Observable relations in relativistic cosmology.

By **W. H. Mc Crea.**

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The assumptions underlying the formulation of the line-element of the expanding universe, and the derivation of "world-pictures" in such a universe, are recapitulated. This enables one to know what assumptions are being tested by any particular comparison with observation. Formulae are given for "distance" in an expanding universe as judged by apparent size, apparent luminosity, parallax, rigid measuring rods. Problems of spectral displacement and spectral energy distribution are discussed. The number-density of nebulae, and the information to be obtained from counts of nebulae, are studied. Application is made to the cases of MILNE's "hydrodynamic" universe, and the EINSTEIN-DE SITTER "flat" universe. Observable differences between general relativity and NEWTONIAN models are examined.

1. Introduction. The problem of *spatial distance* in relativity has been extensively studied by E. T. WHITTAKER and his school[1]. Independent studies were made also by TOLMAN[2]. These investigations arose from the fact that any specific astronomical measurement of "distance" supposed carried out in any relativity model of space-time must lead to a result which depends upon the particular operations of measurement, and *not* upon the particular coordinate system used to describe the space-time. The writers named sought therefore to formulate invariants corresponding to various astronomical "distances" depending on observations of apparent magnitude, apparent size, and so on. A comparison of the properties of these invariants with the properties, for example the dependence on time, of the ,,distances" of stars and nebulae in the actual universe would then provide the first step towards a proper observational test of the predictions of relativity theory for large scale phenomena.

The present work was designed to give a simplified version of these calculations adapted to the usual models of the expanding universe. It was prompted by questions put to me by Professor E. A. MILNE[3] as to what "world-pictures", to borrow his own phrase, would be formed by

[1]) E. T. WHITTAKER, Proc. Roy. Soc. London (A) **133**, 93—105, 1931; A. G. WALKER, Monthly Notices, R. A. S. **94**, 159—167, 1934, with references there given. — [2]) R. C. TOLMAN, Proc. Nad. Acad. **16**, 511—520, 1930. — [3]) Forthcoming work by Professor MILNE will discuss "world-pictures" from other points of view. He has in several papers emphasised the importance of calculating "world-pictures", and, writing in December 1933, has stated fully the programme of such calculations needed in the ultimate test and comparison of cosmological theories, Observatory **57**, 24, 1934.

observers in these expanding universes. I am much indebted to him for opportunities to discuss the work with him.

The usefulness of this type of work is twofold. As already pointed out it is necessary for the observational test of the relativity theory of large scale phenomena, and in particular of the expanding universe. In the second place it is necessary for the proper comparison of *different* theories of these phenomena. For example, to compare the general relativity theory and MILNE's theory[1]) it is necessary to enquire if their *observable* consequences are different, and if so which agrees better with experience. Or again, it has been shown by MILNE[2]) that NEWTON's and EINSTEIN's laws of gravitation lead to equations of motion for the expansion of the universe which are identical in form. It will be shown below that the two cases are, however, observationally distinguishable.

Before this paper was ready for publication one dealing with the same subject was published by DE SITTER[3]). There are however certain discrepancies between his results and the present ones, the reasons for which will be indicated. Also, after the present calculations had been performed it was found that many of the results are given in Professor TOLMAN's book[4]) which has just recently appeared. Nevertheless, on account of certain differences of emphasis and application, the publication of the present work appears to be justified. It is difficult to give the new features apart from their setting in the general development. In presenting it the results already given by TOLMAN and others will therefore be derived as briefly as possible, and references given to their work.

2. Metric of the expanding universe. It is well-known that the metric of the general relativity expanding universe is expressible (e. g., TOLMAN, § 149) in the form

$$d s^2 = c^2\, d t^2 - R^2 \frac{d r^2 + r^2\, d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2}{(1 + k r^2/4)^2}, \tag{1}$$

where $R(t)$ is a real function of t only, and k/R^2 is the curvature of the section $t = \text{const}$. The constant k is positive, negative, or zero according as this section represents space of elliptic, hyperbolic, or euclidean type.

[1]) E. A. MILNE, ZS. f. Astrophys. **6**, 1—95, 1933. — [2]) E. A. MILNE, Quart. Journ. of Math. **5**, 64—72, 1934; W. H. MC CREA, and E. A. MILNE, ibid, S. 73—80. — [3]) W. DE SITTER, B. A. N. **7**, 205—215, 1934. This contains a much fuller discussion of the more practical aspects of the comparison with observation. — [4]) R. C. TOLMAN, Relativity Thermodynamics and Cosmology (Oxford 1934), p. 462—482. Results in this book will be quoted by page numbers, equation numbers, or paragraph numbers, thus: TOLMAN, 462—482.

The assumptions upon which the use of this metric rests are of a very general character. It is sufficient to suppose that the four-dimensional continuum is Riemannian, with the usual signature, and that there exists in it a fundamental set of observers having the following properties: one of their world-lines passes through each point of the continuum, and each sees the universe as locally symmetrical about himself[1]).

It follows from the last condition that these observers' "world-lines" are geodesics, which, when the metric is given in the form (1), are the lines $r, \vartheta, \varphi =$ const. From this it follows that all these observers see the same sequence of world-views. Further it is clear that if these observers are supposed attached to free particles then their world-lines will be the natural world-lines of the particles, whatever the laws of motion may be. For each of these particles would see itself as permanently central in the universe and so no law of motion could distinguish any particular direction in which it could start to move in its own frame of reference. Hence it must remain at rest in that frame.

These points are dwelt upon in order to show how the general features of the motion of the fundamental observers depend upon only a few very general assumptions. The only one of these which is characteristic of general relativity, rather than any other kinematic system, is that which leads to the use of Riemannian geometry. (We do not need even to assume that the paths of free particles *in general* are geodesics.) We do not assume any laws of motion, nor any field-equations. Yet the only properties of (1) which remain to be determined are the constant k and the function $R(t)$. The latter can so far be any real function of t. The general theory shows that R, k are finally determined only by a knowledge of the field-equations and the properties of matter in the universe.

We shall now study the world-pictures seen at $t = t_0$ by the observer, A say, whose world-line is $r = 0$. Since from (1) ct measures proper-distance along A's world-line, then according to the usual interpretation of general relativity, t_0 *is the actual reading of his clock,* from the appropriate origin, *at the instant of observation.*

This statement can be further analysed physically, once we have admitted that dt is proportional to a *short* lapse of proper-time for A. For

[1]) A. G. Walker, "On Riemannian spaces with spherical symmetry about a line, and the conditions for isotropy in general relativity". Quart. Journ. of Math. in press. I am indebted to Dr. Walker for allowing me to see this paper before publication.

one then supposes A supplied with some periodic mechanism to serve as a "clock", and enquire if the value of dt for one period is the same at all epochs. That is to say, does the expansion or contraction of the universe affect the rate of A's clock in units of t? We can see that there is no such effect to be taken into account since we can suppose the forces driving the clock to be made arbitrarily large compared with the general field in (1). Indeed the case of the hydrogen atom in an expanding universe studied by McVittie[1]) illustrates this. Using an appropriate wave-equation he finds that the frequencies of the spectral lines in units of t are unaffected by the expansion.

