FIRST ATTEMPT AT A THEORY OF THE ARRANGEMENT AND MOTION OF THE SIDEREAL SYSTEM[1]

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ABSTRACT

First attempt at a general theory of the distribution of masses, forces, and velocities in the stellar system.—(1) *Distribution of stars.* Observations are fairly well represented, at least up to galactic lat. 70°, if we *assume that the equidensity surfaces are similar ellipsoids of revolution*, with axial ratio 5.1, and this enables us to compute quite readily (2) *the gravitational acceleration at various points due to such a system*, by summing up the effects of each of ten ellipsoidal shells, in terms of the acceleration due to the average star at a distance of a parsec. The *total number of stars* is taken as 47.4×10^9. (3) *Random and rotational velocities.* The nature of the equidensity surfaces is such that the stellar system cannot be in a steady state unless there is a general rotational motion around the galactic polar axis, in addition to a random motion analogous to the thermal agitation of a gas. In the neighborhood of the axis, however, there is no rotation, and the behavior is assumed to be like that of a gas at uniform temperature, but with a gravitational acceleration ($G\eta$) decreasing with the distance ρ. Therefore the density Δ is assumed to obey the barometric law: $G\eta=-\bar{u}^2(\delta\Delta/\delta\rho)/\Delta$; and taking the mean random velocity $\bar{u}$ as 10.3 km/sec., the author finds that (4) *the mean mass of the stars* decreases from 2.2 (sun=1) for shell II to 1.4 for shell X (the outer shell), the average being close to 1.6, which is the value independently found for the average mass of both components of visual binaries. In the galactic plane the resultant acceleration—gravitational minus centrifugal—is again put equal to $-\bar{u}^2(\delta\Delta/\delta\rho)/\Delta$, $\bar{u}$ is taken to be constant and the average mass is assumed to decrease from shell to shell as in the direction of the pole. The angular velocities then come out such as to make the linear rotational velocities about constant and equal to 19.5 km/sec. beyond the third shell. If now we suppose that part of the stars are rotating one way and part the other, the relative velocity being 39 km/sec., we have a quantitative explanation of the phenomenon of star-streaming, where the relative velocity is also in the plane of the Milky Way and about 40 km/sec. It is incidentally suggested that when the theory is perfected it may be possible to determine *the amount of dark matter* from its gravitational effect. (5) The *chief defects of the theory* are: That the equidensity surfaces assumed do not agree with the actual surfaces, which tend to become spherical for the shorter distances; that the *position of the center of the system* is not the sun, as assumed, but is probably located at a point some 650 parsecs away in the direction galactic long. 77°, lat. −3°; that the average mass of the stars was assumed to be the same in all shells in deriving the formula for the variation of $G\eta$ with ρ on the basis of which the variation of average mass from shell to shell and the constancy of the rotational velocity were derived—hence either the assumption or the conclusions are wrong; and that no distinction has been made between stars of different types.

1. *Equidensity surfaces supposed to be similar ellipsoids.*—In *Mount Wilson Contribution* No. 188[3] a provisional derivation was given of the star-density in the stellar system. The question was there raised whether the inflection appearing near the pole in the

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[3] *Astrophysical Journal*, **52**, 23, 1920.

equidensity surfaces for small densities is real or not. I have since found that these inflections can be avoided without doing very serious violence to the results of observation. If this is done, the equidensity surfaces become approximately ellipsoids, and not only that, but the data can be represented without exceeding the possible limits of observation error, by assuming the equidensity surfaces to be *concentric, similar revolution ellipsoids, similarly situated.*

2. *Elements of the ellipsoids.*—Taking as unit of star-density that in the neighborhood of the sun, the adopted axes of the ellipsoids, which will be referred to as ellipsoids I, II, ... X and which

TABLE I

EQUIDENSITY ELLIPSOIDS

Ellipsoid	Log Δ	*A*	*B*	*B/A*
		parsecs	parsecs	
I	9.80−10	118	602	5.102
II	9.60	198	1010	5.102
III	9.40	296	1510	5.102
IV	9.20	413	2106	5.102
V	9.00	553	2820	5.102
VI	8.80	717	3656	5.102
VII	8.60	902	4600	5.102
VIII	8.40	1114	5675	5.102
IX	8.20	1365	6960	5.102
X	8.00	1660	8465	5.102

correspond to the values (Δ being the density) $\log \Delta + 10 = 9.8, 9.6,$ . . . . 8.0, are as shown in Table I. The *A*-axis is directed toward the galactic Pole, the *B*-axis lies in the plane of the Milky Way.

For the Milky Way and for the direction toward the Pole this table yields densities which are fairly well represented, for $\rho > 150$ parsecs, by the formulae,

$$\log \Delta = -2.135 + 2.368 \log \rho - 0.593 (\log \rho)^2 \quad \text{(M.W.)}, \qquad (1)$$

$$\log \Delta = -5.356 + 4.890 \log \rho - 1.200 (\log \rho)^2 \quad \text{(Pole)}, \qquad (1a)$$

A section of the equidensity-ellipsoids through the sun (which has been assumed to be the center of the system) at right angles to the plane of the Milky Way is shown in Figure 1.

The agreement of the densities furnished by Table I with those of *Contribution* No. 188 is fairly good for all galactic latitudes up to 65° or 70°. For still higher latitudes it may perhaps still be called tolerable. At least the deviations hardly exceed what would be produced by an error of 0.1 mag. in the photometric scale for these regions.

In the present paper I have substituted these ellipsoids for the surfaces derived directly from observation in *Contribution* No. 188, *not* because I think they are nearer the truth, but simply because they are so enormously more convenient for further computation.

My aim in the present paper is simply to get hold of some approximate information about the real structure and motion of the system, and quantitative accuracy has been considered of secondary

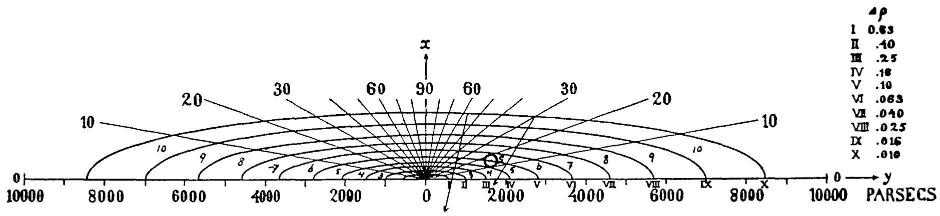

FIG. 1

importance as long as we may hope that the main features are not affected. I trust that this hope will not be disappointed, notwithstanding the many defects—defects that will be duly pointed out—which still attach to the present treatment.

3. *Advantage of the adoption of the ellipsoids.*—The form of the equidensity surfaces thus adopted has the advantage that it calls attention to the possibility of determining with some precision the gravitational attraction of the *whole* of the stellar system on any point inside ellipsoid X, while at the same time it renders the computation of that attraction a relatively easy matter.

