The correction to Z has now become a constant, and may be expressed as a correction to the observed declination of $''\cdot 09$ in a direction to diminish the value of Z. As the corrections to the right ascensions are in the opposite direction, these corrections to Y and Z only slightly affect M, but materially alter D. The original values of M and as corrected for (1) magnitude equation and further corrected for (2) $+''29$ to Y and $-''25$ to Z are

	$6^m\cdot0$–$7^m\cdot8$.	$7^m\cdot9$–$8^m\cdot5$.	$8^m\cdot6$–$8^m\cdot9$.	$9^m\cdot0$–
M	$3''\cdot40$	$2''\cdot19$	$1''\cdot87$	$1''\cdot33$
(1)	3·28	2·23	2·21	1·71
(2)	3·35	2·26	2·24	1·78

On the Radiative Equilibrium of the Stars.

By A. S. Eddington, M.A., M.Sc., F.R.S., Plumian Professor.

1. *Outline of the Investigation.* — The theory of radiative equilibrium of a star's atmosphere was given by K. Schwarzschild in 1906.* He did not apply the theory to the interior of a star; but the necessary extension of the formulæ (taking account of the curvature of the layers of equal temperature) is not difficult. It is found that the resulting distribution of temperature and density in the interior follows a rather simple law.

Taking a star—a "giant" star of low density, so that the laws of a perfect gas are strictly applicable—and calculating from its mass and mean density the numerical values of the temperature, we find that the temperature gradient is so great that there ought to be an outward flow of heat many million times greater than observation indicates. This contradiction is not peculiar to the radiative hypothesis; a high temperature in the interior is necessary in order that the density may have a low mean value notwithstanding the enormous pressure due to the weight of the column of material above.

There is a way out of the difficulty, however, if we are ready to admit that the radiation-pressure due to the outward flow of heat may under calculable conditions of temperature, density, and absorption nearly neutralise the weight of the column, and so reduce the pressure which would otherwise exist in the interior. For the giant stars it is necessary that only a small fraction of the weight should remain uncompensated. (For the dwarf stars, on the other hand, radiation-pressure is practically negligible.)

We thus arrive at the theory that a rarefied gaseous star adjusts itself into a state of equilibrium such that the radiation-pressure very approximately balances gravity at interior points. This condition leads to a relation between mass and density on the one side and effective temperature on the other side, which seems to correspond roughly with observation. The laws arrived at differ considerably from those of Lane and Ritter.

* *Göttingen Nachrichten*, 1906, p. 41.

The principal results are given in §§ 7–10. The theory enables us to estimate the average densities of the giant stars of different spectral types; it shows that the average luminosity will be roughly the same for the different types, and determines this luminosity as compared with the Sun; it determines the maximum effective temperature which a star can attain; and it indicates the extent to which the masses of individual stars are likely to deviate from the mean mass. It is scarcely necessary to say that the conclusions here given are tentative, being based on analysis which is only concerned with obtaining a probable approximation; but there seems to be a satisfactory accordance with observation, so far as is known. The present results also remove an objection which might be urged against the theories of Lane and Ritter, viz. that they require the heat-energy retained in the star to be much greater than that generated by contraction.

The outermost layers of the star are outside the scope of this investigation, and the formulæ here given do not apply to them. In speaking of conditions at the boundary of the star, I refer to a depth negligible compared with the radius, but deep from the point of view of the spectroscopist. Hence the theory has no bearing on the interpretation of spectroscopic results. Frequent reference is made to the effective temperature, because it affords a measure of the total outflow of radiation; we can use this measure without discussing the conditions of the layer which actually possesses the effective temperature.

