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COMMUNICATIONS FROM THE OBSERVATORY AT LEIDEN.

Some further computations regarding non-static universes, by *W. de Sitter.*

The non-static solutions of the field equations of the general theory of relativity, of which the line-element is

$$(1) \qquad ds^2 = - R^2 d\sigma^2 + c^2 dt^2,$$

R being a function of t alone, and $d\sigma^2$ being the line-element of a three-dimensional space of constant positive curvature with unit radius, have been investigated by FRIEDMAN [1]) in 1922 and independently by LEMAÎTRE [2]) in 1927, and have attracted general attention during the last year or so. EINSTEIN has lately [3]) expressed his preference for the particular solution of this kind corresponding to the value $\lambda = 0$ of the "cosmological constant". This solution belongs to a family of oscillating solutions, which were not included in my discussion in *B. A. N.* 193. In the present paper I propose to complete the discussion so as to include all possible solutions for the case of constant mass.

In the actual universe the amount of energy in the form of random motions of material particles and of radiation is extremely small as compared with the material mass. Consequently the assumption that the pressure is zero is a very good approximation to the truth, and from this it follows that the total mass of the universe is constant. The field equations then are reduced to the single equation

$$(2) \qquad \frac{1}{R^2}\left(\frac{dR}{cdt}\right)^2 + \frac{1}{R^2} = \frac{1}{3}\lambda + \frac{R_1}{R^3},$$

where $R_1 = \frac{1}{3}\alpha = \varkappa M/3\pi^2$, M being the total mass of the universe. If we put

$$y = \frac{R}{R_1}, \qquad \mathrm{t} = \frac{ct}{R_1},$$

the equation becomes

$$(3) \qquad \left(\frac{dy}{dt}\right)^2 = \frac{1}{y} - 1 + \gamma y^2,$$

where we have put

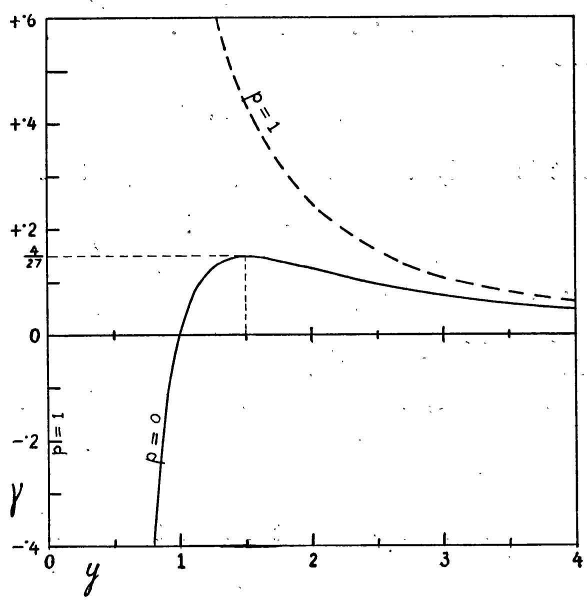
$$\gamma = \frac{1}{3} R_1^2 \lambda.$$

In the discussion of the solutions of the equation (3) the quantity

$$(4) \qquad p = 1 - y + \gamma y^3$$

plays an important part. In the figure 1, in which the abscissa is y and the ordinate γ, the full drawn line is the curve $p = 0$. Below it the value of p is negative, above it positive. Since y is necessarily positive, the

FIGURE 1.



real solutions of (3) occupy the part of the diagram to the right of the axis of γ ($y = 0$) and above the curve $p = 0$. Both R_1 and λ are constants, consequently γ is also a constant, but its value is absolutely unknown. The curve has a maximum for $y = 1{\cdot}5$, $\gamma = \frac{4}{27}$.

If γ exceeds $\frac{4}{27} = 0{\cdot}14815$, there is only one solution, in which y increases from zero to infinity.

[1]) *Zeitschrift für Physik*, X, p. 377.