In virtue of the form of (1) any one of the lines $r, \vartheta, \varphi =$ const. can be transformed into any of the others without a change of t. Hence if an observer, B say has the world-line $r, \vartheta, \varphi = r_1, \vartheta_1, \varphi_1 =$ const., then at the event $(r_1, \vartheta_1, \varphi_1, t_1)$ the reading of his clock will be t_1. The coordinate t determines the *cosmic* time of an event only in the sence of giving the time ascribed to it by that one of the fundamental observers who directly experiences it.

3. Light-tracks. In order now to work out the observable phenomena in the universe described by (1) we must introduce some assumption about light-tracks. According to the standard procedure of general relativity we identify them with the null geodesics.

Now by symmetry, or by actually writing down the equations, it is seen that the null geodesics through $r = 0$ are given by $\vartheta, \varphi =$ const. Also along such a geodesic $ds = 0$, and so

$$dr/(1 + kr^2/4) = \pm\, c\,dt/R\,(t),$$

where R is the positive square root of R^2 in (1). Hence if $(r_1, \vartheta_1, \varphi_1, t_1)$ is the event in B's experience witnessed by A at time t_0, then

$$\int_0^{r_1} \frac{dr}{(1 + kr^2/4)} = \int_{t_1}^{t_0} \frac{c\,dt}{R}. \tag{2}$$

It is this relation which determines how much of the universe can be seen by A at time t_0. Consider for instance the case $k > 0$. Then in (1) we may have $0 \leqq r_1 < \infty$. Also from (1), (2) we may have $t_0 \geqq t_1 > -\infty$, as far as any hitherto imposed conditions go. Now the lefthand-side of (2) is a monotonic increasing function of r_1, which for $k > 0$ tends to $\pi/\sqrt{k}$

[1]) G. C. Mc Vittie, Monthly Notices, R. A. S. **92**, 868—877, 1932.

as $r_1 \to \infty$. Also the right-hand side of (2) increases monotonically as t_1 decreases from t_0 to $-\infty$, but may or may not converge according to the form of $R(t)$. If it converges to a limit l, then l is a function of t_0. If $l < \pi/\sqrt{k}$, then $r_1 \to r_{1l}$, say, as $t_1 \to -\infty$, and r_{1l} determines the furthest (fundamental) particle visible to A at time t_0. The value of r_{1l} depends on l, and so on t_0, and hence the number of (fundamental) particles visible at t_0 depends on t_0. If however $l > \pi/\sqrt{k}$, or if the right hand side of (2) tends to ∞ as $t_1 \to -\infty$, for any values of t_0, then $r_1 \to \infty$ as $t_1 \to t_{1\infty}$, say, where $t_{1\infty} > -\infty$. For these values of t_0 *all* the fundamental particles are visible to A. Corresponding results hold for $k \leqq 0$.

4. Doppler shift. The well-known result for the Doppler shift of light emitted at B and observed by A is

$$\lambda_0/\lambda_1 = R_0/R_1 = D, \text{ say,} \tag{3}$$

where λ_0, λ_1 are the wave-lengths of the light as measured by A, B respectively. (e. g., Tolman, § 155.)

We need also to find the Doppler effect for light emitted at B by a source in motion with respect to B. Let this source be at B at local time t_1, and at $r_1 + \delta r_1$ at local time $t_1 + \delta t_1$. Then the local proper distance in the r-direction between these two points is

$$R_1 \delta r_1/(1 + k r_1^2/4) = v_r \delta t_1,$$

where v_r is the r-component of the local velocity v of the particle relative to B.

Now let t_0, $t_0 + \delta t_0$ be the times of arrival at A of successive wave-crests of light emitted by the moving source at t_1, $t_1 + \delta t_1$. Then by differentiating (2) we have

$$\delta r_1/(1 + k r_1^2/4) = c\delta t_0/R_0 - c\delta t_1/R_1,$$

or, using the previous result,

$$\delta t_0/\delta t_1 = (R_0/R_1)\,(1 + v_r/c).$$

But times δt_0, $\delta t_1'$ must be proportional to the wavelengths of the light as measured by A and as measured at the source, where $\delta t_1'$ for an observer moving with the source corresponds to δt_1 measured by B, and therefore owing to the time-dilatation $\delta t_1 = \delta t_1'/\sqrt{1 - v^2/c^2}$. Hence, using (3) we have

$$\lambda_0/\lambda_1 = D \cdot (1 + v_r/c)/\sqrt{1 - v^2/c^2}. \tag{3'}$$

Thus if we interpret the fundamental Doppler effect D as being due to a radial velocity V, and suppose for the moment $V, v_r \ll c$, then we have from (3′) for the whole apparent Doppler effect $\lambda_0/\lambda_1 \doteqdot (1 + V/c)(1 + v_r/c) \doteqdot 1 + (V + v_r)/c$, which is the ordinary classical additive law. But we see that a more precise way of observing local velocities at B is given by (3′), even if D corresponds to a velocity V *not* small compared with c.

Suppose that B is a nebula whose speed of rotation we wish to determine. We suppose also that the resulting local velocities are small compared with the velocity of light c, so that we may replace $\sqrt{1 - v^2/c^2}$ in (3′) by unity. Then the mean value of λ_0/λ_1, over the whole nebula will give just D, since B is the local centre of the universe so that mean local velocities with respect to B must vanish. Hence to find the r-component v_r of the local velocity at any point in the nebula we have merely to observe λ_0/λ_1 for that point, divide by the mean value of λ_0/λ_1 for the whole nebula, and the resulting ratio is $1 + v_r/c$. In practice, however, the result would be some sort of mean value for the whole thickness of the nebula in the line of sight.

5. *Spatial distance.* Let dl be the lenght of a unit rigid scale placed at B perpendicular to AB in the plane $\varphi = \text{const}$. Then using the property of local proper distance with $dt = dr = d\varphi = 0$, we have

$$dl = \sqrt{-ds^2} = R_1 r_1 d\vartheta/(1 + kr_1^2/4), \tag{4}$$

where $R_1 \equiv R(t_1)$. But now, from the fact that the light tracks to A have $\vartheta = \text{const.}$, $d\vartheta$ is also the angle which dl appears to A to subtend at A. Hence if A adjudges the "distance" of B by the angle subtended by the unit scale he will define it by the ratio

$$\frac{\text{Length of scale}}{\text{Angle subtended at A}} = \frac{dl}{d\vartheta} = \frac{R_1 r_1}{(1 + kr_1^2/4)} = S, \text{ say.} \tag{5}$$

It is natural to term S *distance by apparent size.*

The same result may be got in another simple way by merely noting that the proper area of the sphere $r = r_1$ at $t = t_1$ is by (1)

$$4\pi r_1^2 R_1^2 / (1 + kr_1^2/4),$$

whereas it must of course be $4\pi S^2$ if S is its apparent distance as judged by its area.

The value (5) is also that obtained by A. G. Walker[1]) (equation (40)) and by Tolman (equation (180.3)). It differs from that given by de Sitter[2])

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[1]) A. G. Walker, Monthly Notices, R. A. S. **94**, 159—167, 1934. — [2]) W. de Sitter B. A. N. **7**, 205—215, 1934.

(equation (19)) in having R_1 in place of R_0 $\left(R \equiv (t_0) \right)$. This is due to his having attached a certain amount of physical reality to the "cosmical space" described by $d\sigma^2$, when (1) is written as $ds^2 = c^2 dt^2 - R^2 d\sigma^2$. This space does not in any way represent what is seen as "space" at any instant by any observer. For what he sees at different distances corresponds to *different* values of t.

