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THE HATCHET PLANIMETER

The use of planimeters is fairly common to-day and most students of mathematics and physics are familiar with the theory and practice of the common form of the polar planimeter, known as Amsler's planimeter. The theory is usually given in books on the calculus and handbooks of practical physics.* A lesser known planimeter is that of Prytz, also known as the *hatchet* planimeter. It consists of a metal rod, about a quarter of an inch in diameter, bent twice at right angles to form a long arm about ten inches long and two short arms about three inches long. One arm (fig. 1) is sharpened to a point P, and the other to a convex knife edge, or axe edge, K, in line with P. The distance from the point to the middle of the knife edge is usually made, for convenience, a definite distance, *e.g.* 10 inches or 25 centimetres. The rod must be stout and stiff enough to maintain this distance constant. When the instrument is in use the P-arm is held lightly

**Vide*, Williamson, *Integral Calculus;* Horsburgh, *Modern Instruments of Calculation;* Lipka, *Graphical and Mechanical Calculation.*

in a vertical position with the point P resting on the paper and the long arm stretching out in a horizontal position with the knife edge resting in easy contact upon the paper.

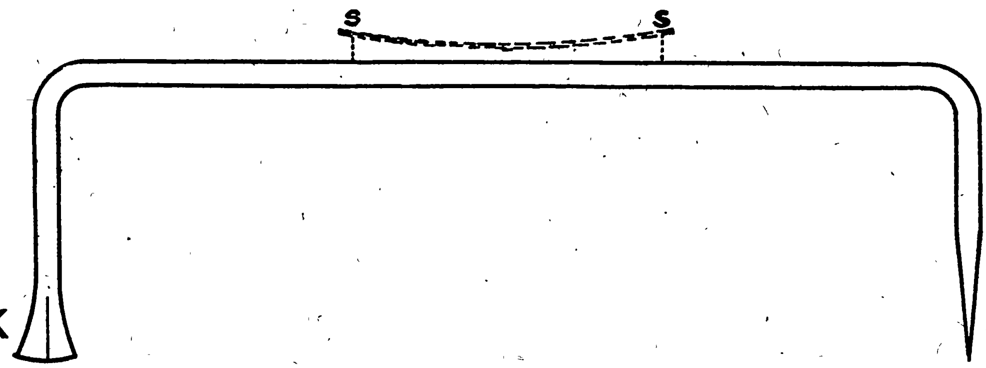

Fig. 1.

When P is made to trace any curve the knife edge K describes a curve such that PK is always tangential to the curve—in other words as P travels along any line K traces a "curve of pursuit."

In finding an area such as A (fig. 2) the position of the centre of area G is first approximately estimated and P placed upon it with the arm of the planimeter in a direction such as GK_1. The knife edge is gently pressed into the paper to form a light dent K_1, P is now made to pass by a straight line GB to the periphery, then

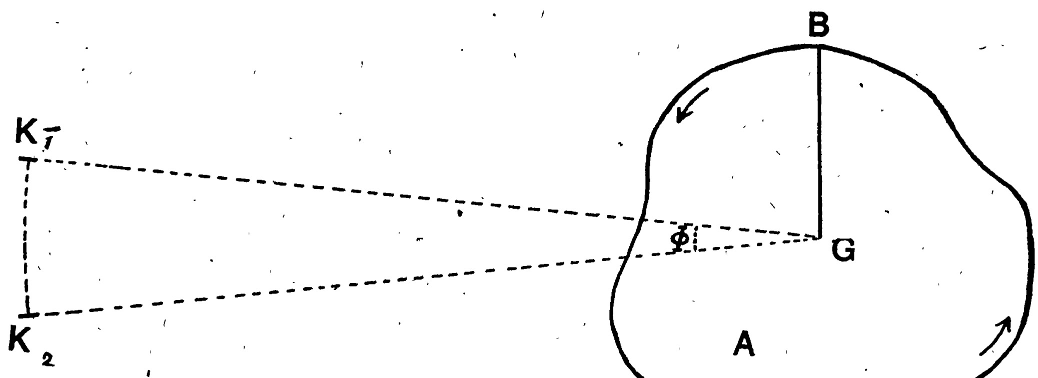

Fig. 2.

anticlockwise around the periphery, back to B and thence to G. The knife edge is then pressed into the paper to form another dent K_2.

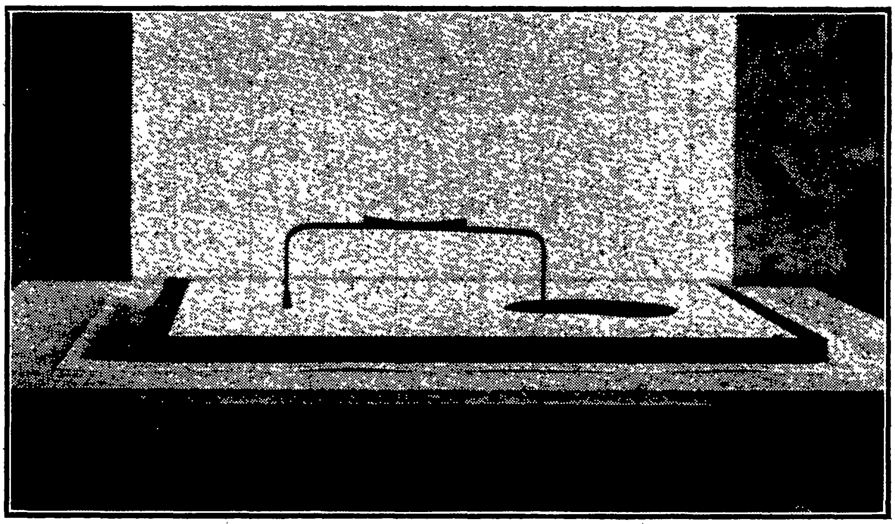

FIG. 3

The required area is given approximately by the product of the arm of the lever and the arc K_1K_2. As a check on K_1 K_2 the procedure may be repeated the tracer point being taken up GB, around the figure clockwise, and back to G. This should bring the knife edge back to K_1. Usually there is a slight discrepancy between the first and third positions and the average should be taken as K_1.

To facilitate the use of the instrument a short graduated scale may be soldered to the long arm of the planimeter so that it can be applied readily to measure K_1 K_2, and if this scale has been bent to an arc whose radius is the length of the planimeter so much the better*. This is indicated by the dotted curve SS in Fig. 1. The complete instrument as used in my laboratory at Toronto is shewn in the photograph (Fig. 3) standing on the drawing board ready for use on the area of the template indicated in black.

*See, however, a later remark on the relative merits of measuring the arc K_1K_2 or the chord K_1 K_2.

The theory of the instrument is not at all simple. It has been given by F. W. Hill in the *Philosophical Magazine* of October, 1894. Hill states that the expression for the area in terms of the length of the arm and the angle K_1GK_2 (Fig. 2) is complex and in general can only be obtained in the form of an infinite series.

His proof is difficult to follow and I have here attempted to simplify it where it seems necessary.

The first step in Hill's proof is: Let the tracer point P move a distance r along a straight line OX (Fig. 4) so that the initial and final positions of the rod are P_1K_1 and P_2K_2.