In another paper[1] van Rhijn and I have tried to show that, as soon as we possess good counts of stars for each interval of magnitude down to apparent magnitude 17 (visual), we shall know with some tolerable approximation the density of the whole region covered by Figure 1, that is, of the whole extent of the stellar system for which the density exceeds one-hundredth of that in the neighborhood of the sun.

[1] *Mt. Wilson Contr.*, No. 229; *Astrophysical Journal*, 55, 242, 1922.

In the near future such counts will be available. They will be furnished by the Mount Wilson "Catalogue of the Selected Areas" (from $\delta = -15°$ to $\delta = +90°$), the discussion of which is in the hands of Seares. A few provisional counts make it probable that this work will in the main confirm the elements used for Table I and Figure 1. I will assume, therefore, that even now the densities are sufficiently well known for the whole of ellipsoid X.

The advantage just alluded to is a consequence of the well-known property that the attraction of an ellipsoidal shell of constant density, bounded by two similar and similarly situated ellipsoids, on an internal point is *zero*. For it is evident by this property that, if in all that part of the system which lies outside ellipsoid X—for which part accurate data are still wanting—the arrangement in similar ellipsoids also holds, the attraction of this outside domain on a point inside ellipsoid X would be *zero*. And as the distribution of density inside ellipsoid X is known, the possibility of computing the attraction of the total system on a point inside of X becomes evident. If on the contrary the same arrangement does not hold outside ellipsoid X, it still seems highly probable a priori that any change in the form of the equidensity surfaces must be gradual, that is, the equidensity surfaces in the neighborhood of X will diverge little from similar ellipsoids, and the greater changes will begin to appear only at more considerable distances. For the consecutive shells, therefore, the attraction on an internal point will begin by being very small, both on account of the near approach to similarity of these shells and their small density and greater distance from the attracted point. For more distant shells the first circumstance will probably diminish in importance with increasing distance, while, on the contrary, the second becomes more and more important. On the whole, therefore, the attraction of all of that part of the system which lies outside X will be small, and its neglect will presumably not prevent us from obtaining fairly exact ideas about the total forces.

4. *Computation of the gravitational forces.*—In ellipsoid I, which for brevity I will call shell 1, and in each of the shells 2, 3, ... 10, between the surfaces of ellipsoids I, II, ... X, the density varies between limiting values which are in the ratio of 1 to 1.585.

In what follows I will assume for each shell a constant average density.

The computation of the gravitational forces has been carried through for:

(1) 10 points in galactic latitude 0° lying in the surfaces of ellipsoids I, II, ... X; these points have been designated by I, 0°; II, 0°; . . . . X, 0°.

(2) 5 points in galactic latitude 30°, situated on the surfaces II, IV, VI, VIII, X; denoted by II, 30°; IV, 30°; . . . . X, 30°.

(3) Similarly for the 5 points II, 57°1; IV, 57°1; . . . . X, 57°1. (57°1 = average latitude between 40° and 90°.)

(4) The 10 points I, 90°; II, 90°; . . . . X, 90°.

As a unit of attraction I have used the attraction on each other of two stars of average mass separated by a distance of 1 parsec.

I first computed the attraction of the full ellipsoids I, II, ... X on the points specified above, on the supposition that they are of a constant density such that every cubic parsec contains a single star. The formulae for this computation are given in the Appendix.

The attraction of the full ellipsoids having been found, simple subtraction gives the attraction of the separate shells 1, 2, ... 10, all supposed to have the density corresponding to one star per cubic parsec. The actual attraction of the shells was obtained by multiplying these results by the number of stars per cubic parsec contained in each shell. For the average densities, expressed in terms of the density in the neighborhood of the sun, I adopted the values corresponding to the logarithms 9.9, 9.7, 9.5, ... 8.1, each minus 10, multiplied by 0.0451, which according to *Contribution* No. 188 (12) is the number of stars per cubic parsec near the sun; this gives the numbers in Table II.

TABLE II

AVERAGE NUMBER OF STARS PER CUBIC PARSEC

Shell	No. Stars	Shell	No. Stars
1	0.0358	6	0.00358
2	.0226	7	.00226
3	.0143	8	.00143
4	.00900	9	.000900
5	0.00568	10	0.000568

Having found the separate attractions, the components of the total attractions parallel to the axes can at once be determined by noting that the attraction of any shell on an internal point is *zero*, and further, by neglecting the attraction of that part of the system outside of ellipsoid X on a point inside this ellipsoid. Instead of the components I have entered in Table III the total forces G and the angles that these forces make with the X-axis. These are of course found by the formulae

$$X=G\cos\phi \qquad Y=G\sin\phi. \tag{2}$$

In addition, I have given the values of $\phi-\psi$, ψ being the angle which the normal to the ellipsoid through the attracted point makes with the X-axis; ψ is determined by

$$\tan\psi=\frac{A^2}{B^2}\frac{\beta}{\alpha} \tag{2a}$$

The angle $\phi-\psi$ is of course the inclination of the direction of the force to the normal.

The interpretation of this table—of the first entry, for example—is: Attraction of the whole stellar system on a body in the point I, 0° is a force equal to the attraction of 33.19 stars of average mass at the distance of 1 parsec from that same body. This force makes an angle of 90° with the X-axis, and an angle of 0° with the normal to the equidensity ellipsoid through the attracted point.

5. *Analytical representation of G for galactic latitudes* 0° *and* 90°.—In trying to represent the force G by an analytical formula, I started from the consideration that, as the density is constant near the center, the attraction must be nearly proportional to the distance ρ for very small values of ρ; further, that for distances very great as compared with the dimensions of the stellar system, the attraction must be practically the same as it would be were the mass of the whole system concentrated in the center. For these latter distances, therefore, G must be proportional to $1/\rho^2$.

The following easily managed formula satisfies both conditions:

