M.	X.						Y.					
	e = 0·51.		e = 0·52.		e = 0·53.		e = 0·51.		e = 0·52.		e = 0·53.	
°												
0	+0·49000	−63	+0·48000	−66	+0·47000	−69	+0·00000	+3064	+0·00000	+3106	+0·00000	+3149
1	+0·48937	−190	+0·47934	−198	+0·46931	−206	+0·03064	+3054	+0·03106	+3094	+0·03149	+3136
2	+0·48747	−315	+0·47736	−328	+0·46725	−341	+0·06118	+3036	+0·06200	+3080	+0·06285	+3119
3	+0·48432	−438	+0·47408	−455	+0·46384	−476	+0·09154	+3017	+0·09280	+3053	+0·09404	+3095
4	+0·47994	−559	+0·46953	−582	+0·45908	−605	+0·12171	+2986	+0·12333	+3022	+0·12499	+3055
5	+0·47435	−675	+0·46371	−702	+0·45303	−730	+0·15157	+2947	+0·15355	+2981	+0·15554	+3016
6	+0·46760	−789	+0·45669	−819	+0·44573	−852	+0·18104	+2905	+0·18336	+2934	+0·18570	+2966
7	+0·45971	−896	+0·44850	−930	+0·43721	−965	+0·21009	+2853	+0·21270	+2881	+0·21536	+2906
8	+0·45075	−1001	+0·43920	−1037	+0·42756	−1075	+0·23862	+2802	+0·24151	+2823	+0·24442	+2847
9	+0·44074	−1098	+0·42883	−1138	+0·41681	−1178	+0·26664	+2740	+0·26974	+2760	+0·27289	+2779
10	+0·42976	−1191	+0·41745	−1232	+0·40503	−1274	+0·29404	+2677	+0·29734	+2694	+0·30068	+2707
11	+0·41785	−1278	+0·40513	−1319	+0·39229	−1364	+0·32081	+2609	+0·32428	+2620	+0·32775	+2633
12	+0·40507	−1360	+0·39194	−1402	+0·37865	−1445	+0·34690	+2542	+0·35048	+2549	+0·35408	+2554
13	+0·39147	−1436	+0·37792	−1479	+0·36420	−1523	+0·37232	+2469	+0·37597	+2474	+0·37962	+2477
14	+0·37711	−1504	+0·36313	−1547	+0·34897	−1591	+0·39701	+2394	+0·40071	+2395	+0·40439	+2393
15	+0·36207	−1570	+0·34766	−1613	+0·33306	−1656	+0·42095	+2321	+0·42466	+2318	+0·42832	+2315
16	+0·34637	−1629	+0·33153	−1670	+0·31650	−1713	+0·44416	+2245	+0·44784	+2239	+0·45147	+2231
17	+0·33008	−1682	+0·31483	−1723	+0·29937	−1865	+0·46661	+2168	+0·47023	+2159	+0·47378	+2149
18	+0·31326	−1732	+0·29760	−1772	+0·28172	−1811	+0·48829	+2093	+0·49182	+2081	+0·49527	+2068
19	+0·29594	−1775	+0·27988	−1814	+0·26361	−1854	+0·50922	+2016	+0·51263	+2004	+0·51595	+1987
20	+0·27819	−1816	+0·26174	−1853	+0·24507	−1888	+0·52938	+1943	+0·53267	+1925	+0·53582	+1907
21	+0·26003	−1852	+0·24321	−1886	+0·22619	−1923	+0·54881	+1865	+0·55192	+1849	+0·55489	+1831
22	+0·24151	−1881	+0·22435	−1916	+0·20696	−1948	+0·56746	+1793	+0·57041	+1772	+0·57320	+1750
23	+0·22270	−1909	+0·20519	−1940	+0·18748	−1973	+0·58539	+1720	+0·58813	+1698	+0·59070	+1676
24	+0·20361	−1933	+0·18579	−1964	+0·16775	−1993	+0·60259	+1647	+0·60511	+1626	+0·60746	+1601
25	+0·18428	−1955	+0·16615	−1983	+0·14782	−2010	+0·61906	+1580	+0·62137	+1555	+0·62347	+1529
26	+0·16473	−1974	+0·14632	−1998	+0·12772	−2024	+0·63486	+1511	+0·63692	+1485	+0·63876	+1460
27	+0·14499	−1986	+0·12634	−2013	+0·10748	−2038	+0·64997	+1443	+0·65177	+1419	+0·65336	+1392
28	+0·12513	−1999	+0·10621	−2022	+0·08710	−2043	+0·66440	+1378	+0·66596	+1351	+0·66728	+1324
29	+0·10514	−2011	+0·08599	−2031	+0·06667	−2052	+0·67818	+1314	+0·67947	+1287	+0·68052	+1259
30	+0·08503	−2017	+0·06568	−2037	+0·04615	−2055	+0·69132	+1251	+0·69234	+1225	+0·69311	+1196
31	+0·06486	−2024	+0·04531	−2041	+0·02560	−2059	+0·70383	+1192	+0·70459	+1163	+0·70507	+1136
32	+0·04462	−2026	+0·02490	−2042	+0·00501	−2057	+0·71575	+1131	+0·71622	+1104	+0·71643	+1076
33	+0·02436	−2028	+0·00448	−2043	−0·01556	−2056	+0·72706	+1075	+0·72726	+1047	+0·72719	+1017
34	+0·00408	−2029	−0·01595	−2043	−0·03612	−2056	+0·73781	+1018	+0·73773	+991	+0·73736	+963
35	−0·01621	−2027	−0·03638	−2038	−0·05668	−2050	+0·74799	+964	+0·74764	+935	+0·74699	+909
36	−0·03648	−2026	−0·05676	−2038	−0·07718	−2045	+0·75763	+911	+0·75699	+884	+0·75608	+856