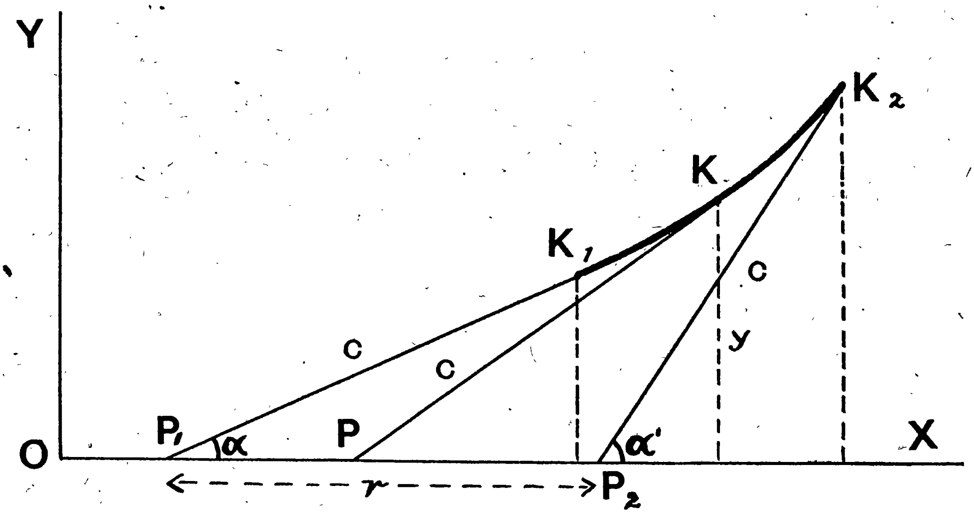

Fig. 4.

Let α, α', be the initial and final inclinations of the rod to OX. Let c be the length of the arm PK. Then shall

$$\tan\frac{\alpha'}{2} = e^{\frac{r}{c}} \tan\frac{\alpha}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

This formula is proved as follows: From the figure we see that in all positions of the rod, $y = c \sin \alpha$, where, for the moment, α is the general value of the slope of the arm to O X.

But $\tan\alpha = \dfrac{dy}{dx}$,

and therefore $= c\cos\alpha\dfrac{d\alpha}{dx}$

$$\therefore dx = c\,\frac{\cos^2\alpha}{\sin\alpha}\,d\alpha$$

$$= c\,\frac{1-\sin^2\alpha}{\sin\alpha}\,d\alpha$$

$$= c\left(\frac{1}{\sin\alpha}-\sin\alpha\right)d\alpha$$

Integrating we get

$$x = c\cos\alpha + c\log\tan\frac{\alpha}{2} + \text{Constant}$$

Going to the limits x, x', corresponding to K_1, K_2 we get

$$x'-x = c\,(\cos\alpha'-\cos\alpha) + c\log\frac{\tan\frac{\alpha'}{2}}{\tan\frac{\alpha}{2}}$$

Now $r = (x'-c\cos\alpha')-(x-c\cos\alpha)$

$$= (x'-x)+c(\cos\alpha-\cos\alpha')$$

$$= c\log\frac{\tan\frac{\alpha'}{2}}{\tan\frac{\alpha}{2}}$$

whence $e^{\frac{r}{c}} = \dfrac{\tan\frac{\alpha'}{2}}{\tan\frac{\alpha}{2}}$

which is equation (1)

Taking two positions of the planimeter near one another, let α, $\alpha+d\alpha$ (Fig. 5) be the inclinations of the planimeter arm to the fixed line along which the tracer point travels, then if dr is the distance between the corresponding two positions of the tracer-point we see that

$$c\,d\alpha = dr\sin\alpha$$

$$\text{or}\quad \frac{d\alpha}{\sin\alpha} = \frac{dr}{c} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

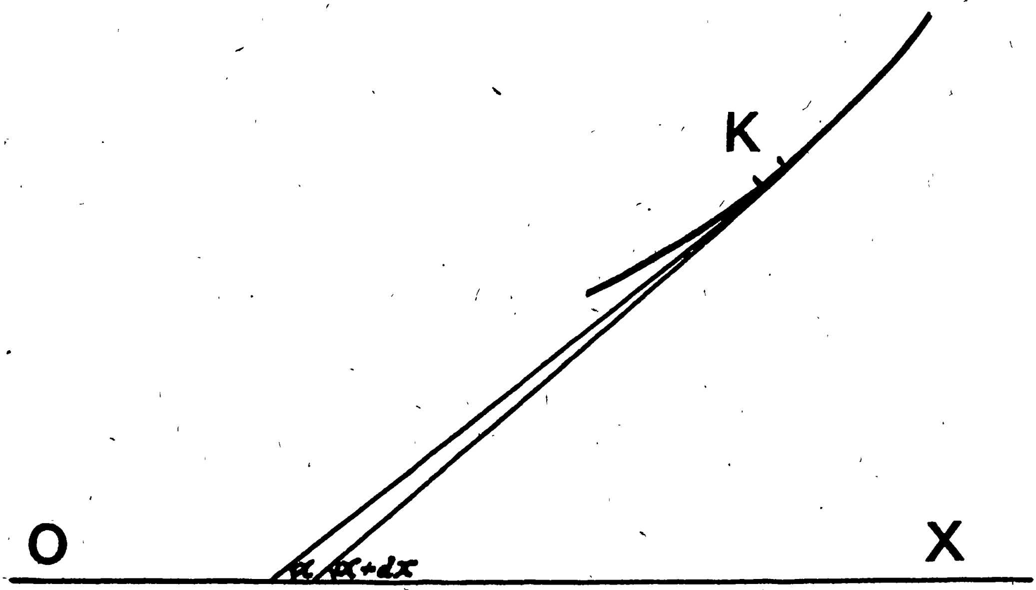

Fig. 5.

Again taking logarithms of the two sides of equation (1) we get

$$\log \tan \frac{\alpha'}{2} = \frac{r}{c} + \log \tan \frac{\alpha}{2}$$

and by differentiating

$$\frac{1}{\tan \frac{\alpha'}{2}} \cdot \sec^2 \frac{\alpha'}{2} \cdot \frac{1}{2} \cdot \frac{d\alpha'}{dr} = \frac{1}{c} + \frac{1}{\tan \frac{\alpha}{2}} \cdot \sec^2 \frac{\alpha}{2} \cdot \frac{1}{2} \cdot \frac{d\alpha}{dr},$$

$$\text{or } \frac{1}{2 \sin \frac{\alpha'}{2} \cos \frac{\alpha'}{2}} \cdot \frac{d\alpha'}{dr} = \frac{1}{c} + \frac{1}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} \cdot \frac{d\alpha}{dr},$$

$$\text{whence } \frac{d\alpha'}{\sin \alpha'} = \frac{dr}{c} + \frac{d\alpha}{\sin \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Divide the area to be measured up into elementary triangles having their vertices at a point O within the area and having elements of the perimeter as their bases. The Triangle OPQ (Fig. 6) is one such triangle. The tracer point may be supposed to move around each of these triangles in turn so that each radius vector (OP, OQ, etc.) is traversed twice, in opposite directions. Starting with the tracer point at O and the rod of the planimeter in position 1,