It is clear that we cannot arrive at much certainty with regard to the conditions in a star's interior, except in so far as the treatment can be based on the most general laws of nature. There are some physical laws so fundamental that we need not hesitate to apply them even to the most extreme conditions; for instance, the density of radiation varies as the fourth power of the temperature, the emissive and absorbing powers of a substance are equal, the pressure of a gas of given density varies as its temperature, the radiation-pressure is determined by the conservation of momentum,—these provide a solid foundation for discussion. The weak link in the present investigation is that I have assumed without much justification that a certain product $k\epsilon$ is constant throughout a star. I have given some evidence that if it is variable the general character of the results would not be greatly altered; and, as a step towards the elucidation of the problem of stellar temperatures, I plead to be allowed provisionally one rather artificial assumption.

2. *Radiative Equilibrium in the case of Spherical Symmetry.*—Schwarzschild's treatment deals with the case when the surfaces of equal temperature are parallel planes. We have to consider the case when they are concentric spheres. The fundamental principle is that the transfer of energy takes place by radiation; convection and conduction are considered negligible.

Let ξ be the distance of a point P from the centre O of a star. At P, let the intensity of radiation travelling outwards in a direction making an angle θ with OP be expressed as a series of zonal harmonics,

$$(A + B\cos\theta + CP_2(\cos\theta) + DP_3(\cos\theta) + \ldots)dSd\omega,$$

where dS is the cross-section of the stream of radiation, and $d\omega$ is the solid angle containing directions near to θ.

Consider a small cylinder of matter of cross-section dS, and with axis ds in the direction θ. The fraction of the radiation absorbed in this cylinder will be proportional to the density ρ, and may be set equal to

$$k\rho ds,$$

where k is a coefficient of absorption and represents the absorption by a cylinder of unit *mass* and unit cross-section. Assuming that each molecule absorbs independently of the others, it will make no difference whether the cylinder is long and of low density, or short and of high density.

Hence the loss of the beam of radiation in traversing the cylinder is

$$k\rho ds(A + B\cos\theta + CP_2(\cos\theta) + \ldots)dSd\omega \quad . \quad . \quad (1)$$

The matter in the cylinder emits energy equally in all directions at a rate proportional to (1) the mass contained, (2) the fourth power of the temperature T, and (3) the specific emissive power of the substance. The last quantity may be set equal to k, since the absorbing and emissive powers of a substance are necessarily equal. Hence the energy emitted from the cylinder in directions included in $d\omega$ is

$$\mu k T^4 \rho ds dS d\omega \quad . \quad . \quad . \quad . \quad (2)$$

where μ is an absolute constant of nature, connecting the units of energy and temperature.

The loss (1) and gain (2) together make up the change of intensity of the radiation in traversing a length ds. Hence

$$\frac{d}{ds}(A + B\cos\theta + CP_2(\cos\theta) + \ldots)dsdSd\omega$$
$$= -k\rho(A + B\cos\theta + CP_2(\cos\theta) + \ldots)dsdSd\omega + k\rho\mu T^4 dsdSd\omega \quad (3)$$

Now $$\frac{d}{ds} = \frac{d\xi}{ds}\frac{d}{d\xi} + \frac{d\theta}{ds}\frac{d}{d\theta}$$
$$= \cos\theta\frac{d}{d\xi} - \frac{\sin\theta}{\xi}\frac{d}{d\theta} \text{ by geometry.}$$

Hence, since A, B, C are functions of ξ only, (3) becomes

$$\cos\theta\frac{dA}{d\xi} + \cos^2\theta\frac{dB}{d\xi} + \cos\theta \,.\, P_2\frac{dC}{d\xi} + \frac{\sin^2\theta}{\xi}B - \frac{\sin\theta}{\xi}\frac{dP_2}{d\theta}C + \ldots$$
$$= -k\rho(A + B\cos\theta + CP_2 + \ldots) + k\rho\mu T^4 \quad (4)$$

Let us choose a unit of length comparable with the radius of the star. Then k is a very large quantity, since radiation must be strongly absorbed in a length small compared with the radius. It is then found that A, B, C rapidly diminish, so that B is of order k^{-1} and C of order k^{-2} compared with A. We shall therefore neglect C on the left side of (4), retaining it on the right, where it is multiplied by the large factor k.