37	−0·05674	−2021	−0·07714	−2029	−0·09763	−2038	+0·76674	+859	+0·76583	+833	+0·76464	+805
38	−0·07695	−2015	−0·09743	−2024	−0·11801	−2032	+0·77533	+810	+0·77416	+782	+0·77269	+756
39	−0·09710	−2013	−0·11767	−2016	−0·13833	−2020	+0·78343	+762	+0·78198	+735	+0·78025	+708
40	−0·11723	−2003	−0·13783	−2009	−0·15853	−2014	+0·79105	+714	+0·78933	+688	+0·78733	+661
41	−0·13726	−1995	−0·15792	−2000	−0·17867	−2001	+0·79819	+667	+0·79621	+643	+0·79394	+616
42	−0·15721	−1987	−0·17792	−1990	−0·19868	−1993	+0·80486	+624	+0·80264	+599	+0·80010	+574
43	−0·17708	−1978	−0·19782	−1979	−0·21861	−1980	+0·81110	+580	+0·80863	+555	+0·80584	+531
44	−0·19686	−1967	−0·21761	−1968	−0·23841	−1970	+0·81690	+540	+0·81418	+515	+0·81115	+491
45	−0·21653	−1958	−0·23729	−1958	−0·25811	−1955	+0·82230	+498	+0·81933	+474	+0·81606	+450
46	−0·23611	−1946	−0·25687	−1945	−0·27766	−1943	+0·82728	+458	+0·82407	+435	+0·82056	+412
47	−0·25557	−1932	−0·27632	−1931	−0·29709	−1930	+0·83186	+420	+0·82842	+398	+0·82468	+375
48	−0·27489	−1921	−0·29563	−1919	−0·31639	−1915	+0·83606	+383	+0·83240	+359	+0·82843	+338
49	−0·29410	−1908	−0·31482	−1905	−0·33554	−1902	+0·83989	+346	+0·83599	+326	+0·83181	+304
50	−0·31318	−1896	−0·33387	−1892	−0·35456	−1887	+0·84335	+311	+0·83925	+289	+0·83485	+269
51	−0·33214	−1882	−0·35279	−1878	−0·37343	−1873	+0·84646	+277	+0·84214	+257	+0·83754	+236
52	−0·35096	−1869	−0·37157	−1863	−0·39216	−1857	+0·84923	+243	+0·84471	+223	+0·83990	+205
53	−0·36965	−1853	−0·39020	−1848	−0·41073	−1842	+0·85166	+211	+0·84694	+193	+0·84195	+173
54	−0·38818	−1842	−0·40868	−1834	−0·42915	−1827	+0·85377	+179	+0·84887	+160	+0·84368	+142
55	−0·40660	−1824	−0·42702	−1818	−0·44742	−1810	+0·85556	+149	+0·85047	+131	+0·84510	+114
56	−0·42484	−1810	−0·44520	−1802	−0·46552	−1795	+0·85705	+118	+0·85178	+101	+0·84624	+84
57	−0·44294	−1797	−0·46322	−1789	−0·48347	−1781	+0·85823	+91	+0·85279	+73	+0·84708	+57
58	−0·46091	−1780	−0·48111	−1774	−0·50128	−1765	+0·85914	+61	+0·85352	+46	+0·84765	+30
59	−0·47871	−1764	−0·49885	−1755	−0·51893	−1748	+0·85975	+35	+0·85398	+18	+0·84795	+3
60	−0·49635	−1749	−0·51640	−1741	−0·53641	−1729	+0·86010	+7	+0·85416	−8	+0·84798	−22
61	−0·51384	−1735	−0·53381	−1723	−0·55370	−1715	+0·86017	−18	+0·85408	−32	+0·84776	−46
62	−0·53119	−1718	−0·55104	−1709	−0·57085	−1699	+0·85999	−45	+0·85376	−58	+0·84730	−72
63	−0·54837	−1703	−0·56813	−1694	−0·58784	−1684	+0·85954	−69	+0·85318	−82	+0·84658	−95
64	−0·56540	−1688	−0·58507	−1678	−0·60468	−1667	+0·85885	−93	+0·85236	−106	+0·84563	−117
65	−0·58228	−1670	−0·60185	−1658	−0·62135	−1648	+0·85792	−115	+0·85130	−128	+0·84446	−140
66	−0·59898	−1656	−0·61843	−1646	−0·63783	−1632	+0·85677	−140	+0·85002	−150	+0·84306	−162
67	−0·61554	−1638	−0·63489	−1626	−0·65415	−1616	+0·85537	−161	+0·84852	−173	+0·84144	−183
68	−0·63192	−1624	−0·65115	−1612	−0·67031	−1600	+0·85376	−183	+0·84679	−194	+0·83961	−203
69	−0·64816	−1605	−0·66727	−1595	−0·68631	−1584	+0·85193	−204	+0·84485	−213	+0·83758	−224
70	−0·66421	−1591	−0·68322	−1579	−0·70215	−1568	+0·84989	−226	+0·84272	−235	+0·83534	−243
71	−0·68012	−1574	−0·69901	−1564	−0·71783	−1550	+0·84763	−244	+0·84037	−254	+0·83291	−262
72	−0·69586	−1559	−0·71465	−1546	−0·73333	−1535	+0·84519	−265	+0·83783	−273	+0·83029	−282
73	−0·71145	−1543	−0·73011	−1530	−0·74868	−1518	+0·84254	−284	+0·83510	−292	+0·82747	−299
74	−0·72688	−1526	−0·74541	−1514	−0·76386	−1503	+0·83970	−303	+0·83218	−310	+0·82448	−317
75	−0·74214		−0·76055		−0·77889		+0·83667		+0·82908		+0·82131	