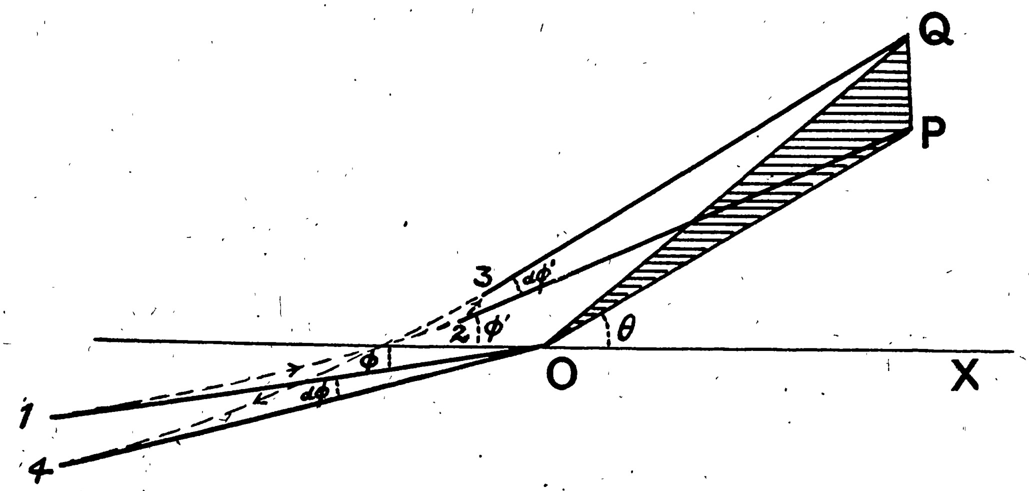

Fig. 6.

the tracer point moves up to P then to Q and back to O the rod taking up successively the positions 1, 2, 3, 4. Taking any fixed line OX let the angles of inclination of OP, OQ, to this line be θ, $\theta + d\theta$ and the angles of inclination of the rod in the positions 1, 2, 3, 4, be ϕ, ϕ', $\phi' + d\phi'$, $\phi + d\phi$.

Let $OP = r$, $OQ = r + dr$

then by equation (1) and Fig. 7 we see that, for the motion from O to P

$$\tan\frac{\theta-\phi'}{2} = e^{-\frac{r}{c}} \tan\frac{\theta-\phi}{2} \quad \ldots\ldots\ldots\ldots (4)$$

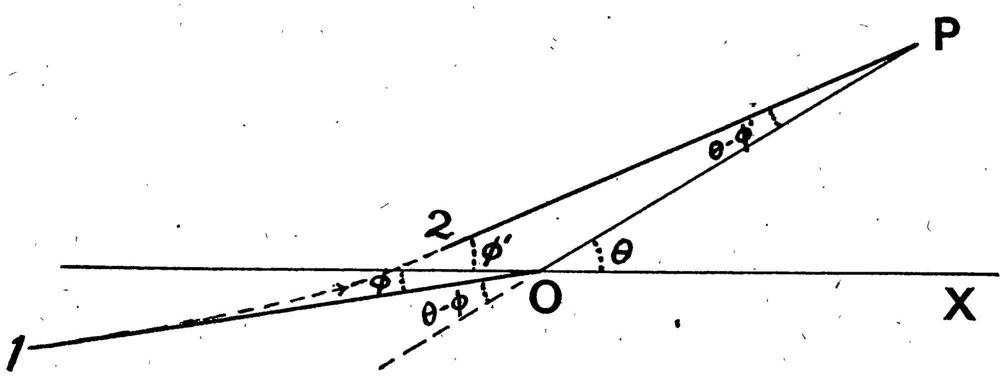

Fig. 7

and for the motion from Q to O

$$\tan\frac{(\theta+d\theta)-(\phi'+d\phi')}{2}=e^{-\frac{r+dr}{c}}\tan\frac{(\theta+d\theta)-(\phi+d\phi)}{2}\ldots\ldots(5)$$

From these equations and equation (3) we get

$$\frac{d\theta-d\phi'}{\sin(\theta-\phi'}=-\frac{dr}{c}+\frac{d\theta-d\phi}{\sin(\theta-\phi)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Take now the motion of the tracer point from P to Q. Let ψ (Fig. 8) be the angle between OP and the element of the perimeter PQ.

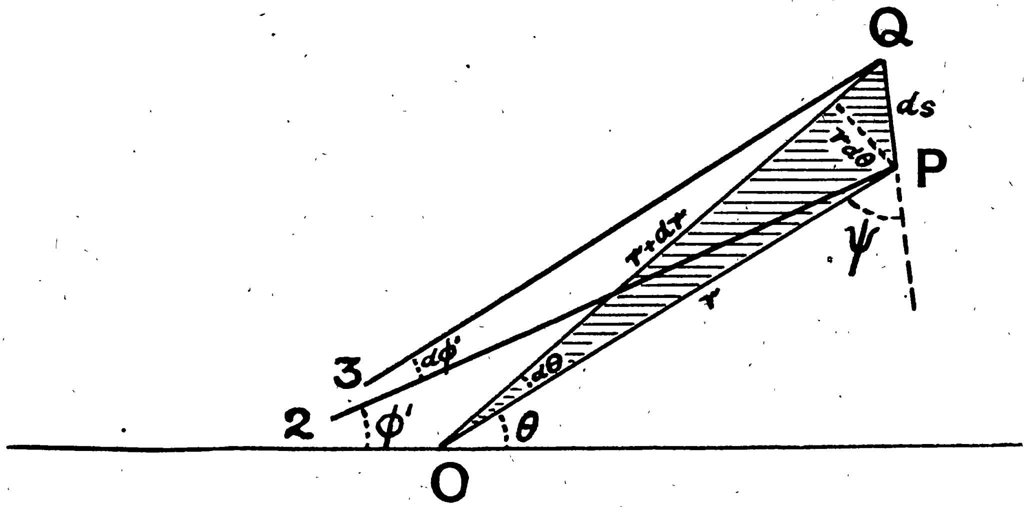

Fig. 8.

meter PQ. When the tracer point is at P the inclination of the planimeter arm to PQ is $\psi+\theta-\phi'$ and when the tracer point is at Q the inclination is $\psi+\theta-\phi'-d\phi'$. Denoting the length of PQ by ds we get by (2)

$$\frac{d\phi'}{\sin(\psi+\theta-\phi')}=\frac{ds}{c}$$

$$\therefore\ c\,d\phi'=ds\left[\sin\psi\cos(\theta-\phi')+\cos\psi\sin(\theta-\phi')\right]$$

But $ds\sin\psi=rd\theta$ and $ds\cos\psi=dr$

$$\therefore\ c\,d\phi'=rd\theta\cos(\theta-\phi')+dr\sin(\theta-\phi')\ldots\ldots\ldots\ldots(7)$$

Writing (6) as

$$\frac{cd\theta - cd\phi'}{\sin(\theta-\phi')} = -dr + \frac{c(d\theta - d\phi)}{\sin(\theta-\phi)}$$

and now substituting for $cd\phi'$ from equation (7) we get

$$\frac{cd\theta - rd\theta\cos(\theta-\phi') - dr\sin(\theta-\phi')}{\sin(\theta-\phi')} = -dr + \frac{c(d\theta-d\phi)}{\sin(\theta-\phi)}$$

$$\text{or}\quad \frac{cd\theta - rd\theta\cos(\theta-\phi')}{\sin(\theta-\phi')} - dr = -dr + \frac{c(d\theta-d\phi)}{\sin(\theta-\phi)}$$