$$G=\frac{A\rho}{1+B\rho+C\rho^2+D\rho^3}. \tag{3}$$

TABLE III

TOTAL ATTRACTIONS OF THE WHOLE SYSTEM

Attr. Point	Log Δ	Co-ordinates Attr. Point		G	φ	φ−ψ	Attr. Point	Log Δ	Co-ordinates Attr. Point		G	φ	φ−ψ
		α	β						α	β			
		parsecs	parsecs						parsecs	parsecs			
I, 0°	9.80−10	0	602	33.19	90°	0°	I, 90°	9.80−10	118	0	40.06	0°	0°
II, 0	9.60	0	1010	37.76	90	0	II, 90	9.60	198	0	54.26	0	0
III, 0	9.40	0	1510	37.95	90	0	III, 90	9.40	296	0	62.45	0	0
IV, 0	9.20	0	2106	35.71	90	0	IV, 90	9.20	413	0	65.95	0	0
V, 0	9.00	0	2820	32.09	90	0	V, 90	9.00	553	0	66.12	0	0
VI, 0	8.80	0	3657	27.71	90	0	VI, 90	8.80	717	0	64.00	0	0
VII, 0	8.60	0	4600	23.67	90	0	VII, 90	8.60	902	0	60.63	0	0
VIII, 0	8.40	0	5675	20.12	90	0	VIII, 90	8.40	1114	0	56.41	0	0
IX, 0	8.20	0	6960	16.59	90	0	IX, 90	8.20	1365	0	51.56	0	0
X, 0	8.00	0	8465	13.53	90	0	X, 90	8.00	1660	0	46.52	0	0
II, 30°	9.60	187	325	52.34	18° 6′	14°18′	II, 57°1	9.60	196	127	53.99	6°54′	5°29′
IV, 30	9.20	391	677	59.26	22 6	18 18	IV, 57.1	9.20	410	265	64.86	8 49	7 24
VI, 30	8.80	679	1176	53.97	25 8	21 20	VI, 57.1	8.80	711	460	62.10	10 33	9 8
VIII, 30	8.40	1055	1827	44.64	28 2	24 14	VIII, 57.1	8.40	1105	715	53.91	12 5	10 40
X, 30	8.00	1572	2723	34.46	30 54	27 6	X, 57.1	8.00	1647	1066	43.85	13 42	12 17

In this formula A/D evidently equals the total number of stars, N, in the stellar system.

A rough estimate of this number can be made by assuming that formula (15) of *Contribution* No. 188, viz.,

$$N_m = e^{a+bm+cm^2}, \tag{4}$$

is true for all values of m. Integration over m gives

$$N = \frac{\sqrt{\pi}}{\sqrt{-c}} e^{a-\frac{b^2}{4c}}. \tag{5}$$

For the values of a, b, c best satisfying the numbers in *Groningen Publications*, No. 27, Table V, which gives number of stars per square degree, I find the following:

Gal. Lat.	a	b	c	Number of Square Degrees
0° to ±20°.	−10.564	+1.5985	−0.0276	14110
±20 to ±40	−10.539	+1.5468	−0.0288	12420
±40 to ±90	−10.815	+1.6592	−0.0414	14730

With these data, computing the total number of stars for each of the three zones, I obtained

Gal. Lat.	No. Stars
0° to ±20°	43.8×10^9
±20 to ±40	3.6×10^9
±40 to ±90	0.043×10^9

whence

$$\frac{A}{D} = N = 47.4 \times 10^9. \tag{6}$$

The remaining constants were so determined that (3) represents the values of G in Table III. I thus found

$$\left.\begin{array}{ll} \text{Gal. Lat.}=0^\circ & \text{Gal. Lat.}=90^\circ \\ A=0.110 & A=0.376 \\ B=1.30 \times 10^{-3} & B=1.83 \times 10^{-3} \\ C=0.657 \times 10^{-6} & C=3.40 \times 10^{-6} \\ D=2.32 \times 10^{-12} & D=7.93 \times 10^{-12} \end{array}\right\} \tag{7}$$

The representation of the tabular values is as follows:

ρ	Gal. Lat. = 0°			ρ	Gal. Lat. = 90°		
	G. Tab.	G. Form.	Tab.−Form		G. Tab.	G. Form.	Tab.−Form
500......	30.0	30.2	−0.2	200......	54.3	50.1	+4.2
1000......	37.8	37.0	+0.8	500......	66.2	68.0	−1.8
2000......	36.0	35.0	+1.0	1000......	58.5	60.3	−1.8
3000......	31.7	30.2	+1.5	1500......	49.1	49.4	−0.3
4000......	26.0	26.0	0.0				
6000......	19.0	19.9	−0.9				
8000......	14.0	16.0	−2.0				

The agreement is not very good, but seems sufficient for our present purpose.

6. *Application of kinetic theory of gases.*—The results thus far obtained rest, it is true, on provisional data, which even now might be materially improved; they further depend on the supposition, not yet fully demonstrated, that, within the distances here considered, there is no appreciable extinction of light in space, but they are, nevertheless, I think, the legitimate outcome of our data.

For what follows I will now introduce some considerations borrowed from the kinetic theory of gases, the applicability of which to the stellar system might be considered doubtful. At all events I do not pretend to have demonstrated this applicability. The results which will be derived cannot lay claim to be demonstrably correct, but they seem to me to be so remarkable that, after a good deal of hesitation, I have resolved to publish them, in the hope that others, better versed in these matters, may furnish us with a more rigorous solution of the problem involved.

Even though it has been shown, in the main by unpublished investigations, that the peculiar motions[1] of the stars with some crude approximation are Maxwellian, the stellar system cannot be treated as a gas at rest; first, because of the existence of stream-

[1] Peculiar velocity is defined as the motion corrected for both the solar and stream-motion. The radial and transverse velocities agree in showing a certain excess of very large motions over the Maxwellian distribution. They are both represented satisfactorily by the sum of *two* Maxwellian distributions. A thorough separate treatment of all the spectral classes is still a great desideratum.

motion; second, because of the form of the equidensity surfaces, which is certainly different from that of the equipotential surfaces of the gravitational force.

That they are different is proved by the fact, among others, that in general the forces are not normal to these surfaces. This is evident enough without further explanation. Moreover, it is clearly brought out by Table III, where the angle with the normal reaches values of more than 27°. Further it is well known that in a gas at rest under its own attraction, the equidensity surfaces are spherical.

The system cannot therefore be in a steady state unless it has a systematic motion. Since the discovery of the star-streams it is clear that such a motion really exists and that it is parallel to the plane of the Milky Way.

It seems rational, therefore, to assume that the system has a sort of rotational motion round the X-axis (see Fig. 1) which is directed toward the pole of the galaxy. The form of the equidensity surfaces found directly in *Contribution* No. 188 as well as that now adopted, strongly indicates some such motion.

This being assumed, the stars along the axis will still have no other motion than their peculiar motions, which, as was just mentioned, are Maxwellian, at least with some approximation. I venture to assume, therefore, that the stars in the immediate neighborhood of this axis are arranged as the molecules of a gas in a quiescent atmosphere.

If:

Δ be the star-density (number of stars per cubic parsec);
u one of the components of the peculiar velocity;
η the acceleration produced by the attraction of a star of average mass at a distance of one parsec, then on the above assumption

$$\overline{u^2}\frac{\delta\Delta}{\Delta}=-G\eta\delta\rho, \tag{8}$$

$\overline{u^2}$ being the average value of u^2.