[2]) *Annales de la Société scientifique de Bruxelles*, XLVII, *Série A, première partie* p. 49, republished in English *M. N.* xci, p. 483. 1931.

[3]) *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 1931, p. 235.

For $0 \leqq \gamma \leqq \frac{4}{27}$ there are two solutions, in one of which y increases from y_2 to infinity, whilst in the other it oscillates between the limits 0 and y_1, y_1 and y_2 being the two points on the curve $p = 0$ where γ has the prescribed value, and $y_1 < 1.5 < y_2$.

For $\gamma < 0$ there is only one solution, in which y oscillates between zero and y_1.

We have thus two families of solutions, viz. the *expanding universes*, which only exist for positive values of γ, and the *oscillating universes*, which exist for values of γ algebraically smaller than $\frac{4}{27}$. In any one solution y is a measure of the instantaneous radius of the universe. As y changes the density ρ changes, since the mass is constant, and the velocity of expansion also changes by (2). It is convenient to express these two parameters by quantities of the dimension of a length, R_A and R_B. Thus

$$R_A^2 = \frac{2}{\varkappa \rho}, \quad R_B = R \bigg/ \frac{dR}{c\,dt},$$

and, following LEMAÎTRE [1]), we can put

$$R_C = \sqrt{\frac{3}{\lambda}}.$$

We can take

$$R_B = 2.10^{27} \pm 2.10^{26} \text{ (p.e.)}.$$

The value of R_A is practically unknown, but it appears probable that the density ρ will not be smaller than 10^{-31} nor larger than, say, 4.10^{-28} [2]). This gives for R_A the limits $1.6.10^{27}$ and 10^{29}.

We have

$$R_1 = \frac{2\,R^3}{3\,R_A^2},$$

and it is easily verified from (1) and (3) that

$$p = \frac{3}{2}\,\frac{R_A^2}{R_B^2} = \frac{1.6.10^{27}}{R_B^2\,\rho} = \frac{4.10^{-28}}{\rho},$$

p being given by (4). The limits of p corresponding to the above limits of ρ are thus $1 < p < 4000$. The curve $p = 1$ consists of the line $y = 0$ and the hyperbolic branch $\gamma y^2 = 1$, which has been entered in fig. 1 as a broken line. In the part of the diagram to the left of this curve p is smaller than 1, and the value of y corresponding to the actual universe at the present moment consequently must be in the part of the diagram to the right of it. This excludes all oscillating universes. It should be borne in mind, however, that the limits adopted for ρ are rather hypothetical.

For the family of expanding universes the co-ordinates z and τ used in *B. A. N.* 193 are more convenient than y and t. We have

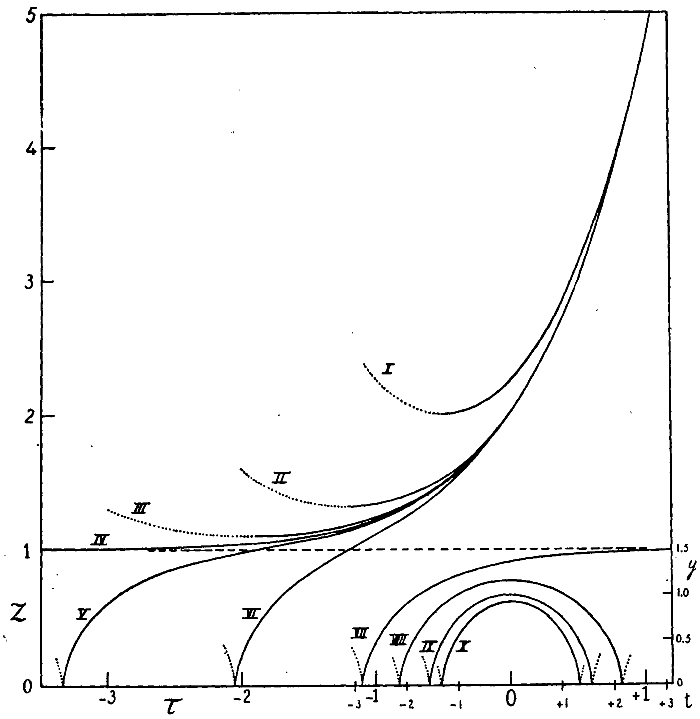
$$z = q\,y \quad , \quad \tau = \sqrt{\gamma}.\,t$$