Cancelling the dr's and transposing we get

$$\frac{d\theta-d\phi}{\sin(\theta-\phi)} = \frac{d\theta - \frac{r}{c}d\theta\cos(\theta-\phi')}{\sin(\theta-\phi')}$$

$$= \frac{d\theta\left(\cos^2\frac{\theta-\phi'}{2}+\sin^2\frac{\theta-\phi'}{2}\right) - \frac{r}{c}d\theta\left(\cos^2\frac{\theta-\phi'}{2}-\sin^2\frac{\theta-\phi'}{2}\right)}{2\sin\frac{\theta-\phi'}{2}\cos\frac{\theta-\phi'}{2}}$$

$$= \frac{d\theta\left(1-\frac{r}{c}\right)\cos^2\frac{\theta-\phi'}{2}}{2\sin\frac{\theta-\phi'}{2}\cos\frac{\theta-\phi'}{2}} + \frac{d\theta\left(1+\frac{r}{c}\right)\sin^2\frac{\theta-\phi'}{2}}{2\sin\frac{\theta-\phi'}{2}\cos\frac{\theta-\phi'}{2}}$$

$$= \frac{d\theta}{2}\left(1-\frac{r}{c}\right)\cot\frac{\theta-\phi'}{2} + \frac{d\theta}{2}\left(1+\frac{r}{c}\right)\tan\frac{\theta-\phi'}{2} \quad\ldots\ldots\ldots(8)$$

Now since by equation (4)

$$\tan\frac{\theta-\phi'}{2} = e^{-\frac{r}{c}}\tan\frac{\theta-\phi}{2}$$

and therefore the reciprocal

$$\cot\frac{\theta-\phi'}{2} = e^{\frac{r}{c}}\cot\frac{\theta-\phi}{2}$$

We therefore get

$$\frac{d\theta-d\phi}{\sin(\theta-\phi)} = \frac{d\theta}{2}\left(1-\frac{r}{c}\right)e^{\frac{r}{c}}\cot\frac{\theta-\phi}{2} + \frac{d\theta}{2}\left(1+\frac{r}{c}\right)e^{-\frac{r}{c}}\tan\frac{\theta-\phi}{2}$$

$$\therefore d\theta - d\phi = 2\sin\frac{\theta-\phi}{2}\cos\frac{\theta-\phi}{2}\left[\frac{d\theta}{2}\left(1-\frac{r}{c}\right)e^{\frac{r}{c}}\cot\frac{\theta-\phi}{2} + \frac{d\theta}{2}\left(1+\frac{r}{c}\right)e^{-\frac{r}{c}}\tan\frac{\theta-\phi}{2}\right]$$

$$= d\theta\left(1-\frac{r}{c}\right)e^{\frac{r}{c}}\cos^2\frac{\theta-\phi}{2} + d\theta\left(1+\frac{r}{c}\right)e^{-\frac{r}{c}}\sin^2\frac{\theta-\phi}{2}$$

$$= d\theta\left(1-\frac{r}{c}\right)e^{\frac{r}{c}}\left(\frac{1+\cos(\theta-\phi)}{2}\right) + d\theta\left(1+\frac{r}{c}\right)e^{-\frac{r}{c}}\left(\frac{1-\cos(\theta-\phi)}{2}\right)$$

$$= \tfrac{1}{2}\,d\theta\left[\left(1-\frac{r}{c}\right)e^{\frac{r}{c}} + \left(1+\frac{r}{c}\right)e^{-\frac{r}{c}}\right] + \tfrac{1}{2}\,d\theta\left[\left(1-\frac{r}{c}\right)e^{\frac{r}{c}} - \left(1+\frac{r}{c}\right)e^{-\frac{r}{c}}\right]\cos(\theta-\phi) \quad \ldots\ldots\ldots\ldots(9)$$

Taking the first expression within the square brackets on the right hand side of this equation we get, if $r<c$

$$\left(1-\frac{r}{c}\right)\left(1+\frac{r}{c}+\frac{r^2}{2c^2}+\frac{r^3}{6c^3}+\frac{r^4}{24c^4}+\ldots\ldots\right) + \left(1+\frac{r}{c}\right)\left(1-\frac{r}{c}+\frac{r^2}{2c^2}-\frac{r^3}{6c^3}+\frac{r^4}{24c^4}-\ldots\ldots\ldots\right)$$

$$= 1+\frac{r}{c}+\frac{r^2}{2c^2}+\frac{r^3}{6c^3}+\frac{r^4}{24c^4}+\frac{r^5}{120c^5}+\frac{r^6}{720c^6}+\ldots\ldots\ldots$$

$$-\frac{r}{c}-\frac{r^2}{c^2}-\frac{r^3}{2c^3}-\frac{r^4}{6c^4}-\frac{r^5}{24c^5}-\frac{r^6}{120c^6}-\ldots\ldots\ldots$$

$$+1-\frac{r}{c}+\frac{r^2}{2c^2}-\frac{r^3}{6c^3}+\frac{r^4}{24c^4}-\frac{r^5}{120c^5}+\frac{r^6}{720c^6}-\ldots\ldots\ldots$$

$$+\frac{r}{c}-\frac{r^2}{c^2}+\frac{r^3}{2c^3}-\frac{r^4}{6c^4}+\frac{r^5}{24c^5}-\frac{r^6}{120c^6}-\ldots\ldots\ldots$$

$$= 2-\frac{r^2}{c^2}-\frac{r^4}{4c^4}-\frac{r^6}{72c^6}+\ldots\ldots$$

$$= 2\left[1-\frac{r^2}{2c^2}-\frac{r^4}{8c^4}-\frac{r^6}{144c^6}-\ldots\ldots\ldots\right]$$

The second expression in square brackets on the right hand side of the equation gives

$$\left(1-\frac{r}{c}\right)\left(1+\frac{r}{c}+\frac{r^2}{2c^2}+\frac{r^3}{6c^3}\cdots\cdots\right)-\left(1+\frac{r}{c}\right)\left(1-\frac{r}{c}+\frac{r^2}{2c^2}-\frac{r^3}{6c^3}+\cdots\cdots\right)=2\left[-\frac{r^3}{3c^3}-\frac{r^5}{30c^5}-\cdots\cdots\cdots\cdots\right]$$

Therefore

$$d\theta-d\phi=d\theta\left[1-\frac{r^2}{2c^2}-\frac{r^4}{8c^4}-\frac{r^6}{144c^6}-\cdots\cdots\right]$$
$$-d\theta\left[\frac{r^3}{3c^3}+\frac{r^5}{30c^5}+\cdots\cdots\right]\cos(\theta-\phi)$$
$$\therefore\ -d\phi=-d\theta\left[\frac{r^2}{2c^2}+\frac{r^4}{8c^4}+\frac{r^6}{144c^6}+\cdots\cdots\right]$$
$$-d\theta\left[\frac{r^3}{3c^3}+\frac{r^5}{30c^5}+\cdots\right]\cos(\theta-\phi)$$

$$\text{or } c^2d\phi=\frac{r^2d\theta}{2}+\frac{r^4}{8c^2}d\theta+\frac{r^6}{144c^4}d\theta+d\theta\left[\frac{r^3}{3c}+\frac{r^5}{30c^3}+\cdots\right]\cos(\theta-\phi) \qquad \cdots\cdots\cdots\cdots(10)$$

The $d\phi$ here given is the change in direction of the planimeter arm when the tracer point has completed the circuit of the elementary triangle OPQ. Repeating the process for all the elementary triangles into which the area is divided we get

$$c^2\int d\phi=\int\tfrac{1}{2}r^2d\theta+\frac{1}{8c^2}\int r^4d\theta+\frac{1}{144c^4}\int r^6d\theta+\cdots\cdots\cdots$$
$$+\frac{1}{3c}\int r^3\cos(\theta-\phi)\,d\theta+\frac{1}{30c^3}\int r^5\cos(\theta-\phi)\,d\phi+\cdots\cdots$$

The integrations of θ extend through two right angles and if Φ is the angle between the initial and final positions of the planimeter arm and A is the area required this equation becomes

$$c^2\Phi=A+\frac{1}{8c^2}\int r^4d\theta+\frac{1}{144c^4}\int r^5d\theta+\cdots\cdots\cdots$$
$$+\frac{1}{3c}\int r^3\cos(\theta-\phi)\,d\theta+\frac{1}{30c^4}\int r^5\cos(\theta-\phi)\,d\theta+\cdots\cdots(11)$$

If the average value of r is very small in comparison with c then all the terms on the right hand after the first become negligible and therefore

$$c^2\Phi = A \qquad \ldots\ldots\ldots\ldots(12)$$

$$\text{or } A = c \times c\Phi = c \times \text{arc } K_1K_2 \qquad \ldots\ldots\ldots(13)$$

which is the formula that was quoted above, and which may, as we see, be employed by using a planimeter whose size is properly related to the dimensions of the figure whose area is to be obtained.