The formula is analogous to that used for barometric determinations of altitude in an atmosphere of constant temperature throughout. On the other hand, we have found empirically formulae such as (1) and (1*a*) (see also *Contribution* No. 188, p. 13 [21]); in other words,

$$\log \Delta = -P + Q \log \rho - R(\log \rho)^3 \quad (\rho > \rho_0), \tag{9}$$

from which, by differentiation

$$\frac{\delta\Delta}{\Delta} = \frac{Q - 2R \log \rho}{\rho} \delta\rho. \tag{10}$$

Comparing the two expressions, (8) and (10), for $\delta\Delta/\Delta$

$$G\eta = -\overline{u^2}\left[\frac{Q-2R\log\rho}{\rho}\right] \quad (\rho > \rho_0). \tag{11}$$

As the motions are supposed to be Maxwellian, the well-known formula used in the theory of least squares gives

$$\overline{u^2} = \frac{\pi}{2}(\bar{u})^2. \tag{12}$$

From observations of the radial velocities at the Lick Observatory, where no choice has been made on the basis of proper motion (*Lick Observatory Bulletin*, **6**, 126), I derived the value[1]

$$\bar{u} = 10.3 \text{ km/sec.} \tag{13}$$

or since

$$\begin{aligned} &1 \text{ kilometer} = 3.25 \times 10^{-14} \text{ parsecs} \\ &1 \text{ parsec} \quad = 3.08 \times 10^{13} \text{ kilometers} \end{aligned} \tag{14}$$

I find, in the units parsec and second, here adopted,

$$\bar{u} = 3.35 \times 10^{-13} \tag{15}$$

$$\overline{u^2} = 1.763 \times 10^{-25} \tag{16}$$

so that (11) becomes

$$G\eta = -1.763 \times 10^{-25} \frac{Q - 2R \log \rho}{\rho} \quad (\rho > \rho_0) \tag{17}$$

Finally, for galactic latitude 90°, we obtain from equation (1*a*) the values:

$$Q = +4.890 \quad R = +1.200 \tag{18}$$

$$\eta = \frac{(-8.620 + 4.229 \log \rho) \times 10^{-25}}{G\rho} \quad (\rho > 150). \tag{19}$$

[1] There is a mistake in the derivation of this value. The true value is certainly somewhat lower. From considerations given below I have not deemed it necessary for the present paper to repeat the computations with an improved value of $\bar{u}$.

For small values of ρ, formula (9) does not hold. According to *Contribution* No. 188, and particularly according to *Contribution* No. 229, it represents the observations excellently for values of ρ well beyond the maximum (which in the present case lies near $\rho=110$ parsecs). For values of ρ below the maximum the density is nearly constant. The differential-quotient $\delta\Delta/\delta\rho$ thus becomes very small and $\delta\Delta/\Delta\delta\rho$ very unreliable. In the present case it will probably be well not to rely on the formula below, say, 150 parsecs. This limit was adopted in (19).

I have computed the values of η from (19) both on the supposition that G has the values found directly in Table III and that it has the values yielded by formula (3). The former were adopted (Table IV).

TABLE IV

VALUES OF η AND $\overline{m}$

Point	ρ	η Form. (3)	η Adopted	$\overline{m}$ (Sun=1)
	parsecs			
II, 90°	198	11.1×10^{-30}	10.2×10^{-30}	2.2
IV, 90	413	$8.9\quad10^{-30}$	$9.0\quad10^{-30}$	2.0
VI, 90	717	$7.3\quad10^{-30}$	$7.5\quad10^{-30}$	1.7
VIII, 90	1114	$6.6\quad10^{-30}$	$6.8\quad10^{-30}$	1.5
X, 90	1660	$6.5\quad10^{-30}$	$6.5\quad10^{-30}$	1.4

The quantity η is, as stated above, the acceleration per second, in parsecs, produced by the attraction of a star of average mass on a body at a distance of one parsec. The acceleration which the sun would produce, expressed in the same units, is

$$\text{Acceleration by sun}=4.53\times10^{-30}. \tag{20}$$

This enables us to find the average mass $\overline{m}$ of a star expressed in the mass of the sun as a unit. The values of $\overline{m}$ thus found have been inserted in the last column of Table IV.

These values agree surprisingly well with what has been found by totally different considerations. In a recent paper[1] Jackson and Furner find for visual binary stars, as the best average

$$\frac{1}{\sqrt[3]{m_1+m_2}}=0.855.$$

[1] *Monthly Notices*, **81**, 4, 1920.

Consequently $m_1+m_2=1.60$, which agrees with Table IV if we suppose that the combined mass of the two components and not that of a single component is comparable with the mass of a single star, and especially if we further consider that there are theoretical grounds for expecting that the average mass will decrease for increasing distance.[1]

Remark. Dark matter. It is important to note that what has here been determined is the total mass within a definite volume, divided by the number of luminous stars. I will call this mass the average effective mass of the stars. It has been possible to include the luminous stars completely owing to the assumption that at present we know the luminosity-curve over so large a part of its course that further extrapolation seems allowable.

Now suppose that in a volume of space containing l luminous stars there be dark matter with an aggregate mass equal to Kl average luminous stars; then, evidently the effective mass equals $(l+K)\times$average mass of a luminous star.

We therefore have the means of estimating the mass of dark matter in the universe. As matters stand at present it appears at once that this mass cannot be excessive. If it were otherwise, the average mass as derived from binary stars would have been very much lower than what has been found for the effective mass.

7. *Angular velocities* (ω) *in the plane of the galaxy.*—Ignoring for an instant the fact that the stars in the Milky Way cannot be systematically at rest and treating the stars near this plane in the same way as those near the axis, I am led by a formula analogous to (17) to values of η which are not quite half those given in Table IV. I suppose that the difference must be wholly due to the centrifugal force induced by the rotational motions. In fact, I assume that the average mass is the same throughout the whole system, at least for points on the same equidensity surface.

If, therefore, ρ and ρ' represent the distances from the center, of two points on the same equidensity surface, the first in the direction of the Pole, the second in the Milky Way, for which points

[1] Jeans, *Problems of Cosmogony and Stellar Dynamics* (1919), p. 239.

the total attractive forces of the system are respectively G and G', formula (11) gives

For the Pole $$G\eta = -\overline{u^2}\,\frac{Q-2R\log\rho}{\rho}\,, \tag{21}$$

For the Milky Way $$G'\eta-\rho'\omega^2 = -\overline{u^2}\,\frac{Q'-2R'\log\rho'}{\rho'}\,, \tag{22}$$

where ω represents the angular velocity for the point in the Milky Way, and Q' and R' the constants of equation (9) for that same plane. The two equations determine η and ω. The equation for the latter is

$$\omega^2 = \frac{G'}{\rho'}\cdot\overline{u^2}\left[\frac{Q'-2R'\log\rho'}{G'\rho'} - \frac{Q-2R\log\rho}{G\rho}\right], \tag{23}$$

in which, for $\overline{u^2}$ see (16); for G' and G see Table V; and for Q and R see (18), and where finally, by comparing (9) and (1),

$$Q' = 2.368 \text{ and } R' = 0.593 \tag{24}$$

The maximum of Δ according to formula (9) lies, for the Milky Way at about $\rho = 100$, for the direction toward the Pole at $\rho = 110$. As has been mentioned, the formula ceases to be correct below these values. I assume, as before, that the limit of validity is $\rho = 150$.

8. *Angular velocity for stars not in the galaxy.*—For the regions not in galactic latitudes 0° or 90° I determine the angular velocity by the condition that the resultant of the attractive and centrifugal forces must be at right angles to the equidensity surfaces.