Arrange the left side in zonal harmonics by using

$$\cos^2\theta = \tfrac{1}{3} + \tfrac{2}{3}P_2(\cos\theta),$$
$$\sin^2\theta = \tfrac{2}{3} - \tfrac{2}{3}P_2(\cos\theta),$$

and equate coefficients.

The constant terms give

$$\tfrac{1}{3}\frac{dB}{d\xi} + \tfrac{2}{3}\frac{B}{\xi} = k\rho(-A + \mu T^4) \quad . \quad . \quad . \quad (5)$$

The coefficients of $\cos\theta$ give

$$\frac{dA}{d\xi} = -k\rho B \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The coefficients of $P_2(\cos\theta)$ give

$$\tfrac{2}{3}\frac{dB}{d\xi} - \tfrac{2}{3}\frac{B}{\xi} = -k\rho C,$$

which verifies the statement made above that C is small compared with B. We shall therefore neglect C henceforth.

Equation (5) can be written

$$\frac{1}{\xi^2}\,\frac{d}{d\xi}(B\xi^2) = 3k\rho(-A + \mu T^4) \quad . \quad . \quad . \quad (7)$$

To interpret the quantity B, we remark that the total radiation flowing outwards across unit surface transverse to the radius is obtained by integrating $(A + B\cos\theta)\cos\theta d\omega$ over the corresponding hemisphere. The factor $\cos\theta$ is required, since the cross-section of a beam of radiation flowing obliquely through a surface dS is equal to $dS\cos\theta$. The result is

$$\pi A + \tfrac{2}{3}\pi B \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Similarly the amount flowing outwards is

$$\pi A - \tfrac{2}{3}\pi B \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

Thus the net flow outwards is $\tfrac{4}{3}\pi B$ per unit area, or $\tfrac{16}{3}\pi^2 B\xi^2$ across the sphere of radius ξ.

In a strictly steady state the total energy between two boundaries must be constant, and therefore the net outward flow across all boundaries must be the same. Thus $B\xi^2$ is constant, and by (7)

$$A = \mu T^4.$$

In an actual star the stream of energy flowing outwards is supplied by slow changes occurring within the star. The simplest theory results if we suppose that the energy is produced by radioactive processes. Let the amount thus liberated per unit mass be $4\pi\epsilon$. Then in the steady state the outward flow of heat across the outer boundary of a spherical shell will exceed that across the inner boundary by the amount of heat generated in the shell. That is,

$$d(\tfrac{16}{3}\pi^2 B\xi^2) = 4\pi\epsilon \,.\, 4\pi\rho\xi^2 d\xi.$$

Hence

$$\frac{1}{\xi^2}\frac{d}{d\xi}(B\xi^2) = 3\rho\epsilon \qquad . \quad . \quad . \quad . \quad (9)$$

Imagine the successive spherical layers to be expanded or contracted until the whole star has a uniform density τ. Let x be the radius in the uniform star, which corresponds to ξ in the original star. Then

$$\rho\xi^2 d\xi = \tau x^2 dx \qquad . \quad . \quad . \quad . \quad (10)$$

Hence from (6) and (9)

$$\frac{1}{\tau}\frac{dA}{dx} = -kB\frac{x^2}{\xi^2} \qquad . \quad . \quad . \quad . \quad (11)$$

$$\frac{1}{\tau}\frac{d}{dx}(B\xi^2) = 3\epsilon x^2 \qquad . \quad . \quad . \quad . \quad (12)$$

Integrating (12),

$$B = \tau\epsilon\frac{x^3}{\xi^2} \qquad . \quad . \quad . \quad . \quad (13)$$

and

$$\frac{1}{\tau}\frac{dA}{dx} = -k\tau\epsilon\frac{x^5}{\xi^4} \qquad . \quad . \quad . \quad . \quad (14)$$