If the length of the planimeter is not large in comparison with the breadth of the figure it is important to inquire into the relative magnitudes of the other terms on the right hand side of equation (11) and to see if any of them may be decreased by a proper choice of origin and initial positions of the first radius vector and arm of the planimeter.

Taking the term $\frac{1}{8c^2}\int r^4 d\theta$ this may be written $\frac{1}{2c^2}\int \frac{1}{2}r^2 d\theta.\ \frac{r^2}{2}$ which is $\frac{Ak^2}{2c^2}$ where Ak^2 is the moment of inertia about an axis through O at right angles to the area. This is the so-called polar moment of inertia. Its value is least when the axis goes through the centre of surface of the area or the centre of gravity of a lamina occupying the area, which point is also called the centroid.

Taking the term $\frac{1}{144c^4}\int r^6 d\theta$, this is seen to be less than $\frac{2}{144c^4}\int a^4.\ \frac{r^2 d\theta}{2}$ where a is the greatest value of r. So that this term is less than $\frac{2A}{144}\left(\frac{a}{c}\right)^4$ which, if $\frac{a}{c}$ is a little less than unity, is less than one per cent. of A.

It now remains to evaluate the terms containing $\cos(\theta-\phi)$. The first term is the most important. If the area of the curve is less than that of a square whose side is the length of the planimeter arm the greatest value of Φ is one radian (see equations 11 and 12). Expanding the first cosine term we get

$$\frac{1}{3c}\int r^3 \cos(\theta-\phi)\, d\phi = \frac{1}{3c}\int r^3 \left[\cos\theta \cos\phi + \sin\theta \sin\phi\right] d\theta$$

$$= \frac{1}{3c}\int r^3 \left[\cos\theta\left(1-2\sin^2\frac{\phi}{2}\right)+\sin\theta\sin\phi\right]d\theta$$

$$= \frac{1}{3c}\int r^3\cos\theta\, d\theta - \frac{2}{3c}\int r^3\cos\theta\sin^2\frac{\phi}{2}\, d\theta + \frac{1}{3c}\int r^3\sin\theta\sin\phi\, d\theta$$

$\sin\phi$ is less than ϕ so that the last term will not exceed $\frac{1}{3c}\int r^3\phi\sin\theta\, d\theta$ and $\sin\frac{\phi}{2}$ will be less than $\frac{\phi}{2}$ so that the second term will be less than $\frac{1}{6c}\int r^3\phi^2\cos\theta\, d\theta$.

Therefore

$$\frac{1}{3c}\int r^3\cos(\theta-\phi)\, d\theta < \frac{1}{3c}\int r^2\cos\theta\, d\theta + \frac{1}{3c}\int r^3\phi\sin\theta\, d\theta - \frac{1}{6c}\int r^3\phi^2\cos\theta d\theta$$

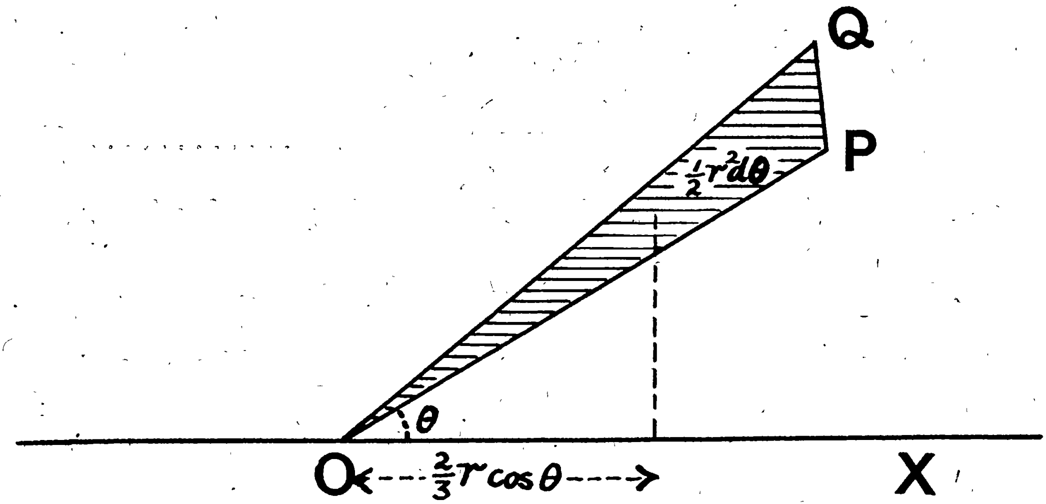

Fig. 9.

Let the initial position of the planimeter arm be taken as the axis of x then taking an elementary triangle as shown in Fig. 9 it is evident that

$$\tfrac{1}{3}r^3\cos\theta\, d\theta = \frac{r^2 d\theta}{2}\cdot\frac{2}{3}\, r\cos\theta$$

= area of triangle OPQ multiplied by the distance of its centre of area from O measured parallel to the axis of x. Extending this to the whole figure we get

$$\frac{1}{3c}\int r^3\cos\theta\, d\,\theta = \frac{1}{c}\, A\bar{x}$$

where $\bar{x}$ is the x co-ordinate of the centroid of the figure measured from an origin at O. The second and third terms diminish as $\left(\frac{a}{c}\right)$ diminishes and as they are also oscillating quantities their integrals may be neglected.

Therefore we write

$$c^2 \Phi = A + \frac{Ak^2}{2c^2} + \frac{A\bar{x}}{c} + \frac{P}{c^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

where P is very small

If the vertex O is also the centroid, $\bar{x}$ is zero, and if the moment of inertia about the axis through the centroid at right angles to the area is denoted by $A\ \rho^2$, $A\ \rho^2$ is the least value of Ak^2.

∴ in this case

$$\therefore c^2 \Phi = A + \frac{A\ \rho^2}{2c^2} \text{ very nearly}$$

$$= A\left(1 + \frac{\rho^2}{2c^2}\right) \quad \text{,,} \qquad \ldots\ldots\ldots\ldots (15)$$

so that the error made in the area by taking $c^2 \Phi$ for its value is an overestimate of the order of $\frac{\rho^2}{2c^2} \times 100$ per cent.

