517	-0.00476	-0.00433	-0.00391	+0.00285	+0.00266	+0.00247	+0.00228
1.00	-0.00509	-0.00471	-0.00432	-0.00391	+0.00279	+0.00262	+0.00244	+0.00227
1.10	-0.00501	-0.00466	-0.00429	-0.00391	+0.00272	+0.00257	+0.00241	+0.00225
1.20	-0.00491	-0.00460	-0.00426	-0.00391	+0.00265	+0.00251	+0.00237	+0.00223
1.30	-0.00480	-0.00453	-0.00423	-0.00390	+0.00256	+0.00245	+0.00233	+0.00220
1.40	-0.00468	-0.00445	-0.00418	-0.00389	+0.00246	+0.00237	+0.00227	+0.00217
1.50	-0.00454	-0.00435	-0.00413	-0.00387	+0.00234	+0.00228	+0.00221	+0.00212
1.60	-0.00438	-0.00424	-0.00406	-0.00385	+0.00222	+0.00218	+0.00214	+0.00207
1.70	-0.00420	-0.00412	-0.00399	-0.00381	+0.00207	+0.00207	+0.00205	+0.00201
1.80	-0.00399	-0.00397	-0.00389	-0.00376	+0.00192	+0.00195	+0.00195	+0.00194
1.90	-0.00377	-0.00380	-0.00378	-0.00370	+0.00174	+0.00180	+0.00184	+0.00186
2.00	-0.00351	-0.00361	-0.00365	-0.00362	+0.00155	+0.00164	+0.00171	+0.00176
2.10	-0.00323	-0.00340	-0.00350	-0.00353	+0.00134	+0.00147	+0.00157	+0.00165
2.20	-0.00292	-0.00315	-0.00332	-0.00341	+0.00112	+0.00127	+0.00141	+0.00152
2.30	-0.00257	-0.00288	-0.00311	-0.00327	+0.00087	+0.00106	+0.00122	+0.00137
2.40	-0.00220	-0.00257	-0.00287	-0.00310	+0.00061	+0.00082	+0.00102	+0.00119
2.50	-0.00178	-0.00223	-0.00260	-0.00290	+0.00033	+0.00057	+0.00080	+0.00100
2.60	-0.00133	-0.00185	-0.00229	-0.00267	+0.00003	+0.00030	+0.00055	+0.00079
2.70	-0.00085	-0.00143	-0.00195	-0.00239	+0.00028	+0.00001	+0.00029	+0.00055
2.80	-0.00032	-0.00097	-0.00150	-0.00208	+0.00060	+0.00030	+0.00000	+0.00029
2.90	+0.00024	-0.00047	-0.00113	-0.00172	+0.00093	+0.00061	+0.00030	+0.00001
3.00	+0.00083	+0.00007	-0.00065	-0.00132	+0.00126	+0.00094	+0.00061	+0.00029
3.10	+0.00146	+0.00065	-0.00013	-0.00087	+0.00158	+0.00126	+0.00094	+0.00060
3.20	+0.00211	+0.00127	+0.00043	-0.00036	+0.00189	+0.00159	+0.00126	+0.00093
3.30	+0.00277	+0.00191	+0.00104	+0.00019	+0.00217	+0.00190	+0.00159	+0.00126
3.40	+0.00345	+0.00259	+0.00169	+0.00079	+0.00242	+0.00218	+0.00190	+0.00159
3.50	+0.00413	+0.00328	+0.00237	+0.00144	+0.00263	+0.00243	+0.00219	+0.00190
3.60	+0.00479	+0.00397	+0.00307	+0.00212	+0.00277	+0.00263	+0.00243	+0.00219
3.70	+0.00542	+0.00465	+0.00378	+0.00284	+0.00283	+0.00277	+0.00263	+0.00244
3.80	+0.00599	+0.00531	+0.00449	+0.00358	+0.00281	+0.00282	+0.00276	+0.00264
3.90	+0.00649	+0.00591	+0.00517	+0.00431	+0.00267	+0.00277	+0.00280	+0.00276
4.00	+0.00687	+0.00643	+0.00580	+0.00503	+0.00241	+0.00261	+0.00273	+0.00278
4.10	+0.00711	+0.00684	+0.00635	+0.00570	+0.00201	+0.00231	+0.00254	+0.00269
4.20	+0.00717	+0.00710	+0.00679	+0.00629	+0.00147	+0.00187	+0.00221	+0.00247
4.30	+0.00700	+0.00716	+0.00707	+0.00676	+0.00076	+0.00127	+0.00171	+0.00208
4.40	+0.00657	+0.00699	+0.00714	+0.00706	+0.00009	+0.00051	+0.00105	+0.00153
4.50	+0.00582	+0.00653	+0.00696	+0.00714	+0.00108	+0.00041	+0.00021	+0.00080
4.60	+0.00471	+0.00572	+0.00646	+0.00694	+0.00218	+0.00147	+0.00078	+0.00012
4.70	+0.00320	+0.00452	+0.00558	+0.00639	+0.00336	+0.00264	+0.00191	+0.00119