M.	X.						Y.					
°	c = 0·51.		c = 0·52.		c = 0·53.		c = 0·51.		c = 0·52.		c = 0·53.	
76	−0·74214	−1510	−0·76055	−1498	−0·77889	−1484	+0·83667	−320	+0·82908	−327	+0·82131	−334
76	−0·75724	−1493	−0·77553	−1480	−0·79373	−1467	+0·83347	−338	+0·82581	−344	+0·81797	−350
77	−0·77217	−1476	−0·79033	−1466	−0·80840	−1454	+0·83009	−356	+0·82237	−362	+0·81447	−367
78	−0·78693	−1463	−0·80499	−1448	−0·82294	−1437	+0·82653	−372	+0·81875	−379	+0·81080	−383
79	−0·80156	−1445	−0·81947	−1434	−0·83731	−1419	+0·82281	−390	+0·81496	−393	+0·80697	−399
80	−0·81601	−1430	−0·83381	−1417	−0·85150	−1405	+0·81891	−406	+0·81103	−411	+0·80298	−415
81	−0·83031	−1412	−0·84798	−1399	−0·86555	−1386	+0·81485	−420	+0·80692	−425	+0·79883	−428
82	−0·84443	−1396	−0·86197	−1384	−0·87941	−1374	+0·81065	−437	+0·80267	−440	+0·79455	−444
83	−0·85839	−1381	−0·87581	−1369	−0·89315	−1356	+0·80628	−451	+0·79827	−455	+0·79011	−458
84	−0·87220	−1365	−0·88950	−1353	−0·90671	−1338	+0·80177	−466	+0·79372	−469	+0·78553	−471
85	−0·88585	−1350	−0·90303	−1335	−0·92009	−1324	+0·79711	−481	+0·78903	−483	+0·78082	−485
86	−0·89935	−1334	−0·91638	−1320	−0·93333	−1306	+0·79230	−495	+0·78420	−497	+0·77597	−497
87	−0·91269	−1317	−0·92958	−1307	−0·94639	−1294	+0·78735	−509	+0·77923	−510	+0·77100	−513
88	−0·92586	−1303	−0·94265	−1290	−0·95933	−1277	+0·78226	−521	+0·77413	−523	+0·76587	−524
89	−0·93889	−1287	−0·95555	−1273	−0·97210	−1261	+0·77705	−536	+0·76890	−536	+0·76063	−537
90	−0·95176	−1270	−0·96828	−1257	−0·98471	−1243	+0·77169	−547	+0·76354	−549	+0·75526	−548
91	−0·96446	−1252	−0·98085	−1241	−0·99714	−1230	+0·76622	−559	+0·75805	−559	+0·74978	−560
92	−0·97698	−1240	−0·99326	−1225	−1·00944	−1213	+0·76063	−574	+0·75246	−572	+0·74418	−571
93	−0·98938	−1222	−1·00551	−1211	−1·02157	−1197	+0·75489	−583	+0·74674	−584	+0·73847	−583
94	−1·00160	−1208	−1·01762	−1196	−1·03354	−1184	+0·74906	−596	+0·74090	−595	+0·73264	−594
95	−1·01368	−1193	−1·02958	−1180	−1·04538	−1168	+0·74310	−608	+0·73495	−607	+0·72670	−604
96	−1·02561	−1176	−1·04138	−1163	−1·05706	−1152	+0·73702	−618	+0·72888	−616	+0·72066	−615
97	−1·03737	−1161	−1·05301	−1148	−1·06858	−1136	+0·73084	−630	+0·72272	−627	+0·71451	−626
98	−1·04898	−1145	−1·06449	−1134	−1·07994	−1122	+0·72454	−640	+0·71645	−638	+0·70825	−635
99	−1·06043	−1130	−1·07583	−1118	−1·09116	−1106	+0·71814	−650	+0·71007	−648	+0·70190	−646
100	−1·07173	−1116	−1·08701	−1103	−1·10222	−1091	+0·71164	−662	+0·70359	−657	+0·69544	−654
101	−1·08289	−1099	−1·09804	−1087	−1·11313	−1075	+0·70502	−670	+0·69702	−668	+0·68890	−664
102	−1·09388	−1083	−1·10891	−1073	−1·12388	−1061	+0·69832	−679	+0·69034	−677	+0·68226	−673
103	−1·10471	−1067	−1·11964	−1056	−1·13449	−1045	+0·69153	−689	+0·68357	−685	+0·67553	−682
104	−1·11538	−1055	−1·13020	−1041	−1·14494	−1031	+0·68464	−699	+0·67672	−695	+0·66871	−691
105	−1·12593	−1039	−1·14061	−1028	−1·15525	−1014	+0·67765	−708	+0·66977	−705	+0·66180	−698
106	−1·13632	−1024	−1·15089	−1013	−1·16539	−1000	+0·67057	−719	+0·66272	−713	+0·65482	−707
107	−1·14656	−1010	−1·16102	−997	−1·17539	−987	+0·66338	−726	+0·65559	−720	+0·64775	−716
108	−1·15666	−994	−1·17099	−983	−1·18526	−971	+0·65612	−734	+0·64839	−729	+0·64059	−725
109	−1·16660	−977	−1·18082	−968	−1·19497	−956	+0·64878	−743	+0·64110	−737	+0·63334	−731
110	−1·17637	−965	−1·19050	−952	−1·20453	−942	+0·64135	−750	+0·63373	−745	+0·62603	−739
111	−1·18602	−948	−1·20002	−939	−1·21395	−928	+0·63385	−758	+0·62628	−754	+0·61864	−747
112	−1·19550	−934	−1·20941	−923	−1·22323	−913	+0·62627	−767	+0·61874	−761	+0·61117	−755
113	−1·20484	−920	−1·21864	−908	−1·23236	−899	+0·61860	−774	+0·61113	−767	+0·60362	−761
114	−1·21404	−905	−1·22772	−895	−1·24135	−884	+0·61086	−783	+0·60346	−774	+0·59601	−768
115	−1·22309	−890	−1·23667	−878	−1·25019	−868	+0·60303	−788	+0·59572	−782	+0·58833	−775
116	−1·23199	−876	−1·24545	−866	−1·25887	−855	+0·59515	−795	+0·58790	−789	+0·58058	−782
117	−1·24075	−860	−1·25411	−851	−1·26742	−840	+0·58720	−803	+0·58001	−796	+0·57276	−787
118	−1·24935	−845	−1·26262	−836	−1·27582	−826	+0·57917	−807	+0·57205	−802	+0·56489	−794
119	−1·25780	−832	−1·27098	−820	−1·28408	−812	+0·57110	−817	+0·56403	−807	+0·55695	−800
120	−1·26612	−817	−1·27918	−807	−1·29220	−797	+0·56293	−823	+0·55596	−813	+0·54895	−806
121	−1·27429	−803	−1·28725	−793	−1·30017	−783	+0·55470	−830	+0·54783	−821	+0·54089	−813
122	−1·28232	−788	−1·29518	−778	−1·30800	−771	+0·54640	−834	+0·53962	−828	+0·53276	−819
123	−1·29020	−773	−1·30296	−765	−1·31571	−755	+0·53806	−840	+0·53134	−831	+0·52457	−823
124	−1·29793	−759	−1·31061	−751	−1·32326	−741	+0·52966	−846	+0·52303	−839	+0·51634	−830
125	−1·30552	−746	−1·31812	−736	−1·33067	−726	+0·52120	−853	+0·51464	−844	+0·50804	−833
126	−1·31298	−730	−1·32548	−722	−1·33793	−713	+0·51267	−856	+0·50620	−849	+0·49971	−840
127	−1·32028	−717	−1·33270	−708	−1·34506	−698	+0·50411	−865	+0·49771	−854	+0·49131	−844
128	−1·32745	−703	−1·33978	−694	−1·35204	−686	+0·49546	−870	+0·48917	−860	+0·48287	−850
129	−1·33448	−688	−1·34672	−679	−1·35890	−672	+0·48676	−874	+0·48057	−864	+0·47437	−856
130	−1·34136	−674	−1·35351	−666	−1·36562	−658	+0·47802	−878	+0·47193	−870	+0·46581	−861
131	−1·34810	−660	−1·36017	−652	−1·37220	−644	+0·46924	−884	+0·46323	−873	+0·45721	−864
132	−1·35470	−646	−1·36669	−638	−1·37864	−630	+0·46040	−889	+0·45450	−879	+0·44857	−870
133	−1·36116	−632	−1·37307	−624	−1·38494	−616	+0·45151	−893	+0·44571	−884	+0·43987	−871
134	−1·36748	−618	−1·37931	−609	−1·39110	−602	+0·44258	−898	+0·43687	−886	+0·43116	−876
135	−1·37366	−603	−1·38540	−596	−1·39712	−589	+0·43360	−902	+0·42801	−892	+0·42240	−882
136	−1·37969	−590	−1·39136	−583	−1·40301	−575	+0·42458	−906	+0·41909	−897	+0·41358	−886
137	−1·38559	−576	−1·39719	−570	−1·40876	−562	+0·41552	−910	+0·41012	−900	+0·40472	−889
138	−1·39135	−562	−1·40289	−555	−1·41438	−547	+0·40642	−917	+0·40112	−906	+0·39583	−894
139	−1·39697	−548	−1·40844	−540	−1·41985	−535	+0·39725	−919	+0·39206	−907	+0·38689	−896
140	−1·40245	−535	−1·41384	−528	−1·42520	−521	+0·38806	−923	+0·38299	−911	+0·37793	−902
141	−1·40780	−520	−1·41912	−513	−1·43041	−507	+0·37883	−926	+0·37388	−916	+0·36891	−904
142	−1·41300	−506	−1·42425	−500	−1·43548	−495	+0·36957	−929	+0·36472	−917	+0·35987	−907
143	−1·41806	−493	−1·42925	−487	−1·44043	−479	+0·36028	−935	+0·35555	−924	+0·35080	−909
144	−1·42299	−479	−1·43412	−473	−1·44522	−467	+0·35093	−936	+0·34631	−925	+0·34171	−914
145	−1·42778	−465	−1·43885	−459	−1·44989	−453	+0·34157	−940	+0·33706	−929	+0·33257	−918
146	−1·43243	−452	−1·44344	−445	−1·45442	−440	+0·33217	−944	+0·32777	−929	+0·32339	−918
147	−1·43695	−437	−1·44789	−433	−1·45882	−427	+0·32273	−945	+0·31848	−936	+0·31421	−924
148	−1·44132	−425	−1·45222	−419	−1·46309	−414	+0·31328	−951	+0·30912	−937	+0·30497	−926
149	−1·44557	−410	−1·45641	−405	−1·46723	−399	+0·30377	−952	+0·29975	−940	+0·29571	−927
150	−1·44967		−1·46046		−1·47122		+0·29425		+0·29035		+0·28644	