If it is not possible to pick out exactly the centroid as the starting point of the plainmeter the area may be taken first with the arm of the planimeter stretching out to the left of the figure and then repeated with the arm stretching out to the right. This modification reverses the sign of c so that

$$c^2 \Phi_1 = A + \frac{Ak^2}{2c^2} + \frac{A\bar{x}}{c} + \frac{P}{c^3}$$

$$c^2 \Phi_2 = A + \frac{Ak^2}{2c^2} - \frac{A\bar{x}}{c} - \frac{P}{c^3}$$

$$\therefore \frac{c^2}{2}\left(\Phi_1 + \Phi_2\right) = A\left(1 + \frac{k^2}{2c^2}\right) \ldots\ldots\ldots\ldots (16)$$

Instead of reversing the direction of the arm of the lever it may be more convenient to turn the paper carrying the figure through two right angles, and now repeat the process with the arm of the planimeter still out to the left.

It is hardly possible even with the greatest of care to measure the distance between two dents to less than ¼ mm and even if it were so the first and third positions of K_1 (see Fig. 2 and the following text) often differ by more than this amount. The possible error may therefore be taken as 1/8 mm.

The planimeters I use in Toronto have an arm of 250 millimetres, and if there is a possible error in $c\Phi$ (=the distance between K_1 and K_2) of about 1/8 mm the error in $c^2\Phi$ will be about 30 sq. millimetres. If this unavoidable error is greater than $\frac{Ak^2}{2c^2}$ the latter may be neglected. This point will be taken up later.

Two conclusions given by Hill are, with slight variations:

(1) Using a 250 mm planimeter if the origin be near the centre of area and the greatest breadth of the figure be less than 80 mms the formula $c^2\Phi=A$ gives the area as accurately as the nature of the instrument will allow.

(This is obtained by putting $\frac{Ak^2}{2c^2} = 30$ mm² and taking a circle as the type of figure, for a circle is that figure which for the greatest breadth has the greatest area. If R is the radius of the circle $k^2 = \frac{R^2}{2}$ and $A = \pi R^2$

$$\therefore \text{putting } \frac{\pi R^2}{2(250)^2} \cdot \frac{R^2}{2} = 30$$

we get $R=40$ mms. nearly).

(2) If the area does not exceed 142 sq. cms it is not necessary to measure the arc K_1 K_2; the chord may be measured instead. (This is obtained by noting that the arc exceeds the chord by 250 $\left(\phi - 2 \sin \frac{\phi}{2}\right)$mm. If this distance is not to exceed 1/8 mm,, ϕ must not exceed 13° or .227 radian, or the area must not exceed 142 sq. cms.).

PRACTICAL TESTS

In the experimental tests various figures were chosen and their areas found (1) by calculation (if possible) (2) by an Amsler planimeter, (3) by the hatchet planimeter.

The hatchet planimeter had an arm of 24.9 cms. and it carried a steel scale graduated to half mms. and bent to the right curvature.

REGULAR FIGURES

(1) *Circles.* A series of circles of radii nearly 2, 3, 4, 5, 7, 9, and 11 cms. was drawn with a common centre and the areas of the circles found by taking the origin at the centre. The planimeter was used with the arm pointing out to the left. (This is the most convenient direction.) In each case the circle was traced (1) anti-clockwise, (2) clockwise. and the average taken a described above. It is important to hold the P-arm of the planimeter lightly so as not to influence the direction of motion of the hatchet edge. Otherwise just ordinary care was taken.

TABLE I CIRCLES Origin taken at Centre.

RADIUS Cm	Area by calculation Cm²	Area by Amsler's planimeter taken once cm²	Error with Amsler's expressed as percentage	Area by hachet planimeter cm²	Error cm²	Error expressed as percentage	Percentage error calculated by $\frac{k^2}{2c^2}$
2.02	12.82	12.6	−1	12.9	.08	+½	1/6
3.01	28.46	28.3	−½	28.2	−.26	−1	1/3
4.00	50.25	50.6	+7/10	51.0	.75	+1½	2/3
4.98	77.89	78.9	+1/8	79.7	1.8	+2	1
6.98	153.1	154.8	+1	156.9	3.8	+2½	2
9.02	255.6	256.9	+½	265.8	10.2	+4	3
11.00	380.1	382.3	+½	402.2	22.1	+6	5

The agreement of the last two columns is fairly good. If in the case of the larger circles the chord $K_1 K_2$ is measured instead of the arc the error of the hatchet planimeter result is decreased. Thus in the case of the largest circle the chord $K_1 K_2$ multiplied by the arm gives 397 cm².

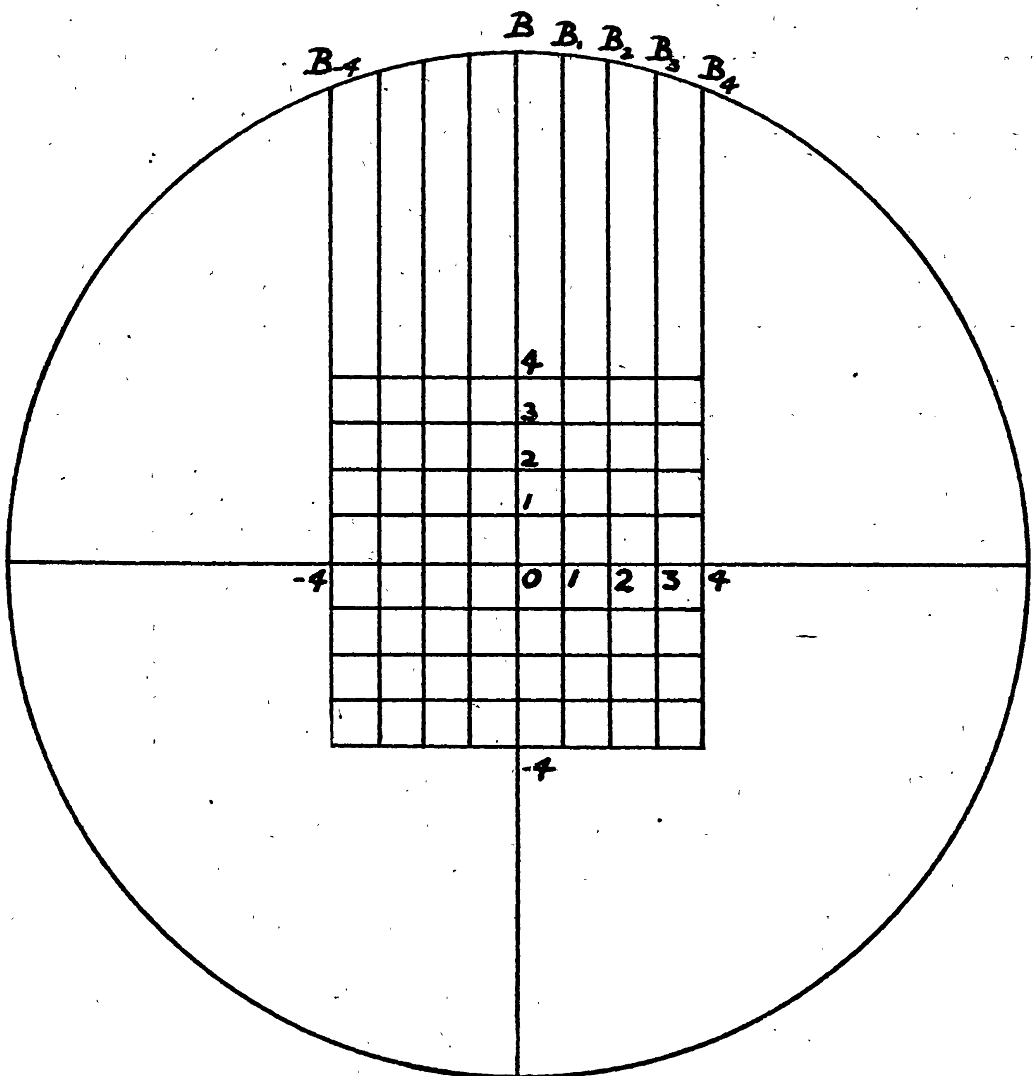

Fig. 10.