6. *Parallax.* The distance S is not in general the same as that determined by parallax. Suppose unit length δl is held at A perpendicular to AB, and let $\delta\vartheta$ be the angle between the apparent directions of B as viewed from the two ends of δl. Then the "distance" P is measured according to the usual astronomical practice by the ratio $\delta l / \delta\vartheta$.

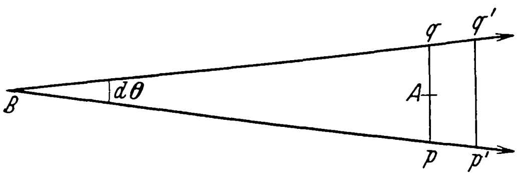

Fig. 1.

Actually this is not an important quantity for the practical study of large scale effects in the universe, since direct parallax measurements are not possible for "large" distances, but for the sake of completeness we shall find an expression for P.

Since we are going to consider light diverging from B we shall in this case transfer the origin O of (r, ϑ, φ) from A to B, so that $r = r_1$, $t = t_0$ for A; $r = 0$, $t = t_1$ for B. Suppose then that the light pulses which meet the extremities of δl at time t_0 leave B at time t_1 in directions inclined to each other at angle $d\vartheta$. We have to calculate $\delta\vartheta$, the apparent angle between their directions as determined by A.

In fig. 1 let p, q be the ends of δl, and let p', q' represent the corresponding points on the light-tracks for a change of r from r_1 to $r_1 + dr$. Then the length pp', qq' measured by A is $R_0 dr/(1 + kr_1^2/4)$, and the difference in time t for the arrival of the signal at p, p' (or q, q') is

$$dt = R_0 dr / c(1 + kr_1^2/4), \tag{6}$$

since the local velocity of light is always c.

As in (4) with t_0, t_1 interchanged, the lenght measured by A is

$$pq = \delta l = R_0 r_1 d\vartheta / (1 + kr_1^2/4). \tag{7}$$

Therefore the lenght $p'q'$ measured by A is

$$\begin{aligned} p'q' &= \delta l + d(\delta l) = \delta l + \frac{\partial(\delta l)}{\partial t_0} dt + \frac{\partial(\delta l)}{\partial r_1} dr \\ &= \delta l + R_0\, dr\, d\vartheta \, (1 - kr_1^2/4 + R_0' r_1/c) \,/\, (1 + kr_1^2/4)^2, \end{aligned} \tag{8}$$

using (6), (7). Further the angle $\delta\vartheta$ defined above is measured by $(p'q' - pq)/pp'$, which from (8) gives

$$\delta\vartheta = \frac{d(\delta l)}{p p'} = \frac{1 - k r_1^2/4 + R_0' r_1/c}{1 + k r_1^2/4} \cdot d\vartheta. \tag{9}$$

Thus, as we should expect, $\delta\vartheta \neq d\vartheta$ in general, but $\delta\vartheta$ reduces to $d\vartheta$ for $r_1 = 0$. Hence, using (7), (9), the *distance by parallax* P of B from A is by definition

$$P = \frac{\delta l}{\delta\vartheta} = \frac{R_0 r_1}{1 - k r_1^2/4 + R_0' r_1/c}. \tag{10}$$

7. Another definition. Again for the sake of completeness we mention the definition of "spatial distance" stated by KERMACK, MC CREA, and WHITTAKER[1]) and shown by RUSE[2]) to be equivalent to his definition of "distance" which provides the closest analogue in general relativity to that determined in ordinary physics by the use of rigid measuring rods. If the equations to the null geodesics are written in the standard form

$$\frac{d^2 x^p}{d\mu^2} + \begin{Bmatrix} q\,r \\ p \end{Bmatrix} \frac{d x^q}{d\mu} \cdot \frac{d x^r}{d\mu} = 0,$$

where a first integral is given by $g_{pq} \frac{d x^p}{d\mu} \cdot \frac{d x^q}{d\mu} = 0$, then the spatial distance between two events on the same null geodesic is taken to be proportional to the parameter μ integrated along it between the two points. The constant of proportionality is chosen so that the resulting distance reduces to ordinary local distance in the neighbourhood of the observer. In the present case it is easily shown that the distance A B so determined is K, say, where

$$K = R_0 \int_0^{r_1} \frac{dr}{(1 + k r^2/4)}. \tag{11}$$

8. Luminosity. We now come on to quantities of greater astronomical significance. In the first place we wish to calculate the apparent brightness of a source of light at B as seen by A. We do this by finding the amount of light from B which falls per unit time on unit area placed at A perpendicular to A B.

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[1]) KERMACK, MC CREA, and WHITTAKER, Proc. Roy. Soc. Edinburgh **53**, 31—47, 1933. — [2]) H. S. RUSE, ibid, p. 79—88.

By Planck's law the energy of a photon is inversely proportional to its wave-length, and so using (3) we have for any photon emitted by B

$$\frac{\text{Energy of photon observed at A}}{\text{Energy of photon emitted at B}} = \frac{\lambda_1}{\lambda_0} = \frac{R_1}{R_0}. \tag{12}$$

Also the same calculation as that which leads to (3) can be interpreted as showing that

$$\frac{\text{Number of photons arriving at A per unit time}}{\text{Number of photons emitted at B per unit time}} = \frac{R_1}{R_0}. \tag{13}$$

Further the angle $d\vartheta$ subtended at B by unit length δl placed at A perpendicular to AB is again given by (7).

Now the energy of the radiation from B which falls in unit time on unit area at A perpendicular to AB is proportional to

(Number of photons arriving per unit time) × (energy per photon) × (solid angle at B into which the quanta are emitted).

Using (12), (13) this is proportional to

$$(R_1/R_0)^2\, d\,\vartheta^2,$$

which using (7) is proportional to

$$R_1^2\,(1 + k r_1^2/4)^2/R_0^4\, r_1^2. \tag{14}$$

If then we define a quantity L by the relation [Tolman (178.10)]

$$L = \frac{R_0^2}{R_1}\,\frac{r_1}{(1 + k r_1^2/4)}, \tag{15}$$

the apparent luminosity of B is proportional to $1/L^2$. Also L reduces to the usual measure of "distance" for small r_1. Hence we call L the "*distance by luminosity*". It is in fact the distance so defined which is derived in practical astronomy by any method which depends upon a comparison of apparent with absolute luminosity. This applies to distances of extra-galactic nebulae deduced from the apparent magnitudes either of the nebulae themselves or of stars contained in them.

There is however one additional consideration which might in some future observations prove to have a measurable consequence. For it must be remenbered that *any* periodic phenomenon in an extra-galactic nebula must show, on the present theory, the same Doppler effect as does the radiation. Thus, if we could observe the periods of Cepheids in the most distant nebulae yet studied, for which the Doppler shift corresponds to about 1/10 of the velocity of light, we should have to ascribe to these stars true periods about 10 per cent shorter than those observed. A similar correction would have to be applied to the period of rotation of the whole nebula, or more precisely, the procedure described at the end of § 4 should be adopted in order to find the period. This consideration, whenever it has an appreciable effect, must be taken into account when these periods are used to derive absolute magnitudes.

