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THE TROJAN GROUP

By Arthur B. Wyse
Lick Observatory

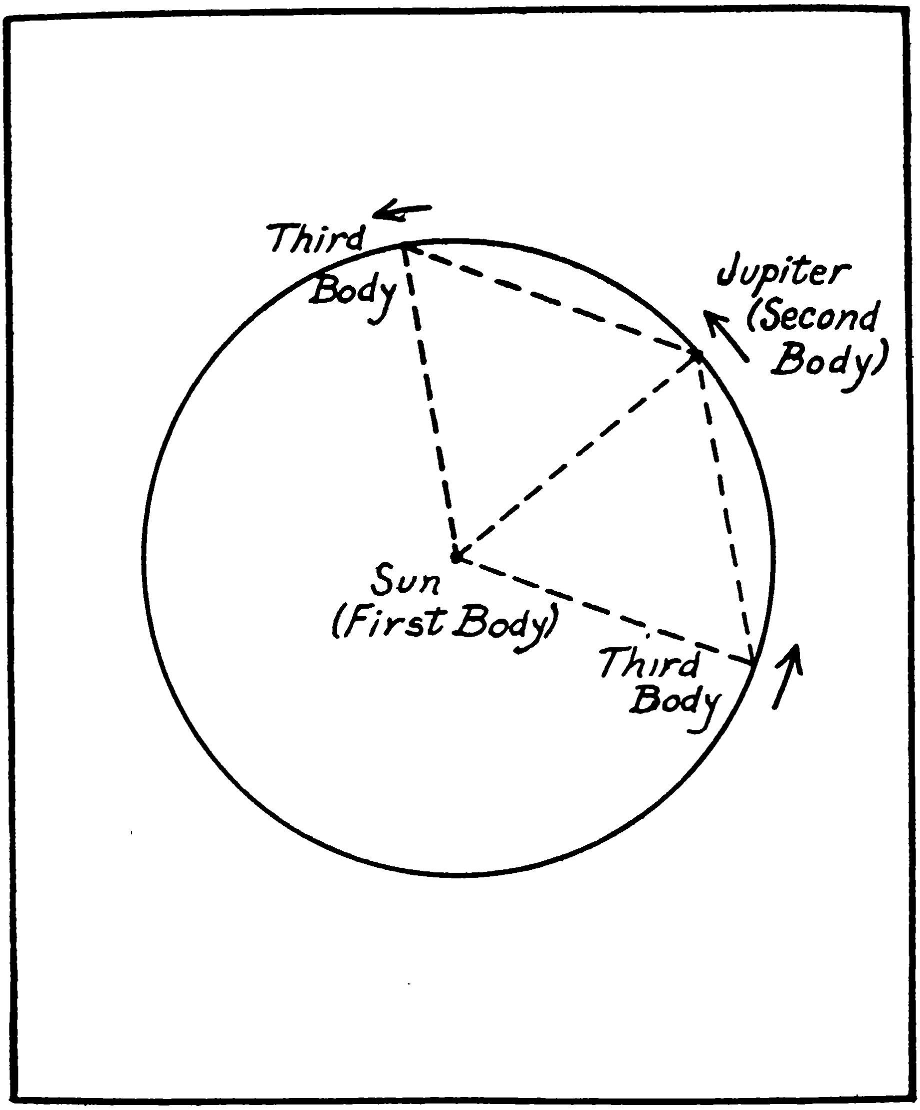
Of THE fourteen hundred-odd catalogued asteroids or minor planets revolving about the Sun, one of the most interesting sets constitutes what is commonly known as the Trojan group. The asteroids, it will be recalled, are small planets ranging in size from giant meteors, a few miles in diameter, to Ceres, with a diameter of some 480 miles. They number in the thousands, the majority probably being too faint to be observed even with a large telescope. They revolve in elliptic orbits about the Sun, except when their motion is disturbed by some external force, such as the gravitational attraction of Jupiter; and most of them pursue paths lying between the orbits of Jupiter and Mars. In examining the nature of the Trojan asteroids, let us turn to the historical background, and the mathematical logic that anticipated their discovery.

It will be remembered that the problem of two bodies was completely solved by Newton in the seventeenth century. Briefly stated, the problem is as follows: Given two spherical bodies of known masses, subject only to their mutual gravitational attractions; and given the positions and velocities of the two bodies at any time; what will be their positions and velocities at any other time? Newton discovered that the attractive force prevailing in nature could be formulated in his inverse-square law of universal gravitation. This concept is so familiar now that it is taught in grammar school; but three hundred years ago it was unknown. On the basis of the simple law of gravitation, Newton found that the two bodies would revolve about each other in elliptic orbits, with the center of gravity at one of the foci of each ellipse. Given the masses and the relative positions and motions of the two bodies at any instant, he could predict the size and shape of the orbits, and the positions and motions at any other instant, past or future. He was able, as a result of his calculations, to interpret Kepler's laws of planetary motion as the result of the law of gravitation.

The problem of three bodies is a simple extension of the two-body problem, but its solution is far from simple. Although the problem has occupied the attention of the most eminent mathematicians since Newton's time, no one has succeeded in finding a usable general solution. There are, however, solutions for certain special cases, and it is in these *particular* solutions that we are interested in connection with the Trojan group.

Suppose, in the three-body problem, that two of the bodies have finite masses, and that the third is of infinitesimal mass; that is to say, the third body is so small that it does not appreciably disturb the motions of the first two, although its own motion is controlled by their attractive forces. And suppose that the two finite masses are revolving about their center of gravity in circular orbits. Imagine, furthermore, that the third body happens to be placed in the plane of the orbit of the first two, in such a position that all three bodies lie at the vertices of an equal-sided, or equilateral triangle. Then, if the above conditions are all fulfilled, the three bodies will continue to move in the same plane, locked forever at the corners of the same equilateral triangle. If an observer stood above the center of gravity of the three-body system, looking downward, and turned at the same rate as that of the system, he would see three bodies all apparently at rest.