4.80	+0.00124	+0.00288	+0.00427	+0.00543	+0.00454	+0.00385	+0.00313	+0.00239
4.90	+0.00118	+0.00175	+0.00247	+0.00398	+0.00564	+0.00504	+0.00437	+0.00366
5.00	-0.00408	-0.00079	+0.00015	+0.00200	+0.00655	+0.00609	+0.00554	+0.00492
5.05	-0.00570	-0.00341	-0.00121	+0.00079	+0.00689	+0.00652	+0.00606	+0.00550
5.10	-0.00742	-0.00505	-0.00272	+0.00057	+0.00712	+0.00687	+0.00650	+0.00603
5.15	-0.00924	-0.00680	-0.00437	+0.00208	+0.00724	+0.00711	+0.00686	+0.00649
5.20	-0.01115	-0.00865	-0.00614	+0.00374	+0.00720	+0.00722	+0.00710	+0.00685
5.25	-0.01311	-0.01060	-0.00804	+0.00554	+0.00700	+0.00717	+0.00720	+0.00700
5.30	-0.01511	-0.01263	-0.01004	+0.00748	+0.00660	+0.00694	+0.00713	+0.00718
5.35	-0.01713	-0.01470	-0.01213	+0.00953	+0.00598	+0.00650	+0.00687	+0.00709
5.40	-0.01912	-0.01679	-0.01428	+0.01109	+0.00512	+0.00583	+0.00639	+0.00679
5.45	-0.02104	-0.01886	-0.01645	+0.01301	+0.00400	+0.00490	+0.00505	+0.00525
5.50	-0.02283	-0.02087	-0.01861	+0.01617	+0.00260	+0.00369	+0.00404	+0.00543
5.55	-0.02444	-0.02274	-0.02070	+0.01843	+0.00090	+0.00218	+0.00333	+0.00432
5.60	-0.02579	-0.02441	-0.02266	+0.02061	+0.00109	+0.00035	+0.00169	+0.00289
5.65	-0.02678	-0.02580	-0.02440	+0.02264	+0.00339	+0.00179	+0.00027	+0.00112
5.70	-0.02731	-0.02681	-0.02584	+0.02445	+0.00596	+0.00424	+0.00257	+0.00101
5.75	-0.02727	-0.02731	-0.02685	+0.02592	+0.00870	+0.00697	+0.00510	+0.00348
5.80	-0.02649	-0.02717	-0.02731	+0.02692	+0.01181	+0.00995	+0.00810	+0.00628
5.85	-0.02483	-0.02622	-0.02704	+0.02729	+0.01495	+0.01311	+0.01123	+0.00936
5.90	-0.02208	-0.02427	-0.02587	+0.02685	+0.01810	+0.01635	+0.01452	+0.01266
5.95	-0.01802	-0.02109	-0.02357	+0.02538	+0.02111	+0.01954	+0.01784	+0.01606
6.00	-0.01240	-0.01643	-0.01987	+0.02262	+0.02379	+0.02248	+0.02100	+0.01940
6.05	-0.00492	-0.00999	-0.01448	+0.01826	+0.02586	+0.02493	+0.02379	+0.02248
6.10	+0.00476	-0.00142	-0.00705	+0.01196	+0.02701	+0.02657	+0.02589	+0.02499
6.15	+0.01700	+0.00966	+0.00282	+0.00329	+0.02681	+0.02700	+0.02691	+0.02656
6.20	+0.03227	+0.02372	+0.01562	+0.00822	+0.02475	+0.02571	+0.02635	+0.02669
6.25	+0.05104	+0.04126	+0.03187	+0.02314	+0.02314	+0.02018	+0.02204	+0.02357
6.30	+0.07389	+0.06289	+0.05219	+0.04211	+0.01231	+0.01520	+0.01777	+0.01996
6.35	+0.10145	+0.08927	+0.07730	+0.06588	+0.00016	+0.00422	+0.00795	+0.01130
6.40	+0.13444	+0.12118	+0.10800	+0.09529	+0.01744	+0.01212	+0.00710	+0.00245
6.45	+0.17370	+0.15948	+0.14523	+0.13135	+0.04192	+0.03527	+0.02887	+0.02284
6.50	+0.22014	+0.20517	+0.19004	+0.17517	+0.07500	+0.06701	+0.05922	+0.05175
6.55	+0.27480	+0.25936	+0.24365	+0.22807	+0.11875	+0.10948	+0.10034	+0.09147
6.60	+0.33887	+0.32334	+0.30742	+0.29151	+0.17564	+0.16528	+0.15496	+0.14483
6.65	+0.41367	+0.39855	+0.38293	+0.36722	+0.24867	+0.23755	+0.22636	+0.21528
6.70	+0.50072	+0.48664	+0.47200	+0.45717	+0.34147	+0.33013	+0.31862	+0.30711
6.75	+0.60183	+0.58960	+0.57681	+0.56376	+0.45875	+0.44804	+0.43709	+0.42605
6.80	+0.71974	+0.71048	+0.70076	+0.69050	+0.60899	+0.60046	+0.59171	+0.58286

