Irregularities of Period of Long-period Variable Stars.

By A. S. Eddington and S. Plakidis.

1. The light fluctuations of long-period variables have a well-marked periodicity complicated by superposed irregularities. There are various possible types of irregularity, and the question arises whether we can infer from the accumulated observational data which types are operative. The following types of irregularity suggest themselves for consideration :—

(1) The individual period (*i.e.* interval between two successive maxima) may differ from the mean period by a purely accidental fluctuation independent of previous fluctuations.

(2) The individual period may be correlated to those preceding it, either

(*a*) positively, so that periods longer than the average tend to occur in groups, or

(*b*) negatively, so that a period longer than the average is likely to be succeeded by a period shorter than the average.

(3) The irregularity may be a fluctuation of phase rather than of period ; that is to say, the primary cause of the light change may have an entirely regular period, but its visible effect may be delayed or accelerated by casual circumstances, so that the date of maximum differs from the ephemeris date by an accidental fluctuation.

(4) The casual " circumstances " in (3) may be long enduring so as to affect a succession of maxima ; the delay of one maximum is then correlated positively to the delay of the preceding maximum.

(5) The irregularity may be due to periodic causes.

The primary aim of the present investigation is to test whether the observed times of maxima are consistent with the hypothesis that the irregularities are of type (1). The stars examined are o Ceti and χ Cygni, for which long-continued observations are available. The feature of type (1) irregularity is that the delays and accelerations of successive periods mount up in the same way as accidental errors. If the star is late, the time lost is written off as irretrievable. There is no tendency to return to the original ephemeris dates, but in the ensuing periods gains or further losses occur as chance may determine.

2. Having determined a mean period from a very long series of observations, we construct a uniform ephemeris and compute the differences of date (observed—calculated) for each observed maximum. Let a_r be the number of days late of the rth maximum compared with the ephemeris, and let

$$u_x(r) = a_{r+x} - a_r \qquad (1)$$

Then $u_x(r)$ is the accumulated delay in x periods which according to hypothesis (1) is the sum of x uncorrelated accidental fluctuations. By the ordinary theory of errors, the probable value of the sum of x fluctuations is $\sqrt{x}$ times the probable value of one fluctuation.

If $\bar{u}_x$ is the average value (without regard to sign) of $u_x(r)$ for as many values of r as the observational material admits, we should have accordingly

$$\bar{u}_x = \epsilon\sqrt{x} \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where $\epsilon = \bar{u}_1$.

By way of contrast, consider hypothesis (3). In this case a_r is a purely accidental deviation, and its probable value is independent of r. If then the average value of $|a_r|$ for all values of r is α, the average value of $|a_{r+x} - a_r|$ is $\alpha\sqrt{2}$. Hence

$$\bar{u}_x = \alpha\sqrt{2} \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

It is easily seen that if fluctuations of both kinds occur, the result will be

$$\bar{u}_x = \sqrt{(2\alpha^2 + x\epsilon^2)} \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

We must in any case expect that the observational data will give an α term, because accidental error in determining the date of maximum is equivalent to accidental fluctuations of type (3). We shall accordingly examine whether the observations satisfy a formula of the form (4).

A correcting factor is required when x is a considerable fraction of the whole number of periods employed in determining the mean period. Let $\epsilon_1, \epsilon_2 \ldots \epsilon_n$ be the accidental fluctuations of successive periods from the true mean period, then the error of the deduced mean period is $(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n)/n$. The deviation of the first period from the *adopted* mean period is therefore

$$\epsilon_1 - (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n)/n.$$

Forming similar expressions for the second, third, . . . periods, and adding the first x of them, we have

$$u_x(0) = \frac{n-x}{n}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_x) - \frac{x}{n}(\epsilon_{x+1} + \epsilon_{x+2} + \ldots + \epsilon_n).$$

Whence by the theory of combination of independent deviations

$$\left.\begin{aligned}\bar{u}_x &= \epsilon\sqrt{\left\{\left(\frac{n-x}{n}\right)^2 x + \left(\frac{x}{n}\right)^2 (n-x)\right\}} \\ &= \epsilon\sqrt{x} \,.\, \sqrt{(1 - x/n)}\end{aligned}\right\} \quad . \quad . \qquad (5)$$

This replaces (2); but in our applications x is small compared with n, and it has not been necessary to use the correcting factor.