M.	X.						Y.					
	$e = 0{\cdot}51.$		$e = 0{\cdot}52.$		$e = 0{\cdot}53.$		$e = 0{\cdot}51.$		$e = 0{\cdot}52.$		$e = 0{\cdot}53.$	
°												
150	−1·44967	− 397	−1·46046	− 391	−1·47122	− 387	+0·29425	− 955	+0·29035	− 942	+0·28644	− 931
151	−1·45364	− 383	−1·46437	− 378	−1·47509	− 373	+0·28470	− 957	+0·28093	− 946	+0·27713	− 932
152	−1·45747	− 369	−1·46815	− 365	−1·47882	− 361	+0·27513	− 961	+0·27147	− 948	+0·26781	− 936
153	−1·46116	− 356	−1·47180	− 352	−1·48243	− 346	+0·26552	− 963	+0·26199	− 951	+0·25845	− 937
154	−1·46472	− 343	−1·47532	− 338	−1·48589	− 334	+0·25589	− 966	+0·25248	− 952	+0·24908	− 941
155	−1·46815	− 329	−1·47870	− 324	−1·48923	− 321	+0·24623	− 966	+0·24296	− 955	+0·23967	− 943
156	−1·47144	− 315	−1·48194	− 311	−1·49244	− 307	+0·23657	− 970	+0·23341	− 957	+0·23024	− 943
157	−1·47459	− 302	−1·48505	− 299	−1·49551	− 293	+0·22687	− 971	+0·22384	− 960	+0·22081	− 946
158	−1·47761	− 289	−1·48804	− 284	−1·49844	− 282	+0·21716	− 975	+0·21424	− 961	+0·21135	− 949
159	−1·48050	− 274	−1·49088	− 271	−1·50126	− 268	+0·20741	− 976	+0·20463	− 962	+0·20186	− 950
160	−1·48324	− 261	−1·49359	− 257	−1·50394	− 253	+0·19765	− 976	+0·19501	− 964	+0·19236	− 951
161	−1·48585	− 248	−1·49616	− 246	−1·50647	− 242	+0·18789	− 978	+0·18537	− 966	+0·18285	− 954
162	−1·48833	− 234	−1·49862	− 231	−1·50889	− 229	+0·17811	− 982	+0·17571	− 968	+0·17331	− 954
163	−1·49067	− 221	−1·50093	− 218	−1·51118	− 215	+0·16829	− 981	+0·16603	− 970	+0·16377	− 957
164	−1·49288	− 208	−1·50311	− 205	−1·51333	− 202	+0·15848	− 983	+0·15633	− 969	+0·15420	− 957
165	−1·49496	− 194	−1·50516	− 191	−1·51535	− 189	+0·14865	− 987	+0·14664	− 972	+0·14463	− 958
166	−1·49690	− 180	−1·50707	− 178	−1·51724	− 176	+0·13878	− 986	+0·13692	− 972	+0·13505	− 959
167	−1·49870	− 168	−1·50885	− 165	−1·51900	− 163	+0·12892	− 987	+0·12720	− 974	+0·12546	− 961
168	−1·50038	− 154	−1·51050	− 152	−1·52063	− 149	+0·11905	− 989	+0·11746	− 975	+0·11585	− 963
169	−1·50192	− 140	−1·51202	− 138	−1·52212	− 137	+0·10916	− 989	+0·10771	− 977	+0·10622	− 962
170	−1·50332	− 127	−1·51340	− 126	−1·52349	− 124	+0·09927	− 990	+0·09794	− 977	+0·09660	− 964
171	−1·50459	− 113	−1·51466	− 112	−1·52473	− 111	+0·08937	− 990	+0·08817	− 976	+0·08696	− 963
172	−1·50572	− 100	−1·51578	− 98	−1·52584	− 97	+0·07947	− 992	+0·07841	− 979	+0·07733	− 966
173	−1·50672	− 88	−1·51676	− 86	−1·52681	− 85	+0·06955	− 992	+0·06862	− 979	+0·06767	− 966
174	−1·50760	− 73	−1·51762	− 73	−1·52766	− 71	+0·05963	− 994	+0·05883	− 981	+0·05801	− 967
175	−1·50833	− 60	−1·51835	− 60	−1·52837	− 59	+0·04969	− 994	+0·04902	− 980	+0·04834	− 966
176	−1·50893	− 47	−1·51895	− 46	−1·52896	− 45	+0·03975	− 994	+0·03922	− 979	+0·03868	− 966
177	−1·50940	− 33	−1·51941	− 32	−1·52941	− 33	+0·02981	− 993	+0·02943	− 981	+0·02902	− 968
178	−1·50973	− 20	−1·51973	− 20	−1·52974	− 20	+0·01988	− 994	+0·01962	− 981	+0·01934	− 966
179	−1·50993	− 7	−1·51993	− 7	−1·52994	− 6	+0·00994	− 994	+0·00981	− 981	+0·00968	− 968
180	−1·51000		−1·52000		−1·53000		+0·00000		+0·00000		+0·00000	