The equilateral triangle solution is a *particular* solution of the problem of three bodies. It was first announced in 1772 by the illustrious mathematician,



Two alternative "particular" solutions of the three-body problem. The gravitational effects of the other planets, such as Saturn and Mars, are disregarded.

Lagrange. It is to be noted that the triangle must be equilateral, regardless of the mass-ratio of the two finite bodies. Intuitively one might expect the third body to remain closer to the more massive of its companions, but such is not the case. Actually, there are two and only two such triangles possible, as may be seen in the figure.

One more fact concerning the equilateral triangle solution: it is, under certain circumstances, a stable solution. In other words, if some momentary external force happens to draw the third body slightly away from the critical point at the vertex of the triangle, the third body will oscillate about that point, but will not recede indefinitely from it. The characteristic of stability is important in nature, since obviously there is no point in space that is entirely free from disturbing gravitational influences.

The question naturally arises whether the circumstances imagined above could be approximately satisfied in the solar system—and the answer is in the affirmative. Jupiter is the largest and most massive of the planets, and revolves about the Sun in an orbit that is nearly circular. If we neglect the disturbances of the other planets, effects that are small in comparison with the attractions of Jupiter and the Sun, we may consider the Sun and Jupiter as the two "finite" bodies in our special three-body problem, and search the third corner of the equilateral triangle to see if an asteroid is to be found there. The asteroid, of course, would be the infinitesimal third body, being too small to have any noticeable effect on the motion of Jupiter or of the Sun.

No fewer than eleven such asteroids have been discovered. They are known as the Trojan group, since they bear the names of heroes that won immortal fame in the Trojan war. Minor planets are designated by number as well as by name. Thus the Trojan asteroids are as follows: (588) Achilles, (617) Patroclus, (624) Hector, (659) Nestor, (884) Priamus, (911) Agamemnon, (1143) Odysseus, (1172) Aeneas, (1173) Anchises, (1208) Troilus, and (1404) Ajax. Of these, Achilles, Hecor, Nestor, Agamemnon, Odysseus and Ajax precede Jupiter, and the others follow it. To the student of classical literature it might seem more appropriate to put the Greek heroes in one camp and the Trojans in the other. This delicate point seems, however, to have been disregarded, since Hector is to be found on the Greek side, and Patroclus on the Trojan.

It will be noted that since the mass of Jupiter is less than one thousandth that of the Sun, the Sun-Jupiter center of gravity is very close to the Sun. Consequently the distances of Jupiter and of a Trojan asteroid from the center of gravity are nearly the same. This would not be the case if Jupiter had, let us say, half the Sun's mass.

None of the eleven Trojan asteroids is situated exactly at the vertex of the equilateral triangle. They are all, however, *near* one of the two critical points, i. e., within some twenty degrees as seen from the Sun, and their orbits are presumably such that each asteroid oscillates about one of the critical points, and will continue to do so indefinitely.

As was mentioned above, this *particular* solution of the three-body problem is stable under certain conditions. Specifically, if one of the finite bodies has less than one twenty-sixth the mass of the other, the solution is stable. Since Jupiter's mass is less than one thousandth that of the Sun, the condition for stability is satisfied. It is therefore reasonable to suppose that, unless some unusually powerful disturbing force should appear, the Trojan asteroids will continue to enjoy one another's distinguished company long after the characters they symbolize have been forgotten.

In addition to the equilateral triangle solution of the three-body problem, Lagrange discovered other particular solutions. Although these do not strictly belong to the subject of this Leaflet, one of them is of interest in that it has an application in the solar system. Suppose again that the two finite bodies are revolving about each other in circular orbits. Then there are three "straight-line solutions". That is to say, there are three critical points in the imaginary line that lies in the direction of the two finite bodies; if an infinitesimal third body is placed at any one of these points, it will remain in the same relative position. One of the critical points lies between the two finite bodies; one lies in the direction of, and beyond, the first body; and the other is located in the direction of, and beyond, the second body. In other words, if the three bodies are on a straight line, there is a particular solution for each of the three orders, 1-3-2, 3-1-2, and 1-2-3. The relative separations that satisfy the solutions depend upon the mass-ratio of bodies 1 and 2.

The straight-line solutions, unlike the triangular solution, are unstable. A very slight disturbance will upset the balance, and will cause the third body to be removed indefinitely from the critical point. Under certain circumstances, however, it is possible for the third body, after the disturbance, to make one or more oscillations around the critical point before going very far away.

One of the straight-line solutions mentioned above has an application to the Gegenschein, a very faint, luminous area of the sky, which can be seen under the most favorable conditions, directly opposite the Sun. Let the Sun be the first body, and the earth the second; the earth revolves around the Sun in a nearly circular orbit. Then one of the points for a straight-line solution lies in the direction exactly opposite the Sun, at a distance of about a million miles from the earth. If a meteor happens to pass near that point under suitable conditions of direction and speed, it will make one or more revolutions about the point before leaving. If a large enough number of meteors were to be hovering about the critical point, the visible result would be a faint "counterglow", or Gegenschein. The above may be only a partial explanation of the fact that the Gegenschein is brighter than the rest of the Zodiacal Band (see Leaflet 81). The meteors opposite the Sun are seen at full phase, so that their light would appear somewhat brighter than those in neighboring regions of the zodiac, seen in gibbous phase. The relative importance of phase on the one hand, and of the straight-line solution of the three-body problem on the other, as they relate to the brightness of the Gegenschein, is not known.