3. The observed dates of maximum for o Ceti and χ Cygni have been taken from *Harvard Annals*, **55**, supplemented up to the year 1927 inclusive from the *Reports of the V.S.S. of the B.A.A.** Wherever in the former source more than one date for the same maximum is given, the mean has been adopted. From these data mean periods of 331·4

* These last data were kindly supplied by Mr. F. de Roy, Director of the V.S.S.

days for *o* Ceti and 406·5 days for χ Cygni were deduced, and uniform ephemerides were computed with the formulæ

$$M = 2402985 + 331{\cdot}4\ E$$
$$M = 2389524 + 406{\cdot}5\ E$$

respectively.

Former discussions of these stars have been made by T. E. R. Phillips and W. J. Luyten. The former (*J.B.A.A.*, **27**, No. 1, 1916), giving the date curve of maxima, remarks that "both show very large irregularities, and it looks rather as if in these stars the time of maximum is affected by continuous but non-periodic change." Dr. Luyten, in his *Observations of Variable Stars* (Leiden, 1921), giving similar diagrams of observed—computed maximum, is "inclined to think that the period of *o* Ceti changes continuously and not abruptly as Professor Turner suggests, although the latter finds remarkable regularities in these changes." As regards χ Cygni, in discussing the results obtained by Professor Turner (*M.N.*, **80**, 282 and 481, 1920), he says that the period of this star being fairly constant in the beginning, from a certain point becomes markedly larger, and at last shows a tendency to decrease.

There are many breaks in the earlier series of observations, certain maxima being unobserved, so that it is a matter of accident whether a particular u, say $u_{10}(25)$, is known and included in $\bar{u}_{10}$ or left out as unknown. This should make no difference if the whole number of observations is very great; but in a limited series it has probably had the effect of making the curves less smooth than they would have been.

4. The curves in fig. 1 show $\bar{u}_x$ as ordinate plotted against x as abscissa. The jagged curves are calculated from the observations of *o* Ceti (I.) and χ Cygni (II.). The smooth curves represent the best fit * with the theoretical formula $\bar{u}_x = \sqrt{(2\alpha^2 + x\epsilon^2)}$, the constants being

For *o* Ceti	$\alpha = 5{\cdot}48$	$\epsilon = 4{\cdot}48$
,, χ Cygni	$\alpha = 1{\cdot}58$	$\epsilon = 5{\cdot}66$

The unit is 1 day.

Although each value of $\bar{u}_x$ is the mean of some 60 to 100 values of $u_x(r)$ the weight of the determinations for large values of x is not very great and the curves become untrustworthy towards the right-hand side of the diagram. The reason is that there is a good deal of overlap of the $u_x(r)$; any change of phase occurring during a few periods will affect all the differences $a_{r+x} - a_r$ which enclose those periods. Thus when x is large, the weight of $\bar{u}_x$ is much less than if it were the mean of 60 to 100 independent values of $u_x(r)$. On the left of the diagram the probable error of any ordinate is about 5 per cent.

The α term turns out to be small and, since it includes observational error in fixing the date of maximum, there is little, if anything, left to attribute to hypothesis (3). The maximum of *o* Ceti is flatter than that of χ Cygni, so that there is greater uncertainty in the observed date;

* Owing to the small weight of the determinations for large values of x (see below), the values of α and ϵ are chosen so as to give the best representation up to about $x=20$.