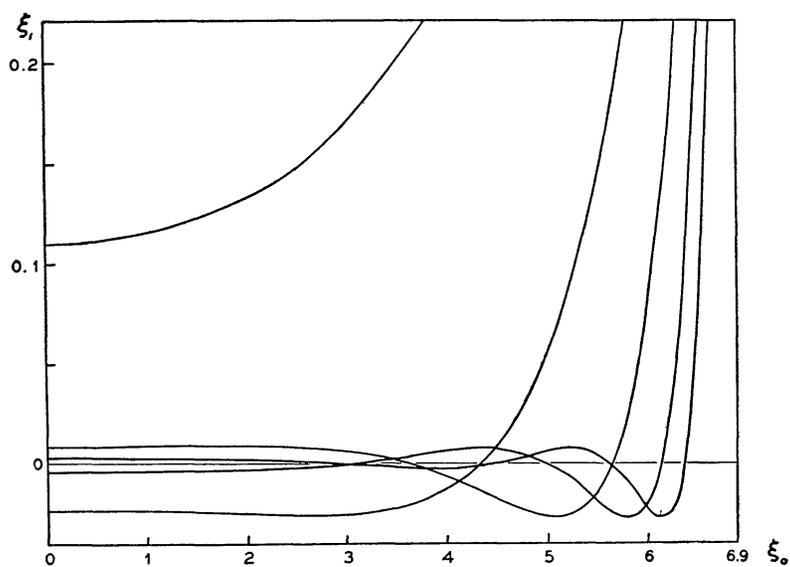


FIG. 1.—The amplitudes  $\xi_i$  of the displacements in the fundamental mode and the first four overtones for  $\alpha = 0.4$ , as functions of the mean distance  $\xi_0$  from the center.

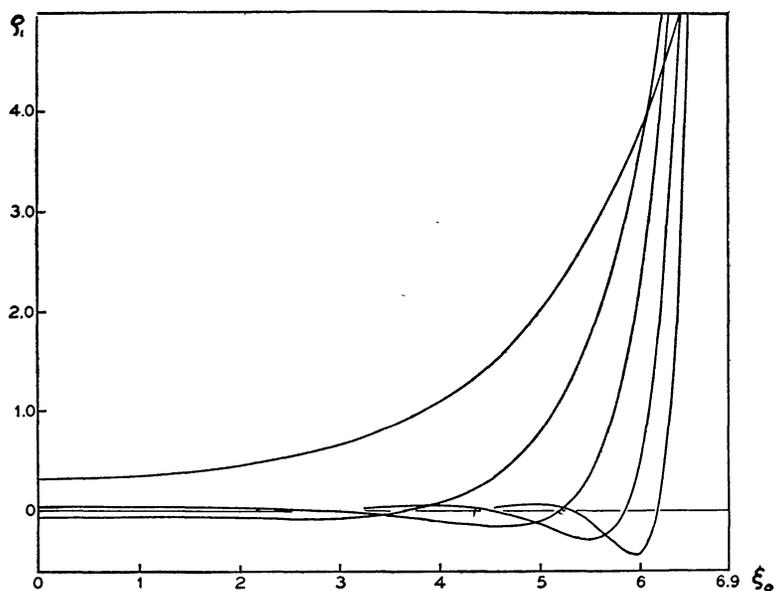


FIG. 2.—The amplitudes  $\rho_i$  of the density variations in the fundamental mode and the first four overtones for  $\alpha = 0.4$ , as functions of the mean distance  $\xi_0$  from the center.