In *B. A. N.* 193 the different solutions are arranged according to the parameter a. The values of γ from 0 to ∞ correspond to the values of a from -1 to $+3$, $\gamma = \frac{4}{27}$ corresponding to $a = 0$. The three equivalent parameters a, q and γ are connected by the relations

$$(5) \qquad q^3 + 2q^2 = 8\gamma, \qquad a = \frac{3q-2}{3q+2},$$
$$\gamma = \frac{4(1+a)}{(3-a)^3}, \qquad q = \frac{2(1+a)}{3-a}.$$

In the figure 2 several solutions have been drawn for different values of γ, or a. For the family of expanding universes the co-ordinates are z and $\tau - \tau_0$.

FIGURE 2.



For the family of oscillating universes the co-ordinates y and t are more convenient than z and τ. In order to make them more easily comparable I have, however, used

$$z' = \frac{2}{3}\,y \quad \text{and} \quad \tau' = \frac{2}{3\sqrt{3}}\,\text{t},$$

$\frac{2}{3}$ and $2/3\sqrt{3}$ being the values of q and $\sqrt{\gamma}$ for $\gamma = \frac{4}{27}$. Scales of y and t have, however, been added in the right and bottom margins.

The values of τ_0 for the expanding series have been so adjusted as to bring the different solutions to coincidence for large values of $\tau - \tau_0$. For the oscillating series the maximum has been placed ad $\tau' - \tau_0' = 0$.

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[1]) *M. N.* xci, p. 498, 1931.

[2]) HUBBLE derived in *Mt. Wilson Contrib.* 324 (1926) $1.5.10^{-31}$ as a lower limit. The value 2.10^{-28} adopted by me for convenience in *B. A. N.* 185 was considered as too large by many astronomers.

For the limiting solution for $\gamma = \frac{4}{27}$, τ_0 has been so chosen as to bring the curve at a convenient place in the diagram.

The four solutions that can be expressed without the help of elliptic functions are those for $\gamma = \frac{4}{27}$ and for $\gamma = 0$. These are:

IV. $\gamma = \frac{4}{27}$: LEMAÎTRE's universe:

$$(6) \qquad \tau - \tau_0 = \cosh^{-1}(z + 1) - \frac{1}{\sqrt{3}} \cosh^{-1} \frac{2z + 1}{z - 1}.$$

VII. $\gamma = \frac{4}{27}$: Limiting member of family of oscillating universes:

$$(7) \qquad \tau - \tau_0 = -\cosh^{-1}(z + 1) + \frac{1}{\sqrt{3}} \cosh^{-1} \frac{2z + 1}{1 - z}.$$

I. $\gamma = 0$: "Solution B", or "empty universe":

$$(8) \qquad z = 2 \cosh(\tau - \tau_0).$$

IX. $\gamma = 0$: Oscillating universe. The co-ordinates y and t can be most conveniently expressed by the intermediary of a parameter ψ:

$$(9) \qquad y = \cos^2 \psi, \quad \mathrm{t} - \mathrm{t}_0 = \psi + \tfrac{1}{2} \sin 2\psi.$$

This last is the solution advocated by EINSTEIN in his paper quoted in the third footnote on page 141. In this case $\gamma = \frac{1}{3} R_1^2 \lambda$ becomes zero by λ being zero, whilst in the case I (empty universe) γ is zero in consequence of the vanishing of R_1, the mass of the universe.

The "solution A", or "static universe", or "EINSTEIN's world" is represented in the diagram by the line $z = 1$.