9. Spectral energy distribution. Suppose now the source B emits energy $E(\lambda_1)d\lambda_1$ in the wave-length range $(\lambda_1, \lambda_1 + d\lambda_1)$ per unit solid angle. Then by (14) this contributes to the energy falling on unit area at A an amount proportional to

$$[R_1^2 (1 + kr_1^2/4)^2/R_0^4 r_1^2]\, E(\lambda_1)\, d\lambda_1. \tag{16}$$

But by (3) the observer A will ascribe to this radiation a wave length $\lambda_0 = \lambda_1 D$. Hence the radiation (16) is

$$[R_1^3 (1 + kr_1^2/4)^2 / R_0^5 r_1^2]\, E(\lambda_0/D)\, d\lambda_0. \tag{17}$$

As far as the *relative* intensity distribution goes, the effect of the red-shift is merely to replace $E(\lambda)$ by $E(\lambda/D)$. The resulting effect on photographic magnitude and colour index, which is of course an effect over and above the effects on *total* luminosity discussed in the last paragraph, has been discussed by Tolman § 177, and more fully by de Sitter (l. c.). The discrepancy between the latter author's determination of distance and the present one has, however, a counterpart in his determination of total luminosity.

10. Relation between "distances". Returning to (5), (15) we now have the result

$$\frac{\text{Apparent brightness of B}}{\text{Apparent area of B}} = \varkappa \frac{S^2}{L^2} = \varkappa \left(\frac{R_1}{R_0}\right)^4 = \frac{\varkappa}{D^4}, \tag{18}$$

where $\varkappa$ is a constant depending only on the intrinsic properties of B at the epoch of emission of the radiation. (Tolman (180.5), (185.8); the result for general space-time was given by A. G. Walker[1]), eq. (5).) Let us apply this to the nebulae, and assume that the average properties of the nebulae at a given distance are independent of that distance. Then if we average over all nebulae for each value of D we should find according to (18)

$$\overline{\log(\text{apparent brightness})} - \overline{\log(\text{apparent area})} = 4 \log D + \text{const.} \tag{19}$$

The bars denote mean values for given D. This result should be susceptible of observational test. Since nebulae can be observed with Doppler shifts

[1]) A. G. Walker, Monthly Notices, R. A. S. **94**, 159—167, 1934.

corresponding to about 1/10 of the velocity of light the formula (18) would predict for these cases a 40 per cent difference between the brightness estimated from a measurement of the total amount of light from them, and that deduced from a measurement of their total area.

Such a test would however provide, not a test of any relativity theory, but of the general correctness of ascribing the red-shift to the Doppler effect of the expansion of the universe. If the shift were due merely to a "degradation" of the energy of the light quanta, and not to the motion of the nebulae, then only the agency taken account of in (12), and not the effect (13) nor the aberration effect of the solid angle calculated by (7), would be effective. Hence in (18) we should have to replace D^4 by D. The difference for the above-mentioned extreme cases so far observed would be about 30 per cent, so that one might hope it could be separated from other effects in analysing observational data.

Uncertainties in conclusions drawn from these tests will arise from the averaging process suggested in (19), both from the difficulty of getting a large enough number of observations to form significant averages, and from the assumption that such an average would be the same at all distances. For we see a nebula B, according to the theory, at an epoch t_1, i. e., when it is a length of time $(t_0 - t_1)$ younger than our own nebula. If this is sufficiently small compared with the "life" of a nebula, then we may expect that its luminosity and size have not appreciably changed in that time, and so the assumption should be legitimate. For this reason it might prove more profitable in this connection to apply the theory to more accurate measures for not too distant nebulae, than to attempt to apply it quite generally. The trouble is that at present we do not know what may be interpreted as "not too distant".

11. Luminosity and Doppler-Shift. It is easy to see that (3), (15) give as a first approximation when $(t_0 - t_1)$ is small

$$(D - 1) \propto L, \tag{20}$$

which of course reproduces just Hubble's law that the apparent velocity of recession is proportional to the distance. This has been discussed fully by other writers.

If then we suppose t_1, r_1 eliminated between (2), (3), (15) we should have a theoretical relation between D, L, for any epoch of observation t_0, to which (20) is a first approximation. This will depend upon the value of k and the analytical form of $R\,(t)$. If also the values of D corresponding to all values of L could be observed for the actual universe and the results expressed as a functional relation between D, L, then a comparison with the theoretical relation would yield a differential equation for $R(t)$ involving k.

This procedure would be quite impracticable. It has been shown by A. G. Walker[1]) however that if we can obtain from observation the next approximation to (20), i. e., the values of coefficients a_1, a_2 (say) in an approximation $L = a_1(D - 1) + a_2(D - 1)^2$, then a comparison with the theoretical law will give values of R_0'/R_0^2, R_0''/R_0^3. Now if we assume the truth of the gravitational field equations, we can express ϱ_0, p_0, the mean proper density and pressure of the contents of the universe at local time t_0, in terms of these ratios, together with k/R_0^2 and the cosmical constant Λ. Hence if we consider the density and pressure as also observable magnitudes, we have an observational means of finding the values of k/R_0^2, Λ for the model universe corresponding most closely with the actual one. The limits within which these values probably lie are discussed by Tolman (§ 183).

The value of extensive observational data on the apparent magnitudes of the nebulae and the corresponding red-shifts is evident. It appears that it is better to express the magnitude in terms of a quantity such as L, rather than in terms of a "corrected" distance. Provided only that in finding the apparent bolometric magnitude m allowance is made for the effect of red-shift on the photographic magnitude, the "distance" L in parsecs of an object of absolute bolometric magnitude M is given by

$$5 \log (L/10) = m - M, \tag{21}$$

in virtue of the relation

$$\text{Apparent luminosity} = \frac{\text{Luminosity at 10 parsecs}}{(L/10)^2}, \tag{22}$$

which defines L, and of the relation between magnitude and luminosity. The quantities L, D may therefore be regarded as convenient "observables" and it is most profitable to express the observational data as empirical relations between such magnitudes. Then one should demand that theory also should state its predictions in terms of relations between these "observables". For this provides not only the best way of comparing these predictions with observations, but also the most direct method of comparing the predictions of different theories.

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[1]) A. G. Walker, Monthly Notices, R. A. S. **94**, 159—167, 1934.

12. Number of nebulae. We shall now once and for all identify the "particles" to which the fundamental observers are attached with the extra-galactic nebulae in our model universe. The world-lines of these particles have r, ϑ, $\varphi =$ const. So the number of these world-lines in a given range dr, $d\vartheta$, $d\varphi$ of these variables is a given constant which, in order to presserve the equivalence of world-views of all the observers must be the same for all equal *coordinate* volume elements. Now from (1) the *proper* volume element for given dr, $d\vartheta$, $d\varphi$ is

$$R^3 r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi \,/\, (1 + kr^2/4)^3,$$

or R^3 times the coordinate volume element. Hence the proper number-density of nebulae n, being equal to the number of intersections of world lines with unit proper volume, is inversely proportional to R^3. Hence we write

$$n = \alpha/R^3. \quad (\alpha = \text{const.}) \tag{23}$$

This result is independent of any gravitational field equations. It is also independent of the existence of mass in the form of matter other than the nebulae, or in the form of radiation, and in fact depends only on the assumption that the nebulae retain their individuality for all values of t. It is however the significant "density" for those observations which consist in *counting* the nebulae in different ranges of apparent magnitude, apparent size, red-shift, etc. We proceed to give the theoretical results of such counting processes.