TABLE 5  
AMPLITUDES OF THE DENSITY VARIATIONS,  $\rho_1$

$\xi_0$	FUNDAMENTAL		FIRST OVERTONE				SECOND OVERTONE				THIRD OVERTONE				FOURTH OVERTONE			
	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.0$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.0$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.0$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.0$	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$
0.0	-0.95	0.14	0.10	0.09	0.07	0.05	0.034	0.030	0.026	0.022	0.016	0.015	0.013	0.012	0.009	0.008	0.008	0.007
0.5	0.34	0.15	0.10	0.09	0.07	0.05	0.034	0.030	0.026	0.022	0.016	0.015	0.013	0.012	0.009	0.008	0.008	0.007
1.0	0.99	0.36	0.15	0.09	0.07	0.05	0.034	0.030	0.026	0.022	0.016	0.015	0.013	0.012	0.009	0.008	0.008	0.007
1.5	1.04	0.39	0.17	0.09	0.08	0.06	0.028	0.027	0.024	0.024	0.011	0.012	0.012	0.011	0.005	0.005	0.006	0.006
2.0	1.10	0.45	0.21	0.08	0.07	0.07	0.019	0.022	0.024	0.024	0.005	0.007	0.008	0.009	0.001	0.002	0.002	0.003
2.2	1.14	0.48	0.23	0.07	0.08	0.07	0.014	0.018	0.021	0.023	0.001	0.004	0.006	0.007	0.002	0.001	0.000	0.001
2.4	1.18	0.51	0.20	0.06	0.07	0.08	0.007	0.013	0.018	0.022	0.002	0.003	0.002	0.005	0.003	0.004	0.002	0.001
2.6	1.23	0.55	0.29	0.04	0.07	0.08	0.002	0.006	0.013	0.019	0.008	0.005	0.002	0.002	0.008	0.006	0.005	0.004
2.8	1.28	0.60	0.33	0.02	0.05	0.08	0.003	0.003	0.006	0.014	0.014	0.011	0.007	0.003	0.011	0.010	0.008	0.007
3.0	1.34	0.66	0.37	0.00	0.04	0.07	0.026	0.014	0.003	0.007	0.021	0.017	0.013	0.009	0.014	0.013	0.012	0.010
3.2	1.41	0.72	0.42	0.04	0.02	0.08	0.042	0.028	0.015	0.002	0.027	0.024	0.020	0.016	0.015	0.015	0.014	0.013
3.4	1.48	0.80	0.49	0.08	0.01	0.04	0.060	0.045	0.030	0.015	0.034	0.031	0.028	0.024	0.015	0.016	0.016	0.016
3.6	1.57	0.88	0.50	0.13	0.05	0.01	0.080	0.065	0.049	0.032	0.038	0.037	0.035	0.032	0.032	0.014	0.015	0.016
3.8	1.66	0.99	0.65	0.20	0.11	0.02	0.100	0.087	0.071	0.053	0.048	0.048	0.040	0.040	0.005	0.008	0.011	0.014
4.0	1.76	1.10	0.76	0.29	0.18	0.07	0.121	0.110	0.096	0.079	0.053	0.038	0.041	0.043	0.006	0.002	0.003	0.007
4.2	1.88	1.23	0.89	0.49	0.27	0.15	0.137	0.132	0.122	0.107	0.107	0.107	0.098	0.091	0.022	0.016	0.011	0.006
4.4	2.00	1.39	1.04	0.54	0.39	0.24	0.144	0.148	0.145	0.136	0.136	0.136	0.119	0.109	0.041	0.036	0.030	0.024
4.6	2.14	1.57	1.22	0.71	0.54	0.38	0.134	0.159	0.159	0.160	0.160	0.160	0.133	0.123	0.050	0.056	0.051	0.046
4.8	2.30	1.77	1.43	0.93	0.74	0.55	0.094	0.128	0.154	0.171	0.171	0.171	0.142	0.132	0.068	0.069	0.068	0.067
5.0	2.47	2.00	1.68	1.19	1.00	0.79	0.067	0.064	0.113	0.152	0.152	0.152	0.128	0.119	0.050	0.064	0.070	0.074
5.1	2.56	2.13	1.82	1.34	1.14	0.93	0.06	0.01	0.07	0.13	0.20	0.20	0.17	0.15	0.04	0.05	0.06	0.07