The other curves in the diagram have been computed by numerical integration, aided by development in series at critical points when necessary.

Once we have decided on a particular value of γ — for which choice there are no empirical data to guide us — everything is determined by the adopted values of R_B and R_A, or R_B and p. We first determine the instantaneous value of z, or y, from

$$(10) \qquad z^3 - (3 - a) z + 2(1 - p)(1 + a) = 0$$

or

$$(11) \qquad 1 - y + \gamma y^3 = p, \quad z = qy.$$

Then R_1 and $R_0 = q R_1$ are given by

$$(12) \qquad R_1^2 = \frac{p}{y^3} R_B^2, \quad R_0^2 = \frac{pq}{z^3} R_B^2,$$

and λ, or R_C, by

$$R_1^2 \lambda = 3\gamma, \quad R_0^2 \lambda = \frac{3}{3 - a},$$

$$R_C = \frac{R_1}{\sqrt{\gamma}} = R_0 \sqrt{(3 - a)}.$$

Then we have for the expanding family

$$R = R_0 z, \quad c(t - t_0) = R_C(\tau - \tau_0)$$

and for the oscillating family

$$R = R_1 y, \quad c(t - t_0) = R_1(\mathrm{t} - \mathrm{t}_0).$$

The chief data for the solutions represented in the diagram are given below. For the expanding series I have taken $p = 10$, corresponding to $\rho = 4.10^{-29}$. For the oscillating series, which does not exist for values of p exceeding unity, I have taken two values of p, viz: $p = 0{\cdot}1$, $\rho = 4.10^{-27}$ and $p = 0{\cdot}5$, $\rho = 8.10^{-28}$.

Expanding family.

Curve	γ	a	q	y_2	z_0	τ_{z_0}	z	R_0/R_B	R_C/R_B	$t_z - t_{z_0}$ (10^9 *yrs*)
I	0	− 1	0	∞	2	− ·491	—	∞	∞	—
II	·10876	− ·10	·5806	2·2773	1·3162	− 1·212	2·935	·479	·8781	1·714
III	·14376	− ·01	·6578	1·6710	1·1	− 1·908	2·994	·495	·8621	2·535
IV	·14815	0	2/3	1·5	1	− ∞	3	·497	·8606	∞
V	·15625	+ ·01	·6756	—	0	− 3·338	3·006	·499	·8595	4·066
VI	·19845	+ ·10	·7586	—	0	− 2·053	3·061	·514	·8438	2·820

z_0 is the minimum value of z, τ_{z_0} the value of $\tau - \tau_0$ when this is reached; R_C/R_B is the factor by which intervals expressed in τ must be multiplied to express them in R_B as unit; to express them in years this must again be multiplied by the value of R_B in lightyears. The last column gives the interval elapsed since the minimum in units of 10^9 years, adopting $R_B = 2.10^{27}$ cm $= 2{\cdot}12.10^9$ lightyears.

In the curve I z is indeterminate. In this case p is not 10, but infinite. The wole analysis is not applicable to this case, since $R_1 = 0$, $R_0 = \infty$.

Oscillating family.

Curve	γ	y_1	T	$p = 0{\cdot}1$				$p = 0{\cdot}5$			
				y	R_1/R_B	$R_B^2\lambda$	$t_z - t_0$	y	R_1/R_B	$R_B^2\lambda$	$t_z - t_0$
VII	$+{\cdot}14815$	1·5	∞	1·094	·276	5·820	0·804	0·521	1·881	·126	1·218
VIII	$+{\cdot}10$	1·15347	4·310	1	·316	3	0·774	0·514	1·921	·081	1·214
IX	0	1	π	0·9	·370	0	0·745	0·5	2	0	1·208
X	$-{\cdot}10$	0·92170	2·690	0·841	·410	$-1{\cdot}782$	0·732	0·488	2·072	$-{\cdot}070$	1·202

T is the period of the oscillation in units of t. To convert intervals expressed in t into R_B as unit we must multiply by R_1/R_B. The column $t_z - t_0$ gives the interval since the minimum value $y = 0$, in units of 10^9 years.