ORBITAL ELEMENTS OF BINARY STARS.

1. It had been remarked long ago by T. N. Thiele (*Astr. Nachr.* **104**, 245), that the usual system of elements of double stars, analogous to those used in the solar system, is not satisfactory.

In the solar system the use of a fundamental plane suggests itself, and though theoretically the invariable plane would seem to be the best, the mean ecliptic has been used for practical reasons.

In a double star the only elements which have a direct meaning for the system itself are P, T, and e. The semi-axis major, expressed in seconds of arc, depends on the distance of the system from our Sun; this is, however, an important element, as we compute the dynamical parallax from it or the mass of the system in case the parallax is known. The remaining elements, i, ω, and Ω, have no intrinsic meaning. Statistical investigations on the distributions of the planes of the orbits have, it is true, been made; however, these planes are, with very few exceptions, not uniquely determined, and, moreover, the inconsistent results reached by various investigators (see Aitken, *The Binary Stars*, p. 218) seem to indicate that a random distribution is the most probable.

The actual practice at present is to determine from a set of " apparent " elements, which are connected with the observed quantities in a very simple way, a set of " true " elements, on which they depend in a complicated way, and to go back again from these true elements to an ephemeris by a similarly awkward process.

Let P and R be the points of the true orbit for which the eccentric anomaly is 0° and 90° and P′ and R′ their projections on the apparent orbit.

The elements a, A, b, and B, introduced by Thiele, are the polar co-ordinates of P′ and R′ with respect to the centre of the (apparent) orbit as an origin. It would have been better to introduce the rectangular co-ordinates instead, say :—

$$A_1 = a \cos A \qquad A_2 = a \sin A$$
$$B_1 = b \cos B \qquad B_2 = b \sin B$$

The co-ordinates of the companion are then :—

$$x' = A_1 \cos E + B_1 \sin E$$
$$y' = A_2 \cos E + B_2 \sin E$$

or, transferring the origin to the primary :—

$$x = \varrho \cos \theta = A_1(\cos E - e) + B_1 \sin E$$
$$y = \varrho \sin \theta = A_2(\cos E - e) + B_2 \sin E$$

In dealing with rectangular co-ordinates it is the custom to take the right ascension as the x co-ordinate. This is, however, inconsequent in double-star astronomy, as we are bound by the previous habit of counting the position-angle from the north to the east. Therefore, the positive x-axis should be directed to the north and the y-axis following, exactly as we take in the true orbit as the axis of abscissae the line of apsides, which is the zero of the anomalies.

The constants A_1, A_2, B_1, and B_2 may be substituted for a, i, ω, and Ω. They represent the motion without ambiguity. Fo mulae have been given by Thiele (*l.c.*) for the change from either system into the other. The well-known method of Zwiers for computing the elements of a double star is essentially the conversion of the elements a, b, A and B, which are measured directly on the apparent orbit, into the true elements.

Dr. Innes, starting from different considerations, worked out independently a system of elements which is very similar to Thiele's, but still more advantageous.

*2. In looking over Mr. Finsen's earlier work on the orbit of α Centaurus, it appeared to me that the process could be shortened in several places. After I had derived my equations, Dr. van den Bos pointed out to me that on every essential point I had been anticipated by Thiele in 1883. My process differs slightly from Thiele's, although, as just said, the underlying ideas are similar, and repetition may not be useless because Thiele's process does not seem to be generally known.

I work backwards, i.e. I start with observation and not theory.

We observe visually position-angle and distance, or photographically rectilinear co-ordinates, namely :—

$$x = \Delta\delta \qquad = \rho \cos \theta$$
$$y = \Delta\alpha \cos \delta = \rho \sin \theta$$

These are connected with the usual orbit-elements by the equations :—

$$\rho \cos (\theta - \Omega) = r \cos (v + \omega)$$
$$\rho \sin (\theta - \Omega) = r \sin (v + \omega) \cos i$$

Let us remove the element Ω, which is mixed up with the observed quantities, to the right-hand side, and separate the variable v from the constant ω; we get :—

$$x = r \cos v(\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i) + r \sin v(-\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i)$$
$$y = r \cos v(\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i) + r \sin v(-\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i)$$