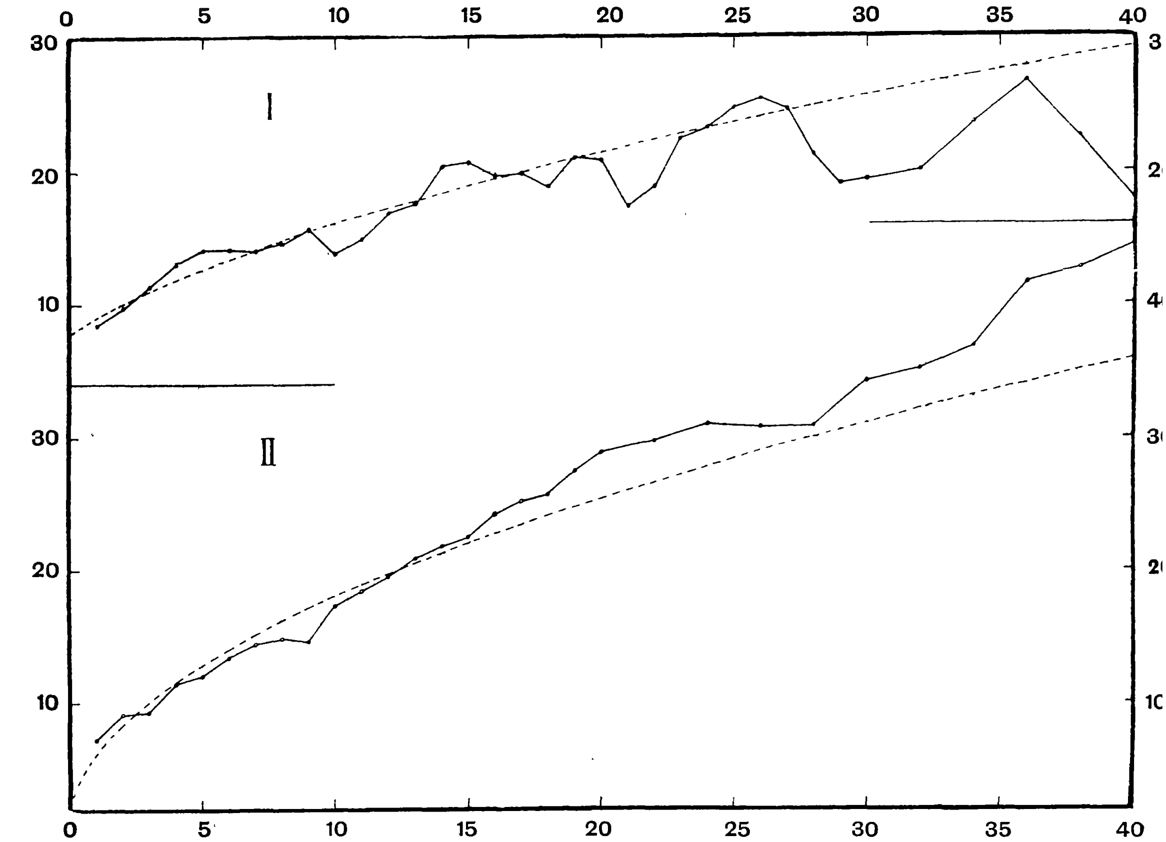

FIG. 1.—Curves of $\bar{u}_x$ for *o* Ceti (I.) and *χ* Cygni (II.).

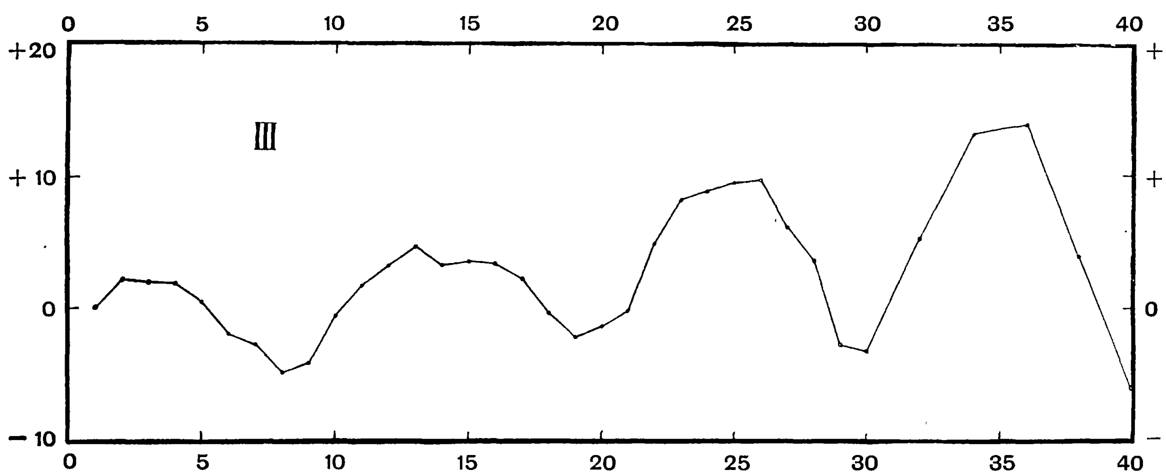

FIG. 2.—Curve for *o* Ceti, showing a spurious periodicity.

and this seems to account adequately for the larger value of α found for the former.