In order that the system may be in the steady state, I assume that the equidensity surfaces are at the same time equipotential surfaces for the resultant of the attractive and centrifugal forces. The above condition is implied in this assumption.

Since

$$X \text{ component of resultant acceleration} = -X\eta,$$
$$Y \text{ component of resultant acceleration} = -Y\eta + \beta\omega^2,$$

and since ψ is the angle between the normal and the X-axis, the equation for ω is

$$\beta\omega^2 - X\eta = -X\eta\tan\psi\,,$$

which becomes slightly more convenient by writing

$$X = G \cos \phi \qquad Y = G \sin \phi .$$

Whence

$$\omega^2 = \frac{G\eta}{\beta} \frac{\sin (\phi - \psi)}{\cos \psi} . \tag{25}$$

For the points in Table V, $\phi - \psi$ and ψ were taken from Table III. The computation was made with the aid of formulae (23) and (25).

TABLE V

VALUES OF ω^2

Point	ω^2	Point	ω^2	Point	ω^2
II, 0°.	0.1757×10^{-30}	II, 30°..	0.4066×10^{-30}	II, 57°.1..	0.4132×10^{-30}
IV, 0 .	.0902 10^{-30}	IV, 30 ..	.2477 10^{-30}	IV, 57.1..	.2835 10
VI, 0 .	.03175 10^{-30}	VI, 30 ..	.1255 10^{-30}	VI, 57.1..	.1606 10
VIII, 0 .	.01237 10^{-30}	VIII, 30 ..	.0683 10^{-30}	VIII, 57.1..	.0949 10
X, 0 .	.00510 10^{-30}	X, 30 ..	.0376 10^{-30}	X, 57.1..	.0568 10

For these same points I have furthermore computed the linear velocities in kilometers per second. They are in Table VI.

TABLE VI

LINEAR VELOCITIES

Point	β	$\beta\omega$	Point	α	β	$\beta\omega$	Point	α	β	$\beta\omega$
		km/sec.				km/sec.				km/sec.
II, 0°..	1010	13.0	II, 30°.	187	325	6.4	II, 57°.1	196	127	2.5
IV, 0 ..	2106	19.5	IV, 30.	391	677	10.4	IV, 57.1	410	265	4.4
VI, 0 ..	3657	20.1	VI, 30.	679	1176	12.8 (13.1)	VI, 57.1	711	460	5.7 (6.3)
VIII, 0 ..	5675	19.4	VIII, 30.	1055	1827	14.7 (15.7)	VIII, 57.1	1105	715	6.8 (9.1)
X, 0 ..	8465	18.6	X, 30.	1572	2733	16.3 (17.9)	X, 57.1	1647	1066	7.8 (11.1)

9. *Explanation of star-streaming.*—According to these numbers the angular velocity is not the same for the same distance β from the axis at different distances α from the plane of the Milky Way. Further on I will explain why the present solution must necessarily be a very crude one. For this reason I am not prepared to maintain the reality of this difference. On the contrary it seems very possible that a more definite solution will finally lead to the conclusion that all the points on a cylinder around the axis of the system move with the same velocity. In fact, if we base our solution on the equidensity surfaces as really derived from the observations,[1] instead

[1] *Mt. Wilson Contr.*, No. 188; *Astrophysical Journal*, **52**, 23, 1920.

of assuming similar ellipsoids, we approach at once much closer to this state of affairs. The numbers in parentheses in Table VI give rough estimates of $\beta\omega$ based on this supposition. In the absence of a more definite solution I will confine myself mainly to points in the plane of the Milky Way, the motions of which, except for small values of β, seem to be somewhat better determined.

The most striking feature brought out by these numbers is undoubtedly the fact that at distances from the axis exceeding 2000 parsecs the linear velocity of the stars is nearly constant, the average being 19.5 km/sec.; that is, the great bulk of the stars must have a motion of 19.5 km in a direction parallel to the plane of the Milky Way. Observation has already proved that there really exists a systematic motion of the stars, that it is exactly parallel to the plane of the Milky Way, and that the motion takes place in two exactly opposite directions, the two streams having a relative velocity of about

$$40 \text{ km/per sec.} \tag{26}$$

Since in the preceding theory the motion is introduced simply to explain certain centrifugal forces, it is at once evident that it supposes nothing about the direction in which the motion takes place. Nothing prevents us from assuming that part of the stars circulate one way, while the rest move in the opposite direction. The relative motion of the two groups will then evidently be

$$2\times 19.5 = 39 \text{ km/sec.} \tag{27}$$

The motion to which our theory leads, besides being in the same plane, has therefore practically the exact value which is known from observation to exist. In fact we are led in the most direct and natural way to a complete explanation of the phenomenon of star-streaming. The circumstance that observation led us to assume two rectilinear streams, whereas we here find the motion to be circular, is probably unimportant. It is of course infinitely probable that the sun must be at a certain distance from the center of the system. If we suppose it to be at the point S (see Fig. 1) then the star-streams are derived from the observed motions of stars within a volume whose dimensions are of the order of those of the sphere around S shown in the figure. As long as the radius of this sphere is small in comparison with the distance of S from the center, the curvature of the stream-lines must be inappreciable.

When we consider that the value (27) has been obtained by a study of the arrangement of stars in space, in which the proper motions play no other part than that of a criterion of distance, while the value (26) has been obtained by a study of the motions themselves, both radial and transverse, the close agreement of the two results seems very significant. It becomes more so through the fact that both theories yield a motion exactly parallel to the plane of the Milky Way. Further, if we take into account the fact that the present theory leads to a value for the average mass of the stars which is in close accordance with what has also been found from utterly different investigations, and if we add a final point, namely, the natural explanation of the different arrangement of the stars of different spectral types, we are led irresistibly to the following conclusion:

The theory here propounded, though it may require considerable modification on account of its defectiveness both as to observational basis and mathematical treatment is probably correct in its main features.

The last point mentioned, which is open to quantitative verification, requires further investigation, but even now promises to be no less significant than the others. It is referred to again in section 14 below.

10. *Accelerations including centrifugal effect.*—If with the aid of Table V we now add the acceleration due to the centrifugal forces to that produced by the attractive force, the resultant will be in the direction of the normal. We find the results in Table VII.