An attempt was made to see what error was introduced if the starting point was not at the centre of gravity. The circle of radius 9 cm. was taken and a gridiron pattern drawn as shown in Fig. 10 served to give starting points at centimetre intervals. In each case for the point chosen the vertical line through that point was traced up to the circumference. The arm of the planimeter was directed to the left. In some cases the readings were not duplicated. The results are given in the Table.

TABLE II Circle (Radius 9.02 cm.)

Starting from centre—Area = 265.8 cm^2.

Starting from:

Points on line drawn up from centre along axis of y		Points on line drawn out from centre at 45° to axes of x and y		Points on line drawn out from centre along axis of x	
Point	Area	Point	Area	Point	Area
x, y		x, y		x, y	
0, 1	264	1, 1	254	1, 0	255
0, 2	263	2, 2	243	2, 0	246
0, 3	263	3, 3	234	3, 0	234
0, 4	263	4, 4	226½	4, 0	224
...	...	−4+4	312	....	...
0, 9	284	−4−4	320	9, 0	182
		+4−4	227		

It will be seen by comparison with Table I that the points 1, 1 and 1, 0 give better results than the centre of the circle. So that an error in locating the centroid made in the direction of these points would improve the result. An error made in the direction of the axis of y introduces no additional error over that obtained from the centre until the origin is taken far out.

The reading for the point −2, 0 corresponds to a reading taken at the point 2, 0 with the arm of the planimeter reversed. Hence the average of readings taken at 2, 0 and −2, 0 should eliminate the x term.

The reading at 2, 0 was 246 sq. cm.
The reading at −2, 0 was 286 sq. cm.

Average = 266 sq. cm.

which is still in excess of the true area but in agreement with the result obtained with the origin at the centre ($Ak^2/2c^2$ for an origin 2 cms from the centre = 9 sq. cm.)

Attention may be called to the results taken at the corner points 4, 4; −4,4; −4−4; 4, −4. The blending of the results at 4.4 and −4, 4 should eliminate the $\bar{x}$ term in equation (14), similarly for the results at −4,−4, and 4,−4. The average of 226½ and 312 is 269 and that of 320 and 227 is 273. Both means are high by about six per cent. However in an area of this size it is hardly likely that the estimated centroid would be farther away from the centre than two cms.

(2) *A Rectangle.* A rectangular tinplate template was tried. It had a hole at the centre of gravity, and could be fastened to the drawing board by drawing-pins through two other holes.

All readings with the hatcher planimeter started from the centr of gravity and the tracer point was kept against the edge of the template as it travelled around it; this keeps the point froe wandering from the right path. Two readings, clockwise, gave arcs of 4.03 and 3.92, and two, anti-clockwise, gave arcs of 4.07 and 3.91. The mean of the four is 3.98 giving an area of 99.1 cm^2. A tracing by the Amsler planimeter gave 95.8 cm^2. However the Amsler tracer needle is thinner than the arm of the hatchet planimeter so that the Amsler point traces a slightly smaller area. On removing the template and taking the Amsler around on the scratch made on the paper by the point of the hatchet planimeter the reading went up to 96.0 cm^2. The area of the rectangle traced by the tracer point was 12.5 cm x 7.7. cm = 96.3 cm^2. The error of the hatchet result was therefore 2.8 cms^2 too great. The formula tells us the excess should be $\frac{Ak^2}{2c^2}$. The moment of inertia of the rectangle about the polar axis through the centroid is

$$A\left(\frac{12.5^2+7.7^2}{12}\right)=A\times 17.9$$

$$\therefore \frac{Ak^2}{2c^2}=\frac{96\times 17{\cdot}9}{2\times(24.9)^2}=1.4\ \text{cm.}^2$$

No error was made on account of the $\bar{x}$ term so that here the P term must be responsible for the excess error.

IRREGULAR FIGURES

(1) *An Indicator Diagram.* A metal template was used. Its shape is shown in Fig. 11, where G is the centre of gravity (found by suspension and a plumb line). The template was fastened down to the drawing board. B G B¹ is a straight line. With

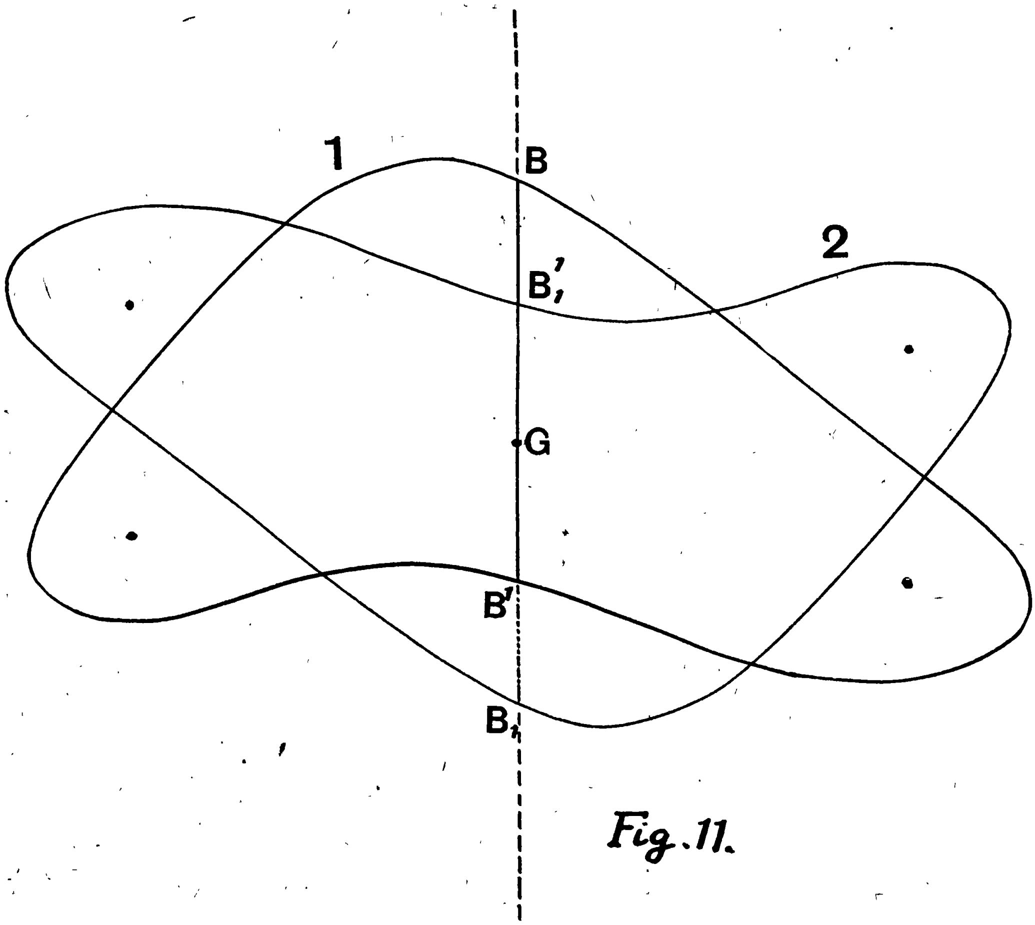

Fig. 11.

the template in position (1) and the planimeter arm extending to the left readings were taken

(*a*) from G to B around clockwise and back to G. Mean arc of two readings = 3.10 cm.