5.2	2.65	2.27	1.98	1.52	1.31	1.00	0.15	0.07	0.01	0.08	0.23	0.22	0.21	0.19	0.01	0.02	0.04	0.05
5.3	2.76	2.42	2.15	1.71	1.50	1.28	0.27	0.17	0.07	0.01	0.20	0.26	0.25	0.23	0.03	0.02	0.00	0.02
5.4	2.86	2.58	2.33	1.92	1.72	1.49	0.42	0.30	0.19	0.09	0.27	0.28	0.28	0.27	0.09	0.07	0.05	0.03
5.5	2.97	2.75	2.53	2.16	1.96	1.73	0.61	0.48	0.35	0.23	0.26	0.28	0.29	0.29	0.16	0.14	0.11	0.09
5.6	3.09	2.93	2.75	2.42	2.23	2.01	0.86	0.71	0.56	0.42	0.21	0.24	0.27	0.30	0.24	0.22	0.20	0.17
5.7	3.21	3.12	2.98	2.71	2.54	2.33	1.16	1.00	0.83	0.66	0.11	0.17	0.22	0.26	0.33	0.31	0.29	0.27
5.8	3.34	3.33	3.24	3.03	2.88	2.68	1.53	1.36	1.18	0.90	0.06	0.02	0.00	0.16	0.30	0.30	0.38	0.36
5.9	3.47	3.55	3.41	3.39	3.26	3.09	1.99	1.81	1.62	1.42	0.34	0.23	0.13	0.03	0.42	0.43	0.43	0.43
6.0	3.61	3.79	3.51	3.79	3.69	3.55	2.55	2.37	2.17	1.97	0.75	0.75	0.75	0.75	0.30	0.39	0.42	0.44
6.1	3.75	4.05	4.14	4.22	4.16	4.07	3.23	3.06	2.87	2.66	1.36	1.20	1.04	0.88	0.14	0.20	0.27	0.32
6.2	3.91	4.32	4.49	4.70	4.70	4.67	4.59	4.06	3.91	3.54	2.21	2.04	1.85	1.67	0.34	0.23	0.13	0.03
6.3	4.07	4.61	4.88	5.24	5.30	5.34	5.34	4.95	4.81	4.64	3.41	3.22	3.03	2.83	1.23	1.08	0.94	0.79
6.4	4.23	4.92	5.30	5.82	5.96	6.09	6.10	6.2	6.1	6.0	5.0	4.9	4.7	4.5	2.8	2.6	2.4	2.2
6.5	4.41	5.26	5.75	6.47	6.70	6.94	7.17	7.7	7.7	7.7	7.2	7.2	7.0	6.8	5.2	5.0	4.8	4.6
6.55	4.50	5.44	5.99	6.82	7.11	7.41	7.71	8.5	8.6	8.7	8.6	8.5	8.4	8.3	7.0	6.8	6.6	6.4
6.60	4.59	5.62	6.24	7.19	7.53	7.91	8.28	9.4	9.6	9.7	10.2	10.2	10.1	10.1	9.1	9.0	8.8	8.6
6.65	4.69	5.80	6.50	7.57	7.98	8.43	8.89	10.4	10.7	10.9	12.0	12.1	12.1	12.1	11.8	11.7	11.6	11.5
6.70	4.79	6.00	6.78	7.97	8.45	8.90	9.55	11.5	11.8	12.2	14.1	14.3	14.4	14.6	15.1	15.1	15.1	15.1
6.75	4.89	6.20	7.06	8.40	8.95	9.59	10.26	12.7	13.1	13.6	16.6	16.9	17.2	17.5	19.5	19.6	19.8	20.0
$\xi_{0r}$	-5.19	-6.82	-7.96	-9.76	-10.58	-11.54	-12.60	-16.8	-17.6	-18.5	-25.8	-26.6	-27.5	-28.5	-36.7	-37.6	-38.5	-39.4

Table 5 gives the amplitudes ( $\rho_i$ ) of the density variations, derived from the  $\xi_i$ 's given in Table 4 with the help of<sup>7</sup>

$$\begin{aligned}\rho_i &= -3\xi_i - \xi_0 \frac{d\xi_i}{d\xi_0} = -\frac{1}{\xi_0^2} \cdot \frac{d}{d\xi_0} (\xi_0^3 \xi_i) \\ &\approx + \frac{1}{2\Delta \cdot \xi_0^2} ([\xi_0^3 \xi_i]_{\xi_0-\Delta} - [\xi_0^3 \xi_i]_{\xi_0+\Delta}).\end{aligned}$$

For  $a = 0.4$  the  $\rho_i$ 's are shown also in Figure 2.

I am indebted to Miss Lillian Feinstein for her most helpful co-operation in the work with the punched-card machines.

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June 1941

<sup>7</sup> Eddington, *op. cit.*, p. 187.