TABLE VII

ACCELERATION, INCLUDING EFFECT OF CENTRIFUGAL FORCE

Point	$G\eta$	Point	$G\eta$	Point	$G\eta$	Point	$G\eta$	G_{Pole}/G_{MW}
II, 0°..	208×10^{-30}	II, 30°..	508×10^{-30}	II, 57°1.	547×10^{-30}	II, 90°..	553×10^{-30}	2.7
IV, 0 ..	$131\quad 10^{-30}$	IV, 30 ..	$495\quad 10^{-30}$	IV, 57.1.	$577\quad 10^{-30}$	IV, 90 ..	$593\quad 10^{-30}$	4.5
VI, 0 ..	$90\quad 10^{-30}$	VI, 30 ..	$367\quad 10^{-30}$	VI, 57.1.	$458\quad 10^{-30}$	VI, 90 ..	$480\quad 10^{-30}$	5.3
VIII, 0 ..	$65\quad 10^{-30}$	VIII, 30 ..	$268\quad 10^{-30}$	VIII, 57.1.	$359\quad 10^{-30}$	VIII, 90 ..	$384\quad 10^{-30}$	5.9
X, 0 ..	$48\quad 10^{-30}$	X, 30 ..	$193\quad 10^{-30}$	X, 57.1.	$277\quad 10^{-30}$	X, 90 ..	$302\quad 10^{-30}$	6.3
ψ	90°		3°49′		1°25′		0°	

11*a*. *Velocity of escape*.—Assuming formula (3) to be valid for all distances, I find for the velocity of escape of a star near the center of the system

$$\left.\begin{array}{ll}\text{In the Milky Way} & 104 \text{ km/sec.}\\ \text{In the direction of Pole} & 98 \text{ km/sec.}\end{array}\right\} \quad (28)$$

As a matter of convenience I used for this computation the constant value

$$\eta = 7.5 \times 10^{-30} \quad (29)$$

11*b*. *Velocity compared with circular velocity*.—The velocity of a star which moves in the plane of the Milky Way in a circular orbit is determined by the formula

$$V_c^2 = G\eta\beta. \quad (30)$$

If for the point II, 0° we take $\eta = 10.2 \times 10^{-30}$, and for the others the value (29), we obtain the results in Table VIII. The linear velocities $\beta\omega$ are thus seen to be well below the critical velocity V_c.

TABLE VIII

LINEAR AND CIRCULAR VELOCITIES IN PLANE OF MILKY WAY

Point	β	G	$\beta\omega$ Table VI	V_c
			km/sec.	km/sec.
II, 0°	1010	37.76	13.0	19.2
IV, 0	2106	35.71	19.5	23.1
VI, 0	3656	27.71	20.1	26.8
VIII, 0	5675	20.12	19.4	28.5
X, 0	8465	13.53	18.6	28.5

12. *Defects of solution*.—The way in which the values ω have been determined has made the resultant of gravitational and centrifugal forces perpendicular to the equidensity surfaces. In order that the surfaces may be really equipotential there is, however, a second condition to be satisfied, viz., that for points on the same surface the total force shall be inversely as the distance of two consecutive surfaces. This condition is *not* satisfied. The difference is shown, in its most extreme form, in the last column of Table VII, which, for the points in the Milky Way and in the direction of the Pole, situated on the same ellipsoid, shows the quotient

$$\frac{\text{Force at Pole}}{\text{Force in Milky Way (including centrifugal force)}}. \quad (31)$$

If the equidensity surfaces were really accurately represented by concentric similar ellipsoids, similarly situated, as assumed thus far, this quotient in every case would be 5.1. In reality this is far from being so.

There are certain facts indicating that actually the equidensity surfaces cannot be similar ellipsoids as hitherto supposed in this article. It is a well-known fact[1] that the stars of large proper motion show no concentration toward the Milky Way, but are distributed over the various galactic latitudes with very approximate uniformity. The meaning of this can hardly be other than that the equidensity surfaces for the smaller distances are spherical. The equipotential surfaces of the gravitational and centrifugal forces, in order to coincide with the equidensity surfaces, therefore must also be spherical for vanishing distances; hence the quotient (31) must approach 1.00. Now this is just what the values of the quotient given in Table VII do. The change in these values is therefore an encouragement toward an attempt at an improved theory rather than otherwise.

13. *Position of the sun relative to the center of the system.*—Before such an attempt can be made with any hope of success it will be necessary, however, to free the data from several imperfections. Foremost among these is the imperfection in the adopted position of the sun. The data used in what precedes rest on the assumption that the sun is at the center. It seems infinitely improbable that this should be the case. Our theory, crude though it necessarily must be, paves the way for overcoming this difficulty:

First, as seen from the sun, the center of the system must lie in a plane at right angles to the true stream-motion. Adopting for the vertex of the relative motion of the two streams $\alpha = 6^h 17^m$, $\delta = +11°9$, or gal. long. $= 167°$, gal. lat. $= 0°$, we find that the center, as seen from the sun, must lie in gal. long. 77° or 257°.

Second, nearly all astronomers who have dealt with the question, though from a very different point of view, agree in assuming for the center a *southerly* galactic latitude.[2]

[1] See for instance in *Verslag. Kon. Ak. v. Wetensch.*, Amsterdam, April, 1893, p. 137 (128).

[2] For instance, Struve, *Etudes d'Astronomie Stellaire*, pp. 61–62; Kapteyn, *Kon. Ak. Amsterd.*, April, 1893, p. 137; Hertzsprung, *Astronomische Nachrichten*, **196**, 207, 1914.

Third, Hertzsprung[1] from Cepheid variables finds the sun to be 38 parsecs north of the central plane of the Milky Way.

Fourth, the most difficult question probably will be that of the distance of the sun from the center. Still I think we can indicate a method which promises well.

For a first approximation, I start from the supposition, for want of a better one, that, even for the smaller distances, the *true* densities are as found in *Contribution* No. 188, which were derived on the basis of the erroneous assumption that the sun is the center. Further, for the moment, I will neglect the distance of the sun from the central plane of the Milky Way, which seems to be small.

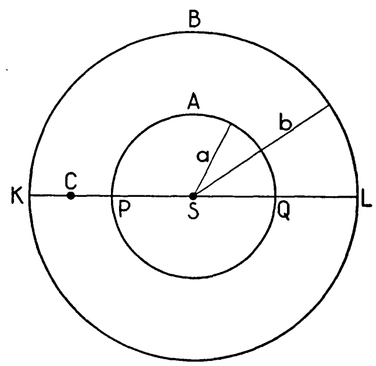

FIG. 2

In Figure 2, S represents the sun, C the center, and $CS=y$, the required distance. Let Δ_ρ represent the true star density at the distance ρ from the center. Now that we give up the erroneous supposition made in *Contribution* No. 188 and the present paper, we must consider the meaning of the densities given in Table VI of *Contribution* No. 188. What this table gives for a specified value of a (see Fig. 2) is in fact Δ', the average of the true densities at all points along the circumference PAQ around S with the radius a. This average will probably be not very different from the mean of the true densities at the points P and Q, and for the present, since our aim is only to arrive at a rough approximation, we will assume that it is exactly the case. Therefore, Δ being the true dens·ty,

$$\left.\begin{aligned}\Delta'_a&=\tfrac{1}{2}(\Delta_{y-a}+\Delta_{y+a}) \text{ if } y<a\\ \Delta'_b&=\tfrac{1}{2}(\Delta_{b-y}+\Delta_{b+y}) \text{ if } y>b\end{aligned}\right\} \qquad (32)$$

[1] *Ibid.*, p. 208.

If now, as a first approximation, we take as the true densities those obtained by formula (1) and assume in succession $y=1500$, 1000, and 500 parsecs, we find values of Δ' corresponding to different values of a as given in Table IX.