The number of particles dN observed by A to lie in the shell of radius r_1, thickness dr, is the number in the corresponding proper volume at time t_1 and is therefore by (23)

$$\begin{aligned} dN &= n(t_1)\cdot 4\pi R_1^3 r_1^2\, dr_1 \,/\, (1 + kr_1^2/4)^3 \\ &= 4\pi\alpha r_1^2\, dr_1 \,/\, (1 + kr_1^2/4)^3, \end{aligned} \tag{24}$$

which is of course constant as it ought to be. This can now be expressed in terms of the corresponding ranges of D, S, L.

From (2), (3) we have respectively, for given t_0,

$$dr_1 \,/\, (1 + kr_1^2/4) = -\,c\,dt_1/R_1, \qquad dD/D = -\,R_1'\,dt_1/R_1,$$

so that

$$dr_1 \,/\, (1 + kr_1^2/4) = c\,dD/R_1' D.$$

Hence from (24)

$$dN = 4\pi\alpha \frac{r_1^2}{(1 + kr_1^2/4)^2}\cdot\frac{c}{R_1'}\cdot\frac{dD}{D}\cdot \tag{25}$$

In this relation r_1, t_1 and so R'_1 are to be considered known in terms of t_0, D in virtue of (2), (3), though the elimination of r_1, t_1 cannot be performed explicitly without a knowledge of the explicit form of $R(t)$.

The relation (25) is therefore one that can in principle be compared directly with observation. For all that is needed for the comparison is a measurement of the redshift for a sufficiently representative number of nebulae and thence an estimate of the total number of nebulae in each range of values of the shift. This would give dN/dD in (25), and hence working backwards from that relation we should be able to find the form of $R(t)$. Thus a knowledge of the functional form of dN/dD would give us the same information as a knowledge of the functional form of the L, D-relation treated in § 11. At first sight it appears moreover to offer a more certain practical method, depending as it does on the measurement of a spectroscopic wave-length change and an operation of counting. Apart from the difficulty of getting the spectra in the first place, and the further difficulties and labour of getting them in sufficient number, it would seem that these observations could be made with any desired degree of accuracy. The observational determination of the L, D-relation on the other hand, requires besides the spectroscopic measurements a measurement of apparent magnitudes and a knowledge of corresponding absolute magnitudes in order to get L values, and these are subject to larger uncertainties. In particular the effect of obscuring matter cannot be accurately allowed for. But it has now to be remembered that we could get some important information from a knowledge of the empirical L, D-relation to a certain order of approximation. The further question then arises as to the order of approximation to which the empirical N, D-relation would have to be known in order to obtain the same amount of information.

We can express $(t_0 - t_1)$ as a series in $\sigma \equiv D - 1$ from (3), and r_1 as a series in $(t_0 - t_1)$ from (2), and hence as a series in σ. Using such series in (25) we find[1])

$$(1+\sigma)\frac{dN}{d\sigma} = 4\pi\alpha\frac{c^3}{R_0'^3}\Big\{\sigma^2 + \frac{2}{R_0'^2}(2R_0R_0'' - R_0'^2)\sigma^3 + \frac{1}{12R_0'^4}(11R_0'^4 - 46R_0'^2R_0R_0'' + 45R_0^2R_0''^2 - 10R_0'R_0'''R_0^2 - 4c^2kR_0'^2)\sigma^4 + \cdots\Big\}. \quad (26)$$

[1]) I am indebted to Mr. K. K. Mitra for checking this calculation.

For sufficiently small σ we have $N \propto \sigma^3$, as we expect from the assumption of local uniform density and the fact that σ is then proportional to ordinary distance. Existing observational data probably provide a check of this first approximation.

Now (26) shows that for sufficiently large shifts σ the terms after the first will become important. If a second approximation can be got observationally it will provide a value of $R_0 R_0''/R_0'^2$ if this ratio is not too small compared with unity. From (27) its value is found to be $(1 + 2\, l^*/k^{*2})$ where l^*, k^* denote the quantities "l", "k" used by Tolman (§ 183). According to his estimates the upper and lower bounds of this function would be (1 ± 3). But now to get values of $k/R_0^2, \Lambda$ as discussed in § 11 it is necessary to know R_0'/R_0^2, R_0''/R_0^3 *separately*. As we should expect, since $dN/d\sigma$ involves in α a constant additional to those occuring in L, the second approximation does not provide this information. Even the next approximation would however not suffice for from (26) it would involve yet another parameter R_0'''/R_0^4. We therefore conclude that the N, D-relation is of less practical value than the L, D-relation.

The N, S-relation corresponding to (26) is similarly found to be given by

$$\frac{dN}{dS} = \frac{4\pi\alpha}{R_0^3}\left(S^2 + 4\frac{R_0'}{c R_0} S^3 \cdots\right). \tag{27}$$

There is an analagous N, L-relation. These show what information can be got from counts of nebulae in different ranges of "distance". This is not what is given directly by observation, for counts are actually made for different ranges of apparent magnitude, or apparent size if this is measured, whereas the L or S values must be derived from a comparison of these quantities with an assumed absolute magnitude, or absolute size, in each separate case. It has however been shown by de Sitter (l. c. § 5) that the actual counts can be compared directly with theory if we assume, say, a normal law of distribution of absolute magnitude. The "dispersion" of the distribution then provides an additional parameter to be derived from the observations.

Special Cases.

The remainder of the paper will be devoted to an application of the foregoing theory to some special cases. Those chosen have the advantage of being among the simplest examples and it proves possible in these cases to write down explicit expressions for most of the relations discussed above. The physical meaning of these expressions can readily be appreciated, and so it becomes easier to know what phenomena to expect in more general cases.

13. Milne's hydrodynamical case. The simplest example of Milne's cosmology[1]) has been shown[2]) to be at any rate kinematically equivalent to the particular case of (1) for which k is negative and R is proportional to t. Changing the units of r we can then write (1) in the form

$$d s^2 = c^2\, dt^2 - \frac{\varkappa^2\, t^2}{(1 - r^2/4)^2}\,(dr^2 + r^2\, d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2), \quad (0 \leqq r < 2) \quad (28)$$

where $\varkappa^2 \propto -1/k$. To get precisely Milne's case we must take $\varkappa = c$.

Equation (2) then becomes

$$\frac{1 + r_1/2}{1 - r_1/2} = \frac{t_0}{t_1}. \quad (29)$$

The whole universe in this case has $0 \leqq r_1 < 2$, and from (29) we see that as r_1 increases from 0 to 2, t_1 decreases from t_0 to 0. Thus the whole universe is in view at any instant t_0, and points close to the boundary $r_1 = 2$ are always seen at local time close to $t_0 = 0$ [3]).