We put :—

(1) $$\frac{r}{a} \cos v = X \qquad \frac{r}{a} \sin v = Y$$

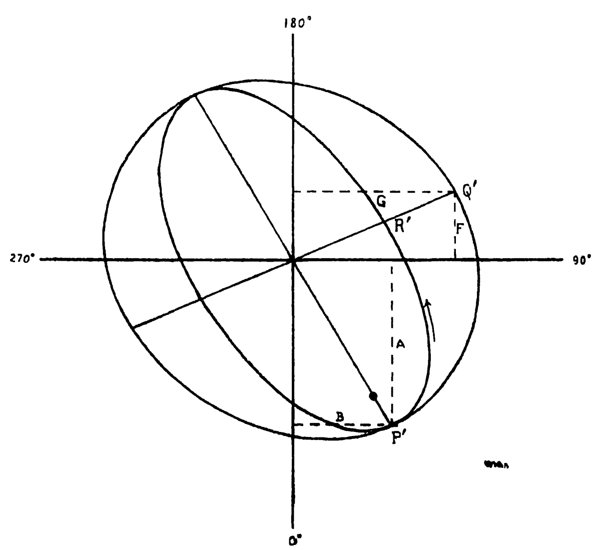
(2) $$a(\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i) = A$$
$$a(\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i) = B$$
$$a(-\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i) = F$$
$$a(-\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i) = G$$

and we have finally :—

(3) $$x = AX + FY$$
$$y = BX + GY$$

3. The analogy between Thiele's and Innes's elements is best seen as follows :—

Instead of P and R and their projections P′ and R′ on the apparent orbit, take P and Q, P′ and Q′, where Q is the point on the auxiliary circle described on the major axis, corresponding to R.



The " natural " elements A, B and F, G are the rectangular co-ordinates of P′ and Q′ referred to the centre of the orbit. The relations are :—

Innes's	equals	Thiele's.
A		$a \cos A$
B		$a \sin A$
F		$b \sec \varphi \cos B$
G		$b \sec \varphi \sin B$
		$(\sin \varphi = e)$

* Sections 2 and 4 are by Dr. Innes.

The advantage of Innes's elements is, that they are independent of the eccentricity, which is not the case with Thiele's h, so that the dynamical and positional elements are completely separated.

X and Y are rectangular co-ordinates in the true orbit referred to the primary, if the semi-axis majo is unity. The bracketed quantities in (2) are the direction cosines of the axes in the true and apparent orbits; hence we recognize the simple equations (3) as nothing but a transformation of co-ordinates.

4. For general application it is desirable that tables of X and Y should be available, and that a table of orbits, giving instead of (or besides) the true elements a, i, ω, and Ω, the natural elements A, B, F, and G should be published.

The latter table has been computed by Dr. van den Bos, and will be published in due time; tables giving X and Y for every degree of mean anomaly from 0° to 180° and for eccentricity 0·01 to 0·99 are in progress at the Union Observatory.

There are several ways of computing this table. We have :—

$$\begin{aligned} X &= \cos E - \sin \varphi \\ Y &= \cos \varphi \sin E \\ M &= E - R^\circ \sin \varphi \sin E \end{aligned} \tag{4}$$

R° radian in degrees.

If we compute the eccentric anomaly for M = 1°, we have X and Y for M = −1°, 0°, and +1°, and we may continue the table by Cowell's process. (See Jackson's paper in the *M.N.*, June, 1924.)

The easiest way is, however, to make use of Åstrand's tables, giving E as a function of M and e, and this is being done here.

5. The formulae

$$\begin{aligned} x &= AX+FY \\ y &= BX+GY \end{aligned} \tag{3}$$

are very advantageous for differential corrections to the elements. We have :—

$$\begin{aligned} \Delta x &= X\Delta A+Y\Delta F+P_{,}\Delta e+Q_{,}n\Delta T+R_{,}\Delta n \\ \Delta y &= X\Delta B+Y\Delta G+P_{,,}\Delta e+Q_{,,}n\Delta T+R_{,,}\Delta n \end{aligned} \tag{5}$$

where

$$\left.\begin{aligned} P_{,} &= A\frac{dX}{de}+F\frac{dY}{de} \\ P_{,,} &= B\frac{dX}{de}+G\frac{dY}{de} \\ Q_{,} &= -A\frac{dX}{dM}-F\frac{dY}{dM} \\ Q_{,,} &= -B\frac{dX}{dM}-G\frac{dY}{dM} \end{aligned}\right\} \left(\frac{dX}{de}, \text{ etc.: partial differentiation is to be inferred.}\right)$$

$$\begin{aligned} R_{,} &= -(t-T)Q_{,} \\ R_{,,} &= -(t-T)Q_{,,} \end{aligned}$$

What renders the use of these formulae so much simpler than the older ones based on the Campbell elements,* is the fact that the partial differential coefficients $\frac{dX}{de}$, $\frac{dY}{de}$, $\frac{dX}{dM}$, and $\frac{dY}{dM}$ can be read off from the X, Y tables at once; then the mean of the preceding and the following first difference in the tables (taken horizontally for de and vertically for dM) differs from the differential coefficient by only one-sixth of the third difference, a quantity which is of no account.

The formulae (5) may be applied if the measures have been made in rectangular co-ordinates (photographic work) as well as in the case where we attach equal weights to position-angles and distances. If the computer prefers to attach different weights, we may use the equations of condition :—

$$\begin{aligned} \varrho\frac{\Delta\theta}{R^\circ} &= \frac{1}{\varrho}(-y\Delta x+x\Delta y) = -\sin\theta\Delta x+\cos\theta\Delta y \\ \Delta\varrho &= \frac{1}{\varrho}(\ x\Delta x+y\Delta y) = \ \ \cos\theta\Delta x+\sin\theta\Delta y \end{aligned} \tag{6}$$

which, as the quantities cos θ and sin θ have already been found in computing the ephemeris, are also easily formed from (5).

6. For computing Innes's elements from Campbell's † we have formulae (2), or adapted to logarithms :—

$$\begin{aligned} A+G &= 2a\cos(\omega+\Omega)\cos^2 i/2 \\ A-G &= 2a\cos(\omega-\Omega)\sin^2 i/2 \\ B-F &= 2a\sin(\omega+\Omega)\cos^2 i/2 \\ -B-F &= 2a\sin(\omega-\Omega)\sin^2 i/2 \end{aligned} \tag{7}$$

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* In his work on the orbit of α Centauri, Mr. Finsen had the opportunity of applying both methods to the same case, and was struck by the great simplification introduced by the new method.

† In all our formulae i is supposed to be between 90° and 180° for retrograde motion.

