Some Computational Problems arising in Eclipses. By L. J. Comrie, M.A., Ph.D.

Summary.—Formulæ are given for obtaining the position angles of the cusps from the Besselian elements. A simple method of obtaining the position and shape of the Moon's shadow on the Earth is given. The preparation of the Ordnance Survey Eclipse Map is described. The geometry of the method of taking long-focus photographs of the eclipsed Sun on a moving plate is developed rigorously. The formula for reducing the diameter of the Moon as measured on a photograph to its geocentric value is investigated. Finally, the corrections to the Besselian elements for the eclipse of 1927 June 29, to bring them into accord with recent observations of the Sun and Moon, are given, together with a recomputation of the central line from the revised elements.

Introduction.—It has fallen to the writer, on more than one occasion, to compute certain results appertaining to the observation of eclipses, after deriving the necessary formulæ. As these formulæ are not to be found in text-books it seems appropriate to put them on record now, at a time when the attention of English astronomers is being directed to the forthcoming total eclipse of June 29. Moreover, it is believed that the formulæ relating to the cusps will prove to be of immediate value to astronomical physicists attempting to photograph the flash spectrum.

Historical.—A century ago the circumstances of eclipses were computed by the so-called method of semi-diameters, which involved heavy interpolation from the ephemeris and the rigorous computation of parallaxes and augmentations. Bessel,* in his masterly *Analyse der Finsternisse*, was the first to simplify the problem, and the principles of his method, so ably expounded and reduced to working form by Chauvenet † more than seventy years ago, remain to this day fundamental in this branch of astronomy. A later work by Buchanan ‡ contributes nothing new to the theory, but purports to be a practical exposition of the predicting practice adopted in the office of the *American Ephemeris and Nautical Almanac.* Its helpfulness is somewhat marred by the extraordinary number of misprints that it contains.

Besselian Elements.—The *Nautical Almanac* gives the "Besselian Elements" of each solar eclipse, and from these the prediction of the circumstances—times, position angles, and magnitude—for a given place is a matter of simple straightforward calculation. For the definitions and geometrical significance of the elements the reader must consult the works already cited (all of which, unfortunately, are out of print), or the *American Ephemeris.* It should be remarked that in the computation of the Besselian elements the so-called "eclipse" semi-diameters of the Sun and Moon are employed, and that a correction is

* *Astronomische Untersuchungen*, II, 95.

† William Chauvenet, *A Manual of Spherical and Practical Astronomy.* (Philadelphia, J. B. Lippincott Company.)

‡ Roberdeau Buchanan, *The Mathematical Theory of Eclipses according to Chauvenet's Transformation of Bessel's Method.* (Philadelphia, J. B. Lippincott Company, 1904.)

applied to the tabular position of the Moon, although, curiously, not to that of the Sun. For the eclipse of June next the correction applied was $+ 7''\cdot 0$ in mean longitude and $- 0''\cdot 5$ in latitude, which becomes $+ 7''\cdot 2$ and $+ 0''\cdot 1$ in actual longitude and latitude respectively, or $+ 0^{s}\cdot 53$ and $- 0''\cdot 2$ in Right Ascension and Declination. This correction, although applied nearly three years ago, is very close to that determined by more recent observations of the Moon.

Position Angles of Cusps.—In determining the position angles of the cusps at a given moment the Besselian computation proceeds to the point where L, m, and M are known. L is the radius of the penumbra on a plane through the observer and parallel to the fundamental plane. M is the position angle of the centre of the Moon referred to the centre of the Sun, or the direction of greatest obscuration, the direction of greatest illumination being $M \pm 180°$.

The apparent semi-diameter of the Moon (including the effect of augmentation) is

$$\frac{0\cdot 5459}{2L - 0\cdot 5459} \times \text{Sun's semi-diameter},$$

or, for the coming eclipse as seen from England,

$$1\cdot 007 \times \text{Sun's semi-diameter}.$$

The distance of the Moon's centre from that of the Sun is

$$\frac{2m}{2L - 0\cdot 5459} \times \text{Sun's semi-diameter}.$$

The maximum angular breadth of the obscured portion of the Sun is

$$\frac{2(L - m)}{2L - 0\cdot 5459} \times \text{Sun's semi-diameter}.$$

The maximum angular breadth of the illuminated portion is

$$\frac{(L + m - 0\cdot 5459)}{2L - 0\cdot 5459} \times \text{Sun's semi-diameter}.$$

The magnitude of the eclipse, or the fraction of the Sun obscured at the given moment, is

$$\frac{L - m}{2L - 0\cdot 5459}.$$

If we define an auxiliary angle A by

$$\tan^2 \frac{A}{2} = \frac{(L - m)(0\cdot 5459 - L + m)}{(L + m)(L + m - 0\cdot 5459)}$$

then the position angles of the cusps, reckoned from the north point through the east in the usual way, will be $M \pm A$. The "overhang" of the Moon $1°$ from the cusp is approximately

$$\frac{2 \sin 1° \, m \sin A}{0\cdot 5459} \times \text{Sun's semi-diameter},$$

or, for this eclipse, $60'' \, m \sin A$.

The angular distance, in degrees, from either cusp to the point where the Moon forms an imaginary cusp with a stratum of the Sun's atmosphere S seconds of arc above the adopted radius is

$$\frac{0{\cdot}5459\,S}{2 \sin 1^\circ\, m \sin A \times \text{Sun's S. D.}}$$

or, for this eclipse,

$$\frac{S}{60\, m \sin A}.$$

This formula, which is a differential one, must not be used if the resulting angular distance is more than 3° or 4°.

Those who wish to prepare diagrams illustrating the appearance of the uneclipsed portion of the Sun at any moment will, of course, regard the Sun's semi-diameter as unity in the formulæ above. The position angles from the vertex are obtained by diminishing those from the north point by C, the parallactic angle, computed, together with the altitude of the Sun, from

$$\begin{aligned} \sin \text{altitude} &= \sin \phi \sin d + \cos \phi \cos d \cos (\mu - \lambda) \\ \sin C &= \cos \phi \sin (\mu - \lambda) \sec \text{altitude} \end{aligned}$$

in which $\cos C$ has the same sign as η. In most cases, however, since a possible error of $0^\circ{\cdot}2$ in C will be considered negligible, we may use

$$\tan C = \frac{\xi}{\eta}$$

while the altitude may be found from

$$\sin \text{altitude} = \zeta$$

For an observer in or near the track of totality it will be found that M and the position angles of the cusps vary very rapidly just before and just after mid-eclipse, so that their computation cannot be checked by differences, even if computed at 5-minute intervals. On the central line these angles will, of course, change 180° or so in the minute containing the mid-eclipse, while even in London the change is as large as 20° in this minute. It will, however, suffice to compute the quantities $x - \xi$, $y - \eta$, and L at intervals of 10^{m} (or 5^{m} if great accuracy is desired) and then to interpolate them to the desired interval—say 1^{m} if a flash spectrum programme is to be attempted. Each individual m, M, and A can then be computed.

Shape of the Moon's Shadow.—Another problem that has arisen is the determination of the position and shape of the Moon's shadow on the Earth at a given moment. The rigorous formulæ for the outline are given by Chauvenet, but a simpler and only slightly less accurate solution will usually suffice. The shadow may be regarded as an ellipse, centred at that point on the central line at which mid-eclipse occurs at the given moment, and whose major axis is directed towards the Sun, the azimuth of which may be found by the usual formula. The length of the semi-minor axis b in miles is

$$3963(l_2 - \zeta \tan f_2),$$