Again (3) becomes

$$D = t_0/t_1 = (1 + r_1/2)\,/\,(1 - r_1/2); \quad (30)$$

(5) becomes

$$S = c t_0 r_1\,/\,(1 + r_1/2)^2 \quad (31)$$

$$= c t_1 r_1\,/\,(1 - r_1^2/4) \quad (32)$$

$$= c\,(t_0^2 - t_1^2)/2\, t_0 \quad (33)$$

using (29); (10) becomes

$$P = c t_0 r_1\,/\,(1 + r_1/2)^2 = S \quad (34)$$

by (31); (11) becomes

$$K = c t_0 \log \left\{(1 + r_1/2)\,/\,(1 - r_1/2)\right\}; \quad (35)$$

(15) becomes

$$L = c t_0 r_1\,/\,(1 - r_1/2)^2, \quad (36)$$

$$= S t_0^2/t_1^2 = c t_0\,(t_0^2 - t_1^2)/2\, t_1^2 \quad (37)$$

using (33).

From (34), (36) we see that as $r_1 \to 2$

$$P = S \to c t_0/2, \quad L \to \infty. \quad (38)$$

[1]) E. A. Milne, ZS. f. Astrophys. **6**, 1—95, 1933. — [2]) W. O. Kermack and W. H. McCrea, Monthly Notices, R. A. S. **93**, 519—529, 1933; H. P. Robertson, ZS. f. Astrophys. **7**, 153—166, 1933. — [3]) E. A. Milne, Monthly Notices, R. A. S. **93**, 668—680 1933.

Thus the distance of the most remote nebulae, when judged by their size or parallax, is always a finite constant, while that judged by their luminosity is infinite. This illustrates the fact that there may be no unambiguous answer to the question: Is the universe spatially infinite? Obviously the answer depends upon the definition of "distance". But this simple model of Milne's shows that two natural definitions of distance, the one depending on the apparent size of an object and the other on its apparent brightness, may lead to a finite extent for the universe measured one way, and an infinite extent when it is measured the other way.

The last result is just that given by Milne when he shows that the luminosity of the most distant nebulae seen is vanishingly small[1]) and we can recover his expression of it. For if we write $\bar{r}$, T for his r, t we have[2])

$$\bar{r} = ctr/(1 - r^2/4), \quad T = t(1 + r^2/4)/(1 - r^2/4). \tag{39}$$

Hence in the first place, if $\bar{r}_1$ corresponds to t_1, r_1 then from (32)

$$\bar{r}_1 = S,$$

so that Milne's coordinate r for a nebulae at the instant when it is seen measures also its distance by apparent size. Substituting (39) in (36) we then find

$$L = T_0 V/(1 - V/c), \tag{40}$$

where $T_0 = t_0$ gives the time of observation, and $V = \bar{r}/T$ is the same as in Milne's work. This relation expresses the same result as Milne's equation (16)[3]).

Again, using (39) in (30) we find, if V_1 applies to (r_1, t_1),

$$D^2 = (1 + V_1/c)/(1 - V_1/c), \tag{41}$$

which is the usual special relativity Doppler effect for radial velocity V_1. We may notice that from (31), (36)

$$\frac{dS}{dt_0} = \frac{S}{t_0} = \frac{T}{T_0}V, \quad \frac{dL}{dt_0} = \frac{L}{t_0} = \frac{V}{1 - V/c}, \tag{42}$$

for any given nebulae, for which r_1 is of course constant. These quantities also define apparent "velocities" of the nebulae, in the sense of rates of change of "distance" with time of observation. But it is seen that they

1) E. A. Milne, ZS. f. Astrophys. **6**, 1—95, 1933, S. 95. — 2) W. O. Kermack and W. H. McCrea, Monthly Notices, R. A. S. **93**, 519—529, 1933. — 3) E. A. Milne, ZS. f. Astrophys. **6**, 1—95, 1933, S. 95.

are not identical with the velocity V. We may note also that the value of D corresponding to any particular nebula, i. e., to a given value of r_1, is a constant independent of the time of observation.

We may now obtain explicit distance-shift relations. From (30), (33) we have

$$S = ct_0 (D^2 - 1)/2 D^2. \tag{43}$$

This by (41) reduces to

$$S = t_0 V/(1 + V/c), \text{ or } V = S/(t_0 - S/c), \tag{44}$$

which is MILNE's[1]) equation (4). Again from (30), (37), (41) we have

$$\begin{aligned} L &= ct_0 (D^2 - 1)/2, \\ &= t_0 V/(1 - V/c), \text{ as in (40).} \end{aligned} \tag{45}$$

In this case it is possible to obtain from (24) explicit expressions for the number of nebulae in given ranges of D, S, L. We find from the expressions (30), (31), (36) for these respective quantities, and the general result (24),

$$dN/dD = \pi \alpha (D^2 - 1)^2/D^3, \tag{46}$$

$$dN/dS = 4\pi\alpha S^2/ct_0 (ct_0 - 2S)^2, \tag{47}$$

$$dN/dL = 4\pi\alpha L^2/ct_0 (ct_0 + 2L)^2. \tag{48}$$

Equation (47) is just that given by MILNE[1]), equation (5′).

If for the moment we suppose all nebulae have the same absolute magnitude, then the total light reaching A from a nebula at "distance" L is proportional to $1/L^2$. Hence from (48) the total light reaching A from the whole universe is proportional to

$$\int_0^\infty \frac{dN}{L^2} = \frac{4\pi\alpha}{ct_0} \int_0^\infty \frac{dL}{(ct_0 + 2L)^2} = \frac{2\pi\alpha}{c^2 t_0^2}. \tag{49}$$

This, as has again been indicated by MILNE, is a finite quantity proportional to $1/t_0^2$, even though the number of nebulae is itself infinite, as is clear from any of the expressions for dN.

It may be of some theoretical interest, though scarcely of practical value, to calculate the spectral distribution of this energy. We shall suppose for this purpose that all the nebulae behave like black bodies of tempera-

[1]) E. A. MILNE, Monthly Notices, R. A. S. **93**, 668—680, 1933.

ture T, and all have the same absolute luminosity. Then by (17) the radiation from B reaching A per unit time in the range $(\lambda_0, \lambda_0 + d\lambda_0)$ is

$$\frac{Q' R_1^3 (1 - r_1^2/4)^2}{R_0^5 r_1^2} \cdot \frac{D^5 \, d\lambda_0}{\lambda_0^5 (e^{h c D/\lambda_0 k T} - 1)}, \tag{50}$$

where Q' is a constant. Then to get the total energy in this range we have to multiply (50) by dN as given by (46) and integrate for all r_1, or D. This gives for the energy a value

$$\frac{Q \, d\lambda_0}{t_0^2 \lambda_0^5} \int_1^\infty \frac{D \, dD}{e^{b D} - 1}, \tag{51}$$

where $Q = 4\pi\alpha Q'/c^2$ and $b = hc/\lambda_0 k T$. If b is sufficiently large the integral is approximately e^{-b}/b and the quantity (51) is proportional to

$$(k T/h c) \, e^{-h c/\lambda_0 k T} \, \lambda_0^{-4} \, d\lambda_0. \tag{52}$$

On the other hand, if b is sufficiently small the integral is approximately $b^{-2}\int_0^\infty x\,dx/(e^x - 1)$, and the quantity (51), is proportional, with the same factor as in (52), to

$$(k T/h c)^2 (\pi^2/6) \, \lambda_0^{-3} \, d\lambda_0.$$

14. Einstein-de Sitter *Universe.* This is the model with $k = 0$. Applying the gravitational field equation to this case, and assuming zero pressure and $\Lambda = 0$, one finds[1])

$$R = a t^{2/3},$$

where a is constant, and t is measured from a suitable origin.