The ϵ terms for the two stars (representing irregularities obeying hypothesis (1)) are very nearly in the same proportion as their periods. Thus the average deviation of an individual period of *o* Ceti from the mean is 1·35 per cent., and that of χ Cygni is 1·39 per cent. It would be interesting to discover whether this equality extends to other long-period variables. On the other hand it is possible that the larger value of ϵ is associated with the larger light-range rather than the longer period of χ Cygni.

5. Before discussing the significance (if any) of the deviations of the observed curves from the theoretical curves, we may indicate briefly the effects of other types of irregularity. Irregularities giving the term α may be called *temporary* since the phase difference at one maximum has no effect on, or correlation with, the phase difference at subsequent maxima. Those giving the term ϵ may be called *permanent* since the phase difference at one maximum is carried forward to all subsequent maxima, although it may be obliterated or enhanced by accidental irregularities incident subsequently. A third class may be called *repeated*, in which the phase difference grows proportionately to the number of periods elapsed except in so far as fresh irregularities supervene; to this class belong discontinuous changes of period, or more generally irregularities of type 2 (*a*) where there is a positive correlation between successive periods. The effect on the $\bar{u}_x$ curve can be found analytically. Since

$$u_x(r) = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_x,$$

where the ϵ's are the fluctuations of successive periods from the mean, we have

$$\{u_x(r)\}^2 = \left.\begin{array}{ll} \epsilon_1{}^2 + \epsilon_2{}^2 + \ldots & (x \text{ terms}) \\ + 2\epsilon_1\epsilon_2 + 2\epsilon_2\epsilon_3 + \ldots & (x - 1 \text{ terms}) \\ + 2\epsilon_1\epsilon_3 + 2\epsilon_2\epsilon_4 + \ldots & (x - 2 \text{ terms}) \\ + \ldots & \end{array}\right\} \quad . \qquad (6)$$

If r_1 is the correlation coefficient between the fluctuations of successive periods, we have (by definition of r_1)

$$\text{mean value of } \epsilon_s\epsilon_{s+1} = r_1\mu^2,$$

where μ is the mean square value of ϵ_s. If r_2 is the correlation coefficient between a period and that next but one preceding, and so on, we have from (6),

$$\text{mean value of } \{u_x(r)\}^2 = \mu^2(x + 2r_1(x - 1) + 2r_2(x - 2) + \ldots).$$

The average values $\bar{u}_x$, ϵ, used in our work, will be related in practically the same way as the mean square values; hence

$$\bar{u}_x = \epsilon\sqrt{x} \,.\, \sqrt{\left\{1 + \frac{x-1}{x}2r_1 + \frac{x-2}{x}2r_2 + \ldots\right\}} \quad . \qquad (7)$$

If the irregularity consists of abrupt changes of period occurring not too frequently, the coefficients r_1, r_2, r_3, . . . will decrease very slowly. If it has more the form of a simple correlation (the correlation between ϵ_s and ϵ_{s+2} being no more than must automatically result from both being correlated to ϵ_{s+1}) the coefficients will decrease rapidly. In either case the curve for sufficiently large values of x approximates to the standard form $\bar{u}_x \propto \sqrt{x}$, the constant coefficient being $\epsilon\sqrt{(1 + 2r_1 + 2r_2 + \ldots)}$; as x diminishes, the value of $\bar{u}_x$ will drop below the $\sqrt{x}$ curve.

A negative correlation will be indicated similarly by the opposite kind of deviation from a $\sqrt{x}$ curve, viz. if a $\sqrt{x}$ curve is fitted for large values of x, the observed $\bar{u}_x$ will rise above the curve as x diminishes. A negative correlation is equivalent to a mixture of "temporary" and "permanent" irregularity, so that it is fairly well covered by our formula (4) which includes both types.