(*b*) from G to B around anticlockwise and back to G. Mean of two readings = 3.37 cm.

(*c*) from G to B¹ around clockwise and back to G. Mean of two readings = 3.17 cm.

(*d*) from G to B^1 around anticlockwise and back to G. Mean of two readings = 3.32 cm.

Keeping G fixed, the template was turned in its own plane through two right angles to position (2). B goes to B_1 and B^1 to B_1'. The readings were again taken. This set corresponds to a repetition with the template in position (1) and the arm pointing out to the right, so that a combination of the two sets should eliminate any error due to inaccuracy in the x component of the centroid.

(*a*) Two readings from G to B_1' and around clockwise gave as mean 3.32 cms.

(*b*) Two readings from G to B_1' and around anticlockwise gave as mean 3.34 cms.

(*c*) Two readings from G to B_1 and around clockwise gave as mean 3.32 cms.

(*d*) Two readings from G to B_1 and around anticlockwise gave as mean 3.52 cms.

The mean of all the readings = 3.31 and 3.31 x 24.9 = 82.4 cm². The Amsler passed around the template twice gave clockwise 81.1 cm², and anticlockwise 81.7 cm². When the template was removed and the Amsler taken around the scratch made by the point P of the hatchet the readings for position (1) were 81.4 cm² and 82.1 cm² and for position (2) 81.3 cm² and 81.8 cm², a mean of 81.6 cm². Comparing the Hatchet result of 82.4 with the Amsler's result of 81.6 we see that the error is less than 1%. The greatest length of the figure was 16.5 cm. and the least 6.6 cms.

(2) *Figure traced from a hand.* As before a metal template was used (Fig. 12) and its centroid G found by suspension. The area was measured twice (*a*) template on paper with thumb to right, mean of two results = 151.3 cm², (*b*) thumb to left, mean of two results = 150.6 cm². Mean 151.0 cm².

Area by Amsler's planimeter = 146.8 cm².

Error = 4 cm².

Calculation of the value of $\frac{Ak^2}{2c^2}$:

To get k^2 a hole O was punched through the tip of the middle finger and the template hung from a needle support and set vibrat-

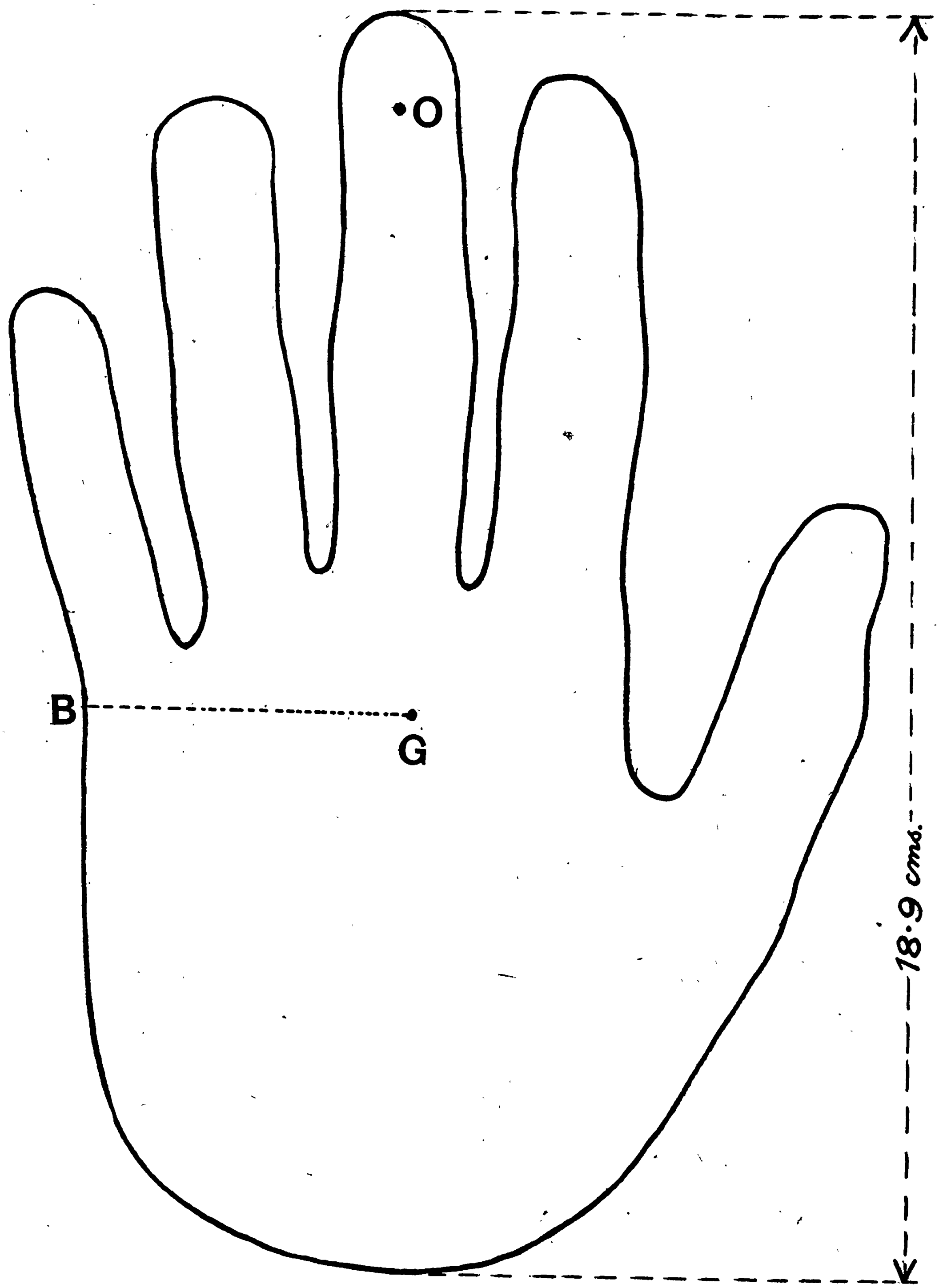

Fig. 12.

ing in its own plane. It made 42 vibrations in 30 seconds. The distance from the point of support to the centroid was 9.15 cm.

$$\text{Therefore } \frac{30}{42} = 2\pi\sqrt{\frac{9.15+\frac{k^2}{9.15}}{981}} \text{ whence } k^2 = 9.15 \times 3.5 \text{ cm}^2$$

$$\therefore \frac{Ak^2}{2c^2} = \frac{147 \times 9.15 \times 3.5}{2 \times (24.9)^2} = 3.7 \text{ cm}^2$$ which considering the slight differences in area which the two instruments trace is a very good confirmation of the 4 cm² error mentioned above.

SUMMARY: It has been shown now that the Hatchet planimeter is an instrument which is quite useful for the estimation of areas.

In the Physics Laboratory at Toronto I have in use about fifty of these planimeters. The students work on areas of maps of countries, areas of indicator diagrams, with subsequent calculation of horse power, areas of ellipses, and areas of various shapes made up of tin-plate templates. The templates are very useful with large classes because the demonstrators can easily keep a record of their areas and they serve as standards. At present we use only one size of planimeter—a 25 cm. arm—but of course if larger areas were wanted a larger instrument would be used. One great merit of the instrument is its cheapness. Without the scale its cost need not exceed 10 cents. With the mounted scale it might be one dollar.