Thus (1) becomes

$$ds^2 = c^2 dt^2 - a^2 t^{4/3} (dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2), \quad (0 \leqq r < \infty).$$

Equation (2) then gives

$$r_1 = 3\,(c/a)\,(t_0^{1/3} - t_1^{1/3}), \tag{53}$$

from which $t_1 \to -\infty$ as $r_1 \to \infty$. Thus, if all values of t_1 in $-\infty < t_1 \leqq t_0$ are allowed, then all parts of space with $0 \leqq r_1 < \infty$ are visible to A at any time t_0, and the local time when they are observed tends to $-\infty$, as the coordinate distance $r_1 \to \infty$. If however we suppose the universe has only a finite past, extending to $t = 0$, say, then only the part with $0 \leqq r_1 \leqq 3\,c t_0^{1/3}/a$ is visible to A at time t_0. Therefore only this region *exists* for A at time t_0.

[1]) Einstein and de Sitter, Proc. Nat. Acad. **18**, 213, 1932; Tolman § 164.

Again (3) becomes

$$D = (t_0/t_1)^{2/3}; \tag{54}$$

(5) becomes

$$S = a t_1^{2/3} r_1 = 3\, c t_1^{2/3} (t_0^{1/3} - t_1^{1/3}), \tag{55}$$

or

$$t_0 = (S/ar_1)^{3/2} + (S/c) + (S a^3 r_1^3/9 c^4)^{1/2} + (a r_1/3 c)^3; \tag{55'}$$

(10) becomes

$$P = a t_0^{2/3} r_1/(1 + 2\, a r_1/3\, c t_0^{1/3}); \tag{56}$$

(11) becomes

$$K = a t_0^{2/3} r_1; \tag{57}$$

(15) becomes

$$L = a t_0^{4/3} t_1^{-2/3} r_1 = 3\, c t_0^{4/3} t_1^{-2/3} (t_0^{1/3} - t_1^{1/3}). \tag{58}$$

The explicit S, D- and L, D-relations are therefore from (54), (55), (58)

$$S = 3\, c t_0 D^{-3/2} (\sqrt{D} - 1), \qquad L = 3\, c t_0 D^{1/2} (\sqrt{D} - 1). \tag{59}$$

Equation (59) may be written for sufficiently small S in the form

$$D = 1 + 2\, S/3\, c t_0 + 7\, S^2/72 c^2 t_0^2 + \cdots. \tag{59'}$$

Also using (53) we can write (54) in the form

$$D = 1/(1 - a r_1/3\, c t_0^{1/3})^2. \tag{54'}$$

Hence for a given nebula, i. e., for fixed r_1, as t_0 increases from $-\infty$, D starts by decreasing from unity to 0 for $t_0 = 0$, then increases to ∞ for $t_0 = (a r_1/3 c)^3$, then decreases again and tends to unity as $t_0 \to \infty$. This last is the result that the apparent velocity of recession decreases with advancing epoch of observation[1]).

In this case further (24) leads to

$$dN = 4 \pi \alpha r_1^2\, dr_1, \tag{60}$$

which gives from (53), (54)

$$dN/dD = 2 \pi \alpha\, (3\, c/a)^3 t_0\, (\sqrt{D} - 1)^2/D^{5/2}, \tag{61}$$

while expressions for dN/dS, dN/dL cannot so easily be expressed explicitly in terms of S, L respectively.

From (55) we see that as $t_1 \to -\infty$, $S \to \infty$ monotonically. On the other hand, from (58)

$$L \to \infty \text{ as } t_1 \to 0,\ L \to 0 \text{ as } t_1 \to -\infty.$$

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[1]) W. H. Mc Crea and E. A. Milne, Quart. Journ. of Math. 5, 73—80 1934.

Thus, if values of t_1 less than 0 are allowed, S is a two-valued function of L. The reason for this is made clear by (54) which gives

$$D \to \infty \text{ as } t_1 \to 0, \; D \to 0 \text{ as } t_1 \to -\infty,$$

and $D = 1$ for $t_1 = \pm t_0$. Thus $D > 1$ for $t_0 > t_1 > -t_0$, and we have a *red*-shift which tends to weaken the apparent total energy of the corresponding radiation, and for the particular value $t_1 = 0$ all the light has been shifted to zero frequency which gives zero luminosity and so infinite "distance by apparent brightness". For $t_1 < -t_0$, on the other hand, $D < 1$ and we have a *violet*-shift which tends to increase the apparent total energy of the corresponding radiation. Since D is small for large negative t_1, the corresponding luminosity is great, and so the distance by apparent brightness is small.

The same effect manifests itself through the total radiation received at A in a given range of wave-length. The radiation from B reaching A per unit time in $(\lambda_0, \lambda_0 + d\lambda_0)$ is, from (17),

$$\frac{Q'}{a^2 t_0^{4/3} r_1^2} \cdot \frac{D^2 \, d\lambda_0}{\lambda_0^5 \left(e^{hcD/\lambda_0 kT} - 1\right)}, \tag{62}$$

making the same assumptions about the radiation of the nebulae as those leading to (40) in Milne's universe. With (60) this gives for the total energy in this range

$$Q \frac{d\lambda_0}{t_0 \lambda_0^5} \left[\int_1^\infty \frac{D^{1/2} \, dD}{(e^{bD} - 1)} + \int_0^\infty \frac{D^{1/2} \, dD}{(e^{bD} - 1)} \right]. \tag{63}$$

where $Q = 6 \pi c \alpha Q' / a^3$.

If b is large this is approximately

$$Q\gamma \, (d\lambda_0 / \lambda_0^{7/2}) \, (kT/hc)^{3/2} / t_0, \tag{64}$$

while if b is small this is approximately

$$2 \, Q\gamma \, (d\lambda_0 / \lambda_0^{7/2}) \, (kT/hc)^{3/2} / t_0, \tag{65}$$

where $\gamma = \int_0^\infty \sqrt{x} \, dx / (e^x - 1)$. Thus (64) shows that for small wave-lengths the energy *increases* as $\lambda^{-7/2}$, whence it follows that the total radiation reaching A per unit time is infinite, though the amount in any range not including $\lambda = 0$ is finite. Actually, of course, this result reveals an inconsistency in the model, since on relativity theory there could not be an infinite energy density at any event where the curvature of space-time is finite. The trouble arises from the fact that in assuming zero pressure we necessarily neglect the effect of radiant energy on the curvature of space-time.

The physical reason for the various novel features of this model is that the nebulae have such large accelerations in the neighbourhood of the singular epoch $t = 0$ that the light which at any instant reaches A from the more distant nebulae left them *before* they reached $t = 0$. Hence those that are sufficiently remote on the r-scale appear to be approaching the observer with velocities which are such that the consequent Doppler increase in the apparent frequency of their radiation makes their apparent luminosity *increase* with increasing r. Thus the background of the sky would appear increasingly bright when observed in increasingly high frequencies.