Such a table then, or something like it, would have been obtained, instead of Table VI (lat. = 0°) of *Contribution* No. 188, if the center were actually at a distance y from the sun. Thus, if y were indeed 1500 parsecs, the considerations of *Contribution* No. 188 would have led to densities which, from distance *zero* up to distance somewhere near 1500 parsecs, would have increased. For $y=1000$ we should still have found an initial increase, and

TABLE IX

VALUES OF Δ'

a (Parsecs)	y in Parsecs		
	1500	1000	500
0	0.25	0.40	0.73
250	.26	.45	.865
500	.29	.49	.70
1000	.43	.59	.49
1500	.55	.43	.29
2000	.30	.245	.19
3000	.145	.115	.10
4000	.079	.062	.055
5000	0.044	0.039	0.035

even for $y=500$ the same would have held. In this last case, however, the increase would have been so small that it might well have been overlooked owing to the errors of observation. Since in reality the observations lead to a regular decrease of the density throughout, I think we must conclude that y cannot have as high a value as 1500 or even 1000 parsecs. As an upper limit, we may take

$$y<700 \text{ parsecs.} \tag{33}$$

We can also find a *lower* limit from the condition that the relative velocity of the two star-streams must not deviate greatly from the value of 40 km/sec. derived from observation.

In the neighborhood of the sun the linear rotatory motion of the system, that is, one-half the relative velocity of the star-streams, is $y\omega$.

If this velocity exceeded the circular velocity of a star at distance y from the center moving under the attraction of the whole system, the system certainly would not be in a steady state. Hence, on the assumption of a steady state,

$$y\omega < V_c , \tag{34}$$

or by formula (30)

$$y\omega < \sqrt{G\eta y} . \tag{35}$$

Assuming that inside ellipsoid II η, has everywhere the constant value 10.2×10^{-30}, we thus find, expressing results in kilometers, for

$y = 1000$ parsecs	$y\omega < 19.1$ km/sec.
700 parsecs	< 18.2 km/sec.
500 parsecs	< 16.5 km/sec.

If, therefore, we admit that $2y\omega$, the relative stream-velocity, cannot well be below 35 km/sec., we obtain as a lower limit

$$y > 600 \text{ parsecs} \tag{36}$$

The two limits (33) and (36) yield, as a first approximation,

$$y = 650 \text{ parsecs} \tag{37}$$

All these results agree fairly well in locating the center, as seen from the sun, at

$$\left.\begin{array}{lll} \text{Gal. long. } 77^\circ & \text{or} & \alpha = 23^{h}10^{m} \\ \text{Gal. lat. } -3^\circ & & \delta = +57^\circ \\ \multicolumn{3}{l}{\text{Distance projected on galactic plane } = 650 \text{ parsecs}} \\ \multicolumn{3}{l}{\text{Distance projected at right angles to that plane } = 38 \text{ parsecs}} \end{array}\right\} \tag{38}$$

The principal remaining uncertainty is perhaps that for the galactic longitude, which instead of 77° might be 257°. Personally I am strongly in favor of adopting the former value, which is in good accordance with the investigations of Herschel, Struve, and myself already quoted. My own result[1] was

$$\alpha = 0^{h}, \quad \delta = +42^\circ .$$

The determination (38) lays claim to no accuracy. To improve it, it will be necessary to carry through a second, and perhaps a third and fourth, approximation, for the determination of the

[1] *Verslag. Kon. Akad. Amsterdam*, January, 1893, p. 129.

lower limit of y; besides, I think we may still improve or corroborate the values of the distance of the sun from the galactic plane.[1] Altogether, the location of the center would seem to be a laborious problem rather than one of great difficulty.

There is, moreover, the possibility of a direct determination. Given sufficient data on numbers of stars and proper motions for the regions around galactic latitude 0° and longitudes 77° and 257°, there would be no difficulty in deriving separately the star-densities at different distances from the sun in these regions. If there is truth in the above theory, these densities must *increase* with the distance up to distance y, in the direction *toward* the center, whereas in the opposite direction they must decrease with increasing distance. The two solutions together must yield a rather crucial test of the whole theory.

14. *Further defects. Separate treatment of the different spectral types.*—There are several further defects in the solution of the present paper. I will enumerate those that occur to me:

a) In the investigation on which the present paper is based the average parallax, as a function of magnitude and proper motion, has been taken from *Groningen Publication*, No. 8. At the present moment we have already available, though not yet published, much improved values, especially for the very small proper motions.

b) The value of $\overline{u^2}$ is not the very best that could have been obtained.

c) The values of G in Table III have been computed on the supposition that the average mass of a star is independent of the distance, whereas later, in section 8, it was found that this is not the case.

d) In this same computation for G, it was further assumed that the attraction of the whole of the system outside ellipsoid X on an internal point is zero. Though reasons were given for admitting that this attraction is small, it may not be negligible. Since according to what precedes it is highly improbable that the

[1] Furthermore the distance of the center from the sun, as here found, is so small that, if special attention is given to the matter, it may not be hopeless to get evidence of the curvature of the stream-lines, particularly for the rather distant stars (faint stars having small proper motion).

equidensity surfaces within X, as well as without, are really similar ellipsoids, the whole computation for G will have to be revised in a more definitive treatment.

e) The present solution is for all the stars together, that is, for a very heterogeneous collection of stars.

Only the last point is of fundamental importance, and I will devote a few lines to it. Just as in the higher, very attenuated parts of an atmosphere, where the molecular encounters become relatively rare and the different gases are, as it were, sorted out, so among the stars we must expect that the different spectral classes will show different arrangements in space, owing to the different values of $\overline{u^2}$ and η (mass) peculiar to these classes. The two cases are not identical, owing to the stream-motion in the stellar system, which is not supposed to exist in the atmosphere. Still there must be analogy. So, to take an extreme example, if there exists a class of stars for which $\overline{u^2}$ is zero, these stars, according to the present theory, would be confined exclusively to the central plane of the galaxy. For evidently a star not in that plane could not have pure stream-motion; that is, it could not move in a perfect circle around the axis and perpendicular to it, because the attractive force of the system on such a point does not lie in that plane. Therefore it would necessarily have a peculiar motion.

This simple consideration makes us understand at once the otherwise astonishing fact that the Wolf-Rayet stars lie with such close approximation in the central plane of the Milky Way. In the present theory this means simply that they can have no other than pure stream-motion (possibly with some peculiar motion *in* the plane of the Milky Way[1]). We thus realize that for the several classes of spectra there must be a very intimate connection between the values of $\overline{u^2}$ and η on the one hand, and the well-known differences in concentration toward the galaxy on the other.