The latter formulae give at once for the inverse process:—

$$(8)\qquad \tan\,(\omega+\Omega) = \frac{B-F}{A+G}$$

$$\tan\,(\omega-\Omega) = \frac{-B-F}{A-G}$$

$$\tan^2 i/2 = \frac{A-G}{A+G}\cdot\frac{\cos\,(\omega+\Omega)}{\cos\,(\omega-\Omega)} = \frac{-B-F}{B-F}\cdot\frac{\sin\,(\omega+\Omega)}{\sin\,(\omega-\Omega)}$$

The first and second of (8) give the angles $\omega+\Omega$ and $\omega-\Omega$ without ambiguity, the quadrants being fixed by (7); but as we may add 360° to either of these, ω and Ω are uncertain by 180°. This uncertainty is removed by taking Ω always below 180°. The third of (8) gives i with $\pm$ affixed, and it seems preferable to take $i>90°$ for retrograde motion, as this avoids the necessity of adding: angles increasing or decreasing, and the fact of retrograde motion is purely a consequence of our position in space relative to the true orbit, and should, therefore, be incorporated in the *positional* element i, and not in the dynamical n, as is frequently done.

Finally, the semi-axis major a may be found from any one of (7).

As stated above, the element a is of more interest to us than the three others. It is desirable, therefore, to have a formula giving a at once from A, B, F, and G. This is easily found. (A, B) and (F, G) are the extremities of conjugate diameters in the auxiliary ellipse, which has, of course, as semi-axes a and $a \cos i$.

The theorems of Appolonius give:—

$$a^2(1+\cos^2 i) = A^2+B^2+F^2+G^2$$
$$a^2 \cos i = AG-BF$$

from which by elimination of $\cos i$ and putting for brevity

$$A^2+B^2+F^2+G^2 = 2p$$
$$AG-BF = q$$

we have

$$(9)\qquad a^2 = p+\sqrt{(p+q)\,(p-q)}$$

This formula may be used for computing the dynamical parallax without deriving the Campbell elements.

The apparent orbit is easily constructed from the conjugate diameters defined by (A, B) and (F cos φ, G cos φ).

7. Before advocating the substitution of these natural elements for the Campbell elements it seems wise to make a thorough comparison between the two sets and to discuss their relative advantages.

1st. Description of the Orbit.

It may be thought at first that the Campbell elements give us a better impression of the characteristics of the orbit. We think this is a consequence of the fact that we are used to them. As far as the true orbit is concerned, the only real characteristic of it is the eccentricity, an element common to both sets. As to the apparent orbit, we are no doubt able, after some experience, to imagine its form and position more or less when looking at the Campbell elements; but it would certainly be strange if the Innes elements, which are based on the apparent orbit directly, were not at an advantage in this respect. In fact, though we are used to visualize an ellipse in the easiest way if we know its axes, there is no reason why we should not be able to imagine it as well from two conjugate diameters.

Now the Innes elements give us at once these data for the projected auxiliary circle, and if we can imagine this ellipse, it is not difficult to see the apparent orbit inside and the position of the primary from the eccentricity.

As I had not thought of this point myself, which was raised in a letter to me from Dr. Aitken, I may be pardoned if I go into the matter in more detail. I tried the following experiment:—On one side of the blackboard I wrote down the elements e, i, ω, and Ω of Σ 1879, on the other side the elements e, A, B, F, and G of OΣ 285. Both orbits are given by Jackson in the Greenwich Catalogue, with diagrams; the apparent orbits are rather similar. Both being northern binaries, and not such prominent ones as Castor, etc., Dr. Innes and Mr. Finsen had no idea what the apparent orbits looked like. I then asked them to try to visualize the apparent orbits and sketch them on the blackboard. The results of this experiment were: both agreed that the two-dimensional problem presented to them by the A, B, F, G elements of OΣ 285 was much easier to think out than the three-dimensional combination of i, ω, and Ω of Σ 1879, and when I showed them Jackson's diagrams, both were struck by the far greater resemblance of the sketch of OΣ 285 than of Σ 1879.

We may agree, therefore, that in this first respect the Innes elements have the advantage.

2nd. Computation of the Orbit.

In a paper on the orbit of 40 Eridani, which I hope to publish soon, I have shown how the methods of Thiele and Zwiers are simplified by the introduction of the natural elements; in fact, Zwiers's method nearly finishes where it starts for the Campbell elements.

3rd. Computation of the Ephemeris.

It is easily seen that the use of formulae (3) and the X, Y tables is much more convenient than the older method, no matter whether Schlesinger's tables of the true, or Åstrand's of the eccentric anomaly are used. The whole computation is done by Crelle's tables or the computing machine (for many binaries the slide rule is sufficient), a trigonometrical table being only wanted for converting the rectangular co-ordinates into polar ones.

Even so long as the X, Y tables have not been published, it will be easier to compute $\dot{X}$ and $\dot{Y}$ by means of (4).

4th. Corrections to the Elements.

These have been dealt with above, and it is very obvious that the natural elements have the advantage over Campbell's.

5th. Radial Velocity.

A great stronghold of the Campbell elements would seem to be the fact that they are adapted to both visual and spectroscopic binaries. Now we are not prepared to urge the desirability of suppressing the Campbell elements altogether. The statistical astronomer, if he wants to investigate the position of the orbit planes, would not like to see i and Ω disappear from our table of orbits.

Nevertheless it will be of interest to see how the natural elements deal with the radial velocity. It must first be remarked, that the spectroscopic observer does not really use Campbell's elements. His elements are γ or V_0 (velocity of the centre of gravity; for an ephemeris the mean velocity $V_1 = \gamma + Ke \cos \omega$ is more convenient), K (semi-amplitude), P, T, e, and ω. The angle ω is not counted in the Campbell way from the node $<180°$, but from the ascending node. The combination $a \sin i$ is only a secondary result derived from K. Here, again, ω does not interest us. It has been the subject of a statistical study, but the result has afterwards been disproved. (See Aitken, *The Binary Stars*, p. 201.)

With these elements the formulae for an ephemeris and for differential corrections as given by Schlesinger are simple and leave nothing to be desired. We shall now derive the radial velocity for our natural elements. We first complete the formulae (3) for the third co-ordinate.

$$z = r \sin (v+\omega) \sin i$$

$$= \frac{r}{a} \cos v(a \sin \omega \sin i) + \frac{r}{a} \sin v(a \cos \omega \sin i)$$

If we put

$$\text{(2')} \qquad \begin{aligned} C &= a \sin \omega \sin i \\ H &= a \cos \omega \sin i \end{aligned}$$

we get

$$\text{(3')} \qquad z = CX+HY$$

Now $\frac{A}{a}, \frac{B}{a}, \frac{C}{a}$ are the direction-cosines of the major axis of the true orbit, and $\frac{F}{a}, \frac{G}{a}, \frac{H}{a}$ those of the minor axis; therefore:—

$$\text{(10)} \qquad \begin{aligned} C^2 &= a^2-A^2-B^2 \\ H^2 &= a^2-F^2-G^2 \\ 0 &= AF+BG+CH \end{aligned}$$

which, in connexion with (9), enable us to compute C and H at once from A, B, F, G. The signs are, of course, unknown (as is the inclination in Campbell's elements), but if the sign of C is fixed, that of H is given by the third equation.