In his Joule Memorial Lecture[1]), Professor MILNE has recently suggested *a priori* reasons for the impossibility of observing anything corresponding to epochs before $t = 0$. It is difficult however to find analogues of his arguments strictly within the framework of general relativity theory. This may be a deficiency in the latter theory, but since at the moment we are concerned with the observational test of models proposed by this theory it remains merely as a matter for observation to decide whether it finds evidence of events before the singular epoch $t = 0$. Such evidence would be provided by the velocities of approach just discussed. Individual nebulae having these velocities could probably not be observed on this model, but evidence might sometime be found for the background of short wave-length radiation.

15. Newtonian Universe. It has been shown by MILNE[2]), and generalised by Mc CREA and MILNE[3]), that there exist models of the universe which are analagous to the general relativity ones but which obey the Newtonian law of gravitation and Newtonian "relativity". It then emerges that the differential equation for the function $R(t)$ is the same on both theories. Therefore the question arises as to whether models with the same $R(t)$ would be observationally different in the two theories. We naturally expect divergences for "local" phenomena; for example in central orbits where general relativity predicts the advance of the perihelion and Newtonian theory does not. But we are here treating only large scale effects in the smoothed out universe, and since then both theories predict the same coordinate velocities any difference must enter in translating these into observed velocities. That is, it must be due to different laws of propagation of light.

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[1]) E. A. MILNE, Mem. and Proc. Manchester Lit. and Phil. Soc. **78**, 1934. — [2]) E. A. MILNE, Quart. Journ. of Math. **5**, 1934. — [3]) W. H. McCREA and E. A. MILNE, Quart. Journ. of Math. **5**, 73—84, 1934.

For definiteness we discuss the Newtonian universe with "parabolic" velocities, which is the analogue of the Einstein-de Sitter universe discussed in § 14. Here the Newtonian velocity v at Newtonian distance r from the observer at time t after the singular epoch is (Milne[1]), equation (5)) $v = 2\,r/3\,t$, giving

$$r = a r_1 t^{2/3}, \tag{66}$$

where $a r_1$ is a constant for a particular nebula written to correspond to the constants used in § 14. Here t is universal Newtonian time.

It will be noted that (55), (57) reduce to (66) in the limiting case of $c \to \infty$. Further in classical theory the Doppler effect is given by

$$D = \lambda_0/\lambda = |1 + v/c|, \tag{67}$$

where λ is the wave-length emitted by a source having radial velocity v away from the observer who then observes wave length λ_0. Applying (67) to (66) we have

$$D = |1 + 2\,a r_1/3 c t^{1/3}| \tag{68}$$

as the analogue of (54′). For small r_1, it agrees with (54′) as far as the first term in r_1, but not in higher terms. For a given nebula, i. e., for fixed r_1, as t increases from $-\infty$, D decreases from unity to 0 at $t = (-2\,a r_1/3\,c)^3$, increases to ∞ for $t = 0$, and decreases again towards unity as $t \to \infty$. This behaviour is qualitatively like that in the Einstein-de Sitter universe, but owing to a different power of t, the epoch of observation, the quantitative rate of change is different and constitutes in principle an observational difference between the two models.

The distance S in this model is ordinary Newtonian distance measured to the point at which the lightsource is actually seen. So using (66) in an elementary calculation the distance S at time t of the nebula identified by parameter r_1 is given by

$$t = (S/a r_1)^{3/2} + (S/c). \tag{69}$$

This is seen to agree with (55′) as far as terms in $1/c$. Also the relation between distances S, L expressed by

$$L = S \cdot D^2 \tag{70}$$

continues to hold in this case.

If now we eliminate r_1 between (68), (69) we find

$$D = |1 + 2\,S/3\,c t^{1/3}\,(t - S/c)^{2/3}| \tag{71}$$

$$= 1 + 2\,S/3\,ct + 4\,S^2/9\,c^2\,t^2 + \cdots, \tag{72}$$

[1]) E. A. Milne, Quart. Journ. of Math. 5, 64—72, 1934.

for sufficiently small S. This is the analogue of (59′) with which it agrees as far as the second term, but again the terms in $1/c^2$ are different. The comparisons are made by taking the observer's time t_0 in the general relativity case to correspond to the Newtonian time t. We must however expect a divergence between the two theories for terms in $1/c^2$ since purely classical theory cannot give unambiguous results to this order. These terms are affected by the choice of the observer with respect to whom the velocity of light is taken to be c.

Is may be noted that in comparing models for which the parameter k is not zero there is an additional difference. For in place of (24) the Newtonian models will always give

$$dN = 4\pi\alpha r_1^2\, dr_1. \tag{73}$$

Milne[1]) has pointed out that all the at present observable large scale properties of the universe could have been predicted on Newtonian mechanics alone. The present paragraph however indicates some of the features which should be studied, if more refined data were available, in order to see which theory ultimately gives the best account of the facts.

16. Discussion. This paper has sought to make clear what particular set of assumptions are being tested when comparison is made with any particular set of observational results. The *broad* conclusions to which we are led are these:

If we compare a relation like that connecting apparent size, apparent brightness, and red-shift we are testing merely the correctness of our interpretation of the observed quantities, and not any particular theory of them.

If we make all possible observations on the "distances", red-shifts, and numbers of the extra-galactic nebulae we are testing the possibility of representing them as the fundamental particles in a universe of the type (1), and the correctness of the derivation of "world-pictures" in such a model.

In order however to choose between such models and classical ones it will in general be necessary to test terms of the order $(1/c^2)$, or, in virtue of equation (73), terms in k in expressions involving the numbers of nebulae.

The observations of "distance", red-shift, and number of nebulae are used to test the theory *without* the assumption of any equations of motion or field equations. The question arises as to whether, *in principle*, it is possible to know from these observations alone if we should take $k < 0$, $k = 0$, or $k > 0$ in the model universe. Clearly we cannot decide on the basis of one type of observation alone. For suppose we could observe the

[1]) E. A. Milne, Quart. Journ. of Math. 5, 64—72 (1934).

distance S as a function of the Doppler effect D. Then we could deduce a function $R(t)$ from the S, D-relation for *any* assumed value of k. This leads us to enquire if the various functions $R(t)$, corresponding to various values of k, would lead to different forms of dN/dD (say). It is fairly evident that this must be so, but it is not easy to prove generally that dN/dD cannot be expressed as a function of $S(D)$ and its derivatives, without explicit mention of k. A simple example however should prove convincing. If in some particular hypothetical model we were to find, say,

$$S = 2 R_0 (D - 1)/(2 D - 1),$$

then, under the assumption $k = 0$, we should find

$$dN/dD = 32 \pi \alpha D^2 (D - 1)^2 (2 D^2 - 2 D + 1)/(2 D - 1)^4,$$

while, under the assumption $k = -1$, we should find a different result given by

$$dN/dD = 32 \pi \alpha D^2 (D - 1)^2/(2 D - 1)^3.$$

This is sufficient to show that observation of *both* distances and numbers of nebulae as functions of their Doppler shift would determine k, and thus show whether we should use models with elliptic, euclidean, or hyperbolic spatial sections. Other sets of observations yielding the same information could be proposed.

The determination of k by this means is of interest in principle, but could not at present be used in practice. The most useful practical method is probably still that proposed first by Walker, and depends upon finding the second approximation in the observed relation between L and D, and upon observing values of the density and pressure in the universe. The use of these latter data does however assume the truth of the gravitational field equations. Conversely the truth of these equations could be tested if we could derive values of k in both ways and compare the results.

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