There remains the possibility of a periodic fluctuation of the time of maximum. This will be indicated by a sinuosity in the curve representing $\bar{u}_x$. For example, if the period of the fluctuation is 10 times the period of the light-change, the fluctuation will cancel out in forming $a_{r+10} - a_r$, $a_{r+20} - a_r$, etc., so that $\bar{u}_{10}$, $\bar{u}_{20}$, . . ., will be lower than neighbouring ordinates.

6. Bearing in mind that there is rather large accidental error for points on the right of the diagrams, the curves for *o* Ceti and χ Cygni appear to be fairly accordant with the hypothesis in § 2, viz. purely accidental fluctuations of period coupled with an α term which probably represents the observational errors in fixing the time of maximum. Applying the criterion found in § 5, there is no evidence of repeated fluctuations (positive correlations).

For *o* Ceti, however, the dip of the curve at $x = 10, 21, 31$, with intermediate maxima, suggests that part of the irregularity may be periodic, with period about 10 times that of the star, or 9 years. Whilst this requires us to make some reservation in claiming that the fluctuations are of the purely accidental type, we do not wish to imply that there are any strong grounds for thinking that this periodicity is genuine. We have in fact rather strong evidence that the sinuosity of curve I. is an accident arising from the incompleteness of the data. Rather by inadvertence, the curve in fig. 2 was plotted, showing the mean value of $u_x(r)$, *with attention to sign*, plotted against x. It is easily seen that for a long and complete series of observations this curve should coincide with the axis of x, since in forming the mean u_x with attention to sign there is a general cancelling out of intermediate fluctuations, leaving only a few fluctuations at each end to contribute to the result. The curve, therefore, merely illustrates the chance effect of incompleteness of data (many maxima being unobserved). Comparison with curve I. in fig. 1 shows that the sinuosities of both curves correspond closely. It is fairly evident, therefore, that those of curve I. are to be attributed to the same chapter of accidents, and can have no physical importance.

7. The problem may perhaps be made clearer by a physical picture.

We may compare the outburst of light at maximum to the breaking of a wave or to a boiling over of the star—phenomena suggestive of some degree of irregularity. The boiling over relieves a strain, and afterwards there is a period of waiting until the strain mounts up sufficiently to cause another outburst. The waiting period must be reckoned as starting from the actual time of outburst (whether early or late), so that the epoch of the next maximum is determined partly by the epoch of the previous maximum and partly by the accidental circumstances which render the period of waiting longer or shorter than the average. The simplest hypothesis is that nothing else enters into the problem, and our curves indicate that this accords reasonably well with the observations. It would not have been surprising if the relief afforded by a quick or premature outburst differed somewhat from that afforded by a delayed outburst so that one or other of the two correlations (2*a*) or (2*b*) might occur. We find no evidence of these in the curves.

A Revision of Newcomb's Occultation Memoir. By H. Spencer Jones, M.A., Sc.D., H.M. Astronomer. (Plate 1.)

In the year 1912 was published Newcomb's long-awaited memoir, "Researches on the Motion of the Moon," Part II., * containing a derivation of the mean motion of the Moon and of other astronomical elements, based upon observations of occultations extending from 1672 to 1908. This memoir, the preparation of which had extended over thirty years, was completed during the author's last illness, and he was unable to see the work through the press.

In this memoir Hansen's Tables were used as a basis for computation, corrections being applied to take account of the principal terms in the longitude of the Moon omitted by Hansen, thereby reducing Hansen's elements to what Newcomb termed "the provisionally accepted theory." Some of the elements used were changed in the course of the reductions, as improved values became available. Newcomb contemplated that at some future date a revision of the work would be required, doubtless having in mind the completion of Brown's Tables of the Moon. His intention was to arrange the memoir in such a manner that no revision of the details of the work would be required, and that the discovery of any errors of computation and their corrections would be as easy as possible. He further stated that "In the reconstruction of the work most of the numbers, especially those pertaining to the equations of condition, can be used without revision" (p. 7), and "It will not be necessary even to repeat the computation of the

* *Astronomical Papers of the American Ephemeris*, **9**, pt. i., 1912.