From (3') we get:—

$$\frac{dz}{dt} = C \frac{dX}{dt} + H \frac{dY}{dt}$$

$$= nC \frac{dX}{dM} + nH \frac{dY}{dM}$$

As C and H are in seconds of arc, $\frac{dz}{dt}$ is given in seconds of arc per annum. To reduce to km/sec we multiply by the quantity λ and have

$$\text{(11)} \qquad \begin{aligned} \text{relative rad. vel.} &= \lambda n C \frac{dX}{dM} + \lambda n H \frac{dY}{dM} \\ &= L \frac{dX}{dM} + N \frac{dY}{dM} \end{aligned}$$

where

$$\lambda = \frac{4 \cdot 737}{\pi_{abs}}$$

and, if we express a in km. and n in degrees *per day*

$$L = \frac{1}{86400} na \sin i \sin \omega$$

$$N = \frac{1}{86400} na \sin i \cos \omega$$

Comparing this with the expression for K

$$K = \frac{1}{86400} \frac{n}{R°} a \sin i \sec \varphi$$

we obtain

$$\text{(12)} \qquad \begin{aligned} L &= KR° \cos \varphi \sin \omega \\ N &= KR° \cos \varphi \cos \omega \end{aligned}$$

If instead of the elements K and ω the elements L and N are given, the computation of an ephemeris by (11) is quite as simple as by the usual formula, as the quantities $\frac{dX}{dM}$ and $\frac{dY}{dM}$ are given by the tables.

If we put

$$\frac{dX}{dM} = X' \qquad \frac{dY}{dM} = Y'$$

we have

(11') $$V = \gamma + LX' + NY'$$

Suppose that tables of X′ and Y′ have been formed (in the same form as those for X and Y, from which they may, indeed, be derived by differencing), then we have for differential corrections a formula similar to (5), namely :—

(5') $$\Delta V = \Delta\gamma + X'\Delta L + Y'\Delta N + P_{,}\Delta e + Q_{,}n\Delta T + R_{,}\Delta n$$

$$\text{where } P_{,} = L\frac{dX'}{de} + N\frac{dY'}{de}$$

$$Q_{,} = -L\frac{dX'}{dM} - N\frac{dY'}{dM}$$

$$R_{,} = -(t-T)Q_{,}$$

This formula again seems simpler than Schlesinger's. We may leave it to the spectroscopic observer to decide which is to be preferred.* The number of visual binaries for which the orbital motion in the line of sight has been observed by the spectroscope is very small, though the interferometer (Capella, Mizar) may increase it sensibly in the future, and some visual binaries would no doubt repay spectroscopic observation in a part of their orbit. Therefore, though we would certainly advise the computer of visual double-star orbits to take the small trouble to compute the Campbell elements by (8), we wish to point out to him that, whatever method he uses, he must of necessity pass a stage where the natural elements are practically derived, and that, by giving them explicitly, he saves a great deal of labour to every one who wants to compare his elements with observation, or correct them differentially.

For the effect of small changes in the elements we have :—

(13)
$$\Delta A = A\frac{\Delta a}{a} + F\Delta\omega - B\Delta\Omega + C\sin\Omega\Delta i$$

$$\Delta B = B\frac{\Delta a}{a} + G\Delta\omega + A\Delta\Omega - C\cos\Omega\Delta i$$

$$\Delta F = F\frac{\Delta a}{a} - A\Delta\omega - G\Delta\Omega + H\sin\Omega\Delta i$$

$$\Delta G = G\frac{\Delta a}{a} - B\Delta\omega + F\Delta\Omega - H\cos\Omega\Delta i$$

$$\Delta C = C\frac{\Delta a}{a} + H\Delta\omega \qquad + C\cot i\Delta i$$

$$\Delta H = H\frac{\Delta a}{a} - C\Delta\omega \qquad + H\cot i\Delta i$$

$$\Delta L = \lambda(n\Delta C + C\Delta n) = \frac{L}{K}\Delta K + N\Delta\omega - L\tan\varphi\Delta\varphi$$

$$\Delta N = \lambda(n\Delta H + H\Delta n) = \frac{N}{K}\Delta K - L\Delta\omega - N\tan\varphi\Delta\varphi$$

For the reverse, introducing for brevity :—

$$\begin{aligned} Q &= A\Delta A + B\Delta B + F\Delta F + G\Delta G \\ S &= -G\Delta A + F\Delta B + B\Delta F - A\Delta G \\ U &= F\Delta A + G\Delta B - A\Delta F - B\Delta G \\ W &= -B\Delta A + A\Delta B - G\Delta F + F\Delta G \end{aligned}$$

we have :—

(14)
$$\begin{aligned} a\sin^2 i\Delta a &= Q + S\cos i \\ a^2\sin i(\sec i - \cos i)\Delta i &= 2Q + S\cos i + S\sec i \\ a^2\sin^4 i\Delta\omega &= (1+\cos^2 i)U - 2\cos i\,W \\ a^2\sin^4 i\Delta\Omega &= (1+\cos^2 i)W - 2\cos i\,U \\ C\Delta C &= (Q + S\cos i)\operatorname{cosec}^2 i - A\Delta A - B\Delta B \\ H\Delta H &= (Q + S\cos i)\operatorname{cosec}^2 i - F\Delta F - G\Delta G \end{aligned}$$

$$R^{\circ}\frac{\Delta K}{K} = \sec\varphi(\sin\omega\Delta L + \cos\omega\Delta N) + R^{\circ}K\tan\varphi\Delta\varphi$$

$$R^{\circ}K\Delta\omega = \sec\varphi(\cos\omega\Delta L - \sin\omega\Delta N)$$

In formulae (13) and (14) $\Delta\omega$, $\Delta\Omega$, Δi, and $\Delta\varphi$ are in radians.

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*If the formulae for the radial velocity and its differential are put into practice, it will be more convenient to remove R° from the expressions (12) for L and N to those for X′ and Y′. Then L and N become of the same order as K, and X′ and Y′ become more convenient numbers.