We have assumed that matter is distributed homogeneously and isotropically in a three-dimensional space of positive curvature, the radius of curvature being a pure function of the time, i. e. we have assumed the line-element (1). For the known part of the universe the distribution is, of course, far from homogeneous, and also not isotropic. That it is homogeneous and isotropic when considered on a much larger scale, is not an empirical fact, but a hypothesis of metaphysical character. If, however, this hypothesis is adopted, and further the absence of pressure and the constancy of the total mass are adopted as sufficiently near approximations, then the evolution of the universe is reigned by the equation (2), or (3). In order to decide which of the many possible universes obeying this equation is ours, we require three data, viz: γ, to decide which particular member of the family of curves is described by the actual universe, y, or t (or z or τ) to determine which point of this curve corresponds to the present moment, and R_1 (or R_0) to fix the scale. For the determination of these three unknowns we have only one precise observational datum, viz: R_B. For two other data we can assign upper or lower limits, viz: for R_A, or p, as has already been explained above, and for $R_1 = \varkappa M/3\pi^2$, for which the total mass of all actually observed material bodies gives a lower limit. Taking the minimum number of extragalactic nebulae to be 3.10^5, and the minimum mass of each $10^8 \odot$, we find $R_1 > 4.10^{18} = 4$ lightyears. This, of course, is a ridiculously low limit, but it is all that we can derive from actual observation without introducing any hypotheses.

The lower limit $p > 1$ excludes all oscillating universes.

From the combination of the upper limit $p < 4000$ and the lower limit of R_1 we find from (12), with the value $R_B = 2.10^{27}$, the upper limit:

(13) $$y < 10^7.$$

This, again, is a purely nominal limit, without any practical significance.

Another datum might be derived from the assertion that the age of the stars and the galactic systems should be smaller than the time elapsed since the radius of the universe passed through its minimum value. This assertion is, of course, of the nature of a metaphysical hypothesis. Its value depends entirely on the interpretation given to the expression (1) of the line-element. Adopting it, we get the condition

(14) $$\int^{R} \frac{cdt}{dR} dR > P.R_B,$$

P being a numerical factor derived from our knowledge of the ages of stars and stellar systems. We can take $P = 500$.

By means of (3) and (12) the condition (14) can be brought in the form

(15) $$\int^{y} \frac{X}{y} dy > P.X,$$

where

$$X^2 = \frac{\gamma y^3}{p} = \frac{\gamma y^3}{1 - y + \gamma y^3}.$$

For large values of y the value of X is practically 1, and as a sufficient approximation we can consequently write for (15)

$$\lg y > P$$

or

$$y > e^P = 10^{232}.$$

Not only is this lower limit irreconcilable with the upper limit (13), but by itself it shows that the time scale of the evolutionary processes can only be formally fitted into the scheme of the expanding universe by means of such a tremendous extrapolation as to deprive the theory of all real meaning. This is only another way of expressing the utter impossibility, which has already repeatedly been pointed out, of reconciling the short time scale of the expanding universe with our ideas regarding the evolution of stars and stellar systems.

It is a consequence of the equations, at once evident from the diagram (fig. 2), that for all solutions, and moderate values of the time (τ) and the radius (z), these are necessarily of the same order, if expressed in corresponding units.

The ratio of the corresponding units of length and time in the theory of relativity is the velocity of light. The rate of evolution of a stellar system on the other hand is connected with the relative velocities of the stars, which are of the order of one ten-thousandth of the velocity of light. In the theory of relativity a year and a lightyear are equivalent, whilst in astronomy 10^9 lightyears is a very large distance, but 10^9 years is a short time.

Appendix.