Such considerations show that for the totality "all stars together" the value of $\overline{u^2}$ will doubtless change with the position in space and in particular with the distance of the stars from the plane

[1] It is noteworthy that, provided the peculiar motion *in* the Milky Way is also *zero* (as it must be when the Maxwellian distribution holds for this limiting case), the parallax of these stars becomes a pure function of the linear velocity, that is, of the quotient, radial velocity divided by sin λ, λ being the angular distance, star-apex.

of the Milky Way. The present investigation, therefore, suffers from the defect that this circumstance was neglected, $\overline{u^2}$ having been taken the same for the whole of the system. I think that this shows sufficiently the absolute necessity of treating the different spectral classes separately.

For such a treatment the necessary data are not yet available, or at least not available in a satisfactory form. Fortunately, however, there exist at present instruments capable of dealing successfully with difficulties which, not so long ago, would have been insurmountable. With their aid we may hope to obtain material for several spectral classes separately, little or not at all inferior to what has already been obtained for the stars as a whole. I think that the determination of the spectral class for some 1000 stars of magnitudes 11 to 12, well distributed over the sky and for which the proper motion is either already sufficiently well known, or may be determined by taking a few additional photographic plates, will go far toward supplying us with what is so urgently wanted.

APPENDIX

Formulae for the computation of the attraction of a homogeneous revolution ellipsoid, of unit density (1 star per cubic parsec), on an exterior point.

Take the axis of revolution as X-axis and let the xy-plane contain the attracted point whose co-ordinates are α and β. Let A and B be the axes of the ellipsoid, the first coinciding with the axis of revolution, and let X and Y represent the components of the required attraction, Z being zero. Then from the well-known formulae,[1] if we put

$$\lambda^2=\left(\frac{B}{A}\right)^2-1 \qquad (h)$$

$$K=\frac{2K}{\lambda^3}\left(\frac{B}{A}\right)^2 \qquad (k)$$

$$p=\lambda\frac{A}{A'}, \qquad (l)$$

[1] I have used the formulae as given by Duhamel, *Cours de mécanique*, V, 1, pp. 321 and 318.

where, if $\alpha=0$, A' is obtained from

$$A'=\sqrt{\beta^2-A^2\lambda^2} \qquad (m)$$

and if $\beta=0$, from

$$A'=\alpha. \qquad (n)$$

For other values of α and β we may write

$$E^2=\frac{\lambda^2}{2}-\frac{\alpha^2+\beta^2}{2A^2} \qquad (o)$$

$$\eta=\lambda\frac{\alpha}{A} \qquad (p)$$

$$\left(\frac{A'}{A}\right)^2=-E+\sqrt{E^2+\eta^2} \qquad (q)$$

With these auxiliary quantities we find

$$X=-\alpha K\phi\ (p), \qquad (r)$$

$$Y=-\beta K\omega\ (p), \qquad (s)$$

in which $\phi(p)$ and $\omega(p)$, or rather their logarithms, have been tabulated in Table X by means of the formulae

$$\phi(p)=2[p-\text{arc tan } p] \qquad (t)$$

$$\omega(p)=\text{arc tan } p-\frac{p}{1+p^2} \qquad (u)$$

TABLE X

VALUES OF LOG ϕ (p) AND LOG ω (p) ACCORDING TO FORMULAE (t) AND (u)

p	Log ϕ	Log ω	p	Log ϕ	Log ω	p	Log ϕ	Log ω
0.000	$-\infty$	$-\infty$	0.05	5.920−10	5.920−10	0.5	8.861−10	8.804−10
1	0.824−10	0.824−10	6	6.157	6.156	0.6	9.076	8.997
2	1.727	1.727	7	6.358	6.357	0.7	9.252	9.149
3	2.255	2.255	8	6.531	6.530	0.8	9.399	9.272
4	2.630	2.630	9	6.685	6.682	0.9	9.524	9.372
5	2.921	2.921	10	6.821	6.819	1.0	9.633	9.455
6	3.158	3.158	11	6.945	6.942	1.1	9.728	9.525
7	3.359	3.359	12	7.058	7.054	1.2	9.812	9.585
8	3.533	3.533	13	7.161	7.157	1.3	9.886	9.635
9	3.687	3.687	14	7.257	7.252	1.4	9.954	9.679
10	3.824	3.824	15	7.346	7.341	1.5	0.015	9.717
11	3.948	3.948	16	7.430	7.423	1.6	0.070	9.750
12	4.061	4.061	17	7.508	7.500	1.7	0.121	9.780
13	4.166	4.166	18	7.581	7.573	1.8	0.168	9.806
14	4.262	4.262	19	7.651	7.642	1.9	0.211	9.828
15	4.352	4.352	20	7.717	7.707	2.0	0.252	9.850
16	4.436	4.436	21	7.779	7.768	2.1	0.289	9.869
17	4.515	4.515	22	7.839	7.827	2.2	0.325	9.885
18	4.590	4.590	23	7.896	7.882	2.3	0.358	9.900
19	4.660	4.660	24	7.950	7.935	2.4	0.389	9.914
20	4.727	4.727	25	8.002	7.986	2.5	0.418	9.927
21	4.790	4.790	26	8.052	8.035	2.6	0.446	9.939
22	4.851	4.851	27	8.100	8.081	2.7	0.472	9.950
23	4.909	4.909	28	8.146	8.126	2.8	0.497	9.960
24	4.964	4.964	29	8.190	8.169	2.9	0.521	9.969
25	5.018	5.017	30	8.233	8.210	3.0	0.544	9.977
26	5.069	5.068	31	8.274	8.250	3.1	0.566	9.985
27	5.118	5.118	32	8.314	8.288	3.2	0.587	9.993
28	5.165	5.165	33	8.352	8.325	3.3	0.607	0.000
29	5.211	5.211	34	8.390	8.361	3.4	0.626	0.006
30	5.255	5.255	35	8.426	8.396	3.5	0.645	0.012
31	5.298	5.297	36	8.461	8.429	3.6	0.663	0.018
32	5.339	5.339	37	8.495	8.461	3.7	0.680	0.023
33	5.379	5.379	38	8.528	8.493	3.8	0.697	0.028
34	5.418	5.418	39	8.560	8.523	3.9	0.713	0.033
35	5.456	5.455	40	8.591	8.552	4.0	0.728	0.038
36	5.492	5.492	41	8.621	8.581	4.1	0.743	0.042
37	5.528	5.528	42	8.651	8.609	4.2	0.758	0.046
38	5.563	5.563	43	8.680	8.635	4.3	0.772	0.050
39	5.597	5.596	44	8.707	8.662	4.4	0.786	0.054
40	5.630	5.629	45	8.735	8.687	4.5	0.799	0.057
41	5.662	5.661	46	8.761	8.712	4.6	0.812	0.060
42	5.693	5.693	47	8.787	8.736	4.7	0.825	0.064
43	5.724	5.723	48	8.813	8.759	4.8	0.837	0.067
44	5.754	5.753	49	8.837	8.782	4.9	0.849	0.070
45	5.783	5.782	0.50	8.861	8.804	5.0	0.861	0.072
46	5.812	5.811						
47	5.840	5.839						
48	5.867	5.866						
49	5.894	5.893						
0.050	5.920	5.920						

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