Since the curves given in the diagram (fig. 2) depend on actual accurate computation, it may be of interest to give the numerical data from which they have been drawn. The curves I, IV, VII and IX were computed by the formulas (6) to (9) given above, or:

$$\text{I.}\quad \tau-\tau_0 = \cosh^{-1}\frac{1}{2}z,$$

$$\text{IV.}\quad \tau-\tau_0 = +\cosh^{-1}(z+1) - \frac{1}{\sqrt{3}}\cosh^{-1}\frac{2z+1}{z-1},$$

$$\text{VII.}\quad \tau-\tau_0 = -\cosh^{-1}(z+1) + \frac{1}{\sqrt{3}}\cosh^{-1}\frac{2z+1}{1-z},$$

$$\text{IX.}\quad t-t_0 = \cos^{-1}\sqrt{y} + \sqrt{y(1-y)}.$$

The others were computed by numerical integration. The curves II, III, V, VI were not continued beyond $z = 2·0$, where they co-incide with the curve IV. The portions in the neighbourhood of the minimum of II and III, of zero for V, VI, VIII and X, and of the maximum for VIII and X, were computed by development in series, since the numerical integration near these points becomes difficult, or impossible.

II		III		V		VI		VIII		X	
z	$\tau-\tau_0$	z	$\tau-\tau_0$	z	$\tau-\tau_0$	z	$\tau-\tau_0$	y	t	y	t
$1+\sqrt{0·1}=$ 1·3162	— 1·212	1·10	— 1·908	0	— 3·338	0	— 2·053	0	0	0	0
1·3202	— 1·112	1·1014	— 1·808	·0676	— 3·329	·0759	— 2·043	·1	·0217	·1	·0217
1·3321	— 1·012	1·1056	— 1·708	·1351	— 3·313	·1517	— 2·024	·2	·0636	·2	·0636
1·34	— ·967	1·1124	— 1·608	·2027	— 3·291	·2276	— 1·999	·3	·1213	·3	·1214
1·3516	— ·912	1·1218	— 1·508	·3	— 3·250	·3	— 1·968	·4	·1945	·4	·1951
1·36	— ·879	1·13	— 1·440	·4	— 3·191	·4	— 1·914	·5	·2843	·5	·2862
1·38	— ·801	1·1334	— 1·408	·5	— 3·113	·5	— 1·847	·6	·3931	·6	·3990
1·40	— ·755	1·14	— 1·373	·6	— 3·013	·6	— 1·762	·7	·5246	·7	·5416
1·5	— ·533	1·15	— 1·316	·7	— 2·876	·7	— 1·657	·8	·6852	·8	·7341
1·6	— ·386	1·16	— 1·264	·8	— 2·682	·8	— 1·528	·9	·8856	·81	·758
1·7	— ·267	1·17	— 1·218	·9	— 2·373	·9	— 1·372	1·0	1·1538	·82	·784
1·8	— ·166	1·18	— 1·176	1·0	— 1·872	1·0	— 1·195	1·02	1·210	·83	·811
1·9	— ·079	1·19	— 1·137	1·1	— 1·356	1·1	— 1·012	1·04	1·272	·84	·840
2·0	0	1·20	— 1·100	1·2	— 1·015	1·2	— ·840	1·06	1·352	·85	·872
		1·3	— ·816	1·3	— ·784	1·3	— ·686	1·08	1·442	·86	·905
		1·4	— ·624	1·4	— ·608	1·4	— ·551	1·10	1·550	·87	·942
		1·5	— ·476	1·5	— ·466	1·5	— ·433	1·1151	1·655	·8932	1·045
		1·6	— ·354	1·6	— ·348	1·6	— ·329	1·1291	1·755	·9091	1·145
		1·7	— ·250	1·7	— ·246	1·7	— ·236	1·1398	1·855	·9186	1·245
		1·8	— ·157	1·8	— ·154	1·8	— ·152	1·1474	1·955	·9217	1·345
		1·9	— ·074	1·9	— ·073	1·9	— ·072	1·1520	2·055		
		2·0	0	2·0	0	2·0	0	1·1535	2·155		