If the outflowing energy is produced by contraction instead of by radioactivity, it is not so easy to give a precise statement, because, strictly speaking, the conditions are changing with the time. We may write down equation (13) as a definition of ϵ; and it will be seen that $4\pi\epsilon$ is then the net flow of radiation across the spherical surface divided by the mass within the surface, *i.e.* the average energy per unit mass generated by contraction. I shall generally take ϵ to be constant, representing roughly a state of affairs such that the energy of stellar radiation comes from processes going on in all parts of the star, and not from a singularity at the centre. It must be noticed that, except near the surface, ϵ is extremely small compared with μT^4, so that in any case $A = \mu T^4$ approximately.

Let g be the value of gravity at ξ

and G the constant of gravitation,

then $$g = \tfrac{4}{3}\pi G\tau x^3/\xi^2,$$

since the numerator gives the total mass within a sphere of radius ξ.

The pressure equation is

$$\frac{1}{\rho}\frac{dp}{d\xi} = -g,$$

whence by (10)

$$\begin{aligned}\frac{1}{\tau}\frac{dp}{dx} &= -g\frac{x^2}{\xi^2}\\ &= -\tfrac{4}{3}\pi G\tau\frac{x^5}{\xi^4}\\ &= \tfrac{4}{3}\frac{\pi G}{k\epsilon}\frac{1}{\tau}\frac{dA}{dx} \quad \text{by (14)} \quad . \quad . \quad (15)\end{aligned}$$

If k and ϵ are constants, integrating and setting $A = \mu T^4$,

$$p = \frac{4\pi\mu G}{3k\epsilon}T^4 \quad . \quad . \quad . \quad . \quad (16)$$

Strictly speaking, there is a small constant of integration, but it can be neglected in comparison with the large values of p in the interior of the star (see § 5 (*b*)).

Up to this point we have not used the gas-equation. For a *gaseous* star obeying Boyle's Law, $p = RT\rho$. Hence from (16), $\rho \propto T^3$ and $p \propto \rho^{\frac{4}{3}}$.

*The distribution of density, temperature, and pressure for radiative equilibrium is identical with that occurring in the adiabatic equilibrium of a mass of gas for which the ratio of the specific heats is $\frac{4}{3}$.**

A gas for which $\gamma = \frac{4}{3}$ can be at the same time in radiative and adiabatic equilibrium; if γ alters, adiabatic equilibrium alters but radiative equilibrium does not.

3. *Intensity of the Outward Radiation.*—The net outward radiation has been found to be $\frac{16}{3}\pi^2 B\xi^2$

$$= \tfrac{16}{3}\pi^2\tau\epsilon x^3 \quad \text{by (13)}.$$

If the effective temperature of the star is T_1, the outflow is equivalent to that due to isotropic radiation of intensity $A_1 = \mu T_1^4$. By (8) this outflow is πA_1 per unit area. Hence

$$\pi\mu T_1^4 \cdot 4\pi\xi^2 = \tfrac{16}{3}\pi^2\tau\epsilon x^3 \quad . \quad . \quad . \quad (17)$$

where x and ξ have their surface values.

Eliminate ϵ between (16) and (17), we obtain

$$\begin{aligned}kT_1^4 &= \tfrac{4}{3}\pi G\tau\frac{x^3}{\xi^2}\frac{T^4}{p}\\ &= \tfrac{4}{3}g_1\frac{T^4}{p} \quad . \quad . \quad . \quad . \quad . \quad (18)\end{aligned}$$

where g_1 is the value of gravity at the surface.

* It may be well to restate the restriction, viz. that $k\epsilon$ must be constant. This point is discussed in § 5.

This equation gives the relation between the constant of absorption and the effective temperature. T and p are any corresponding values of the temperature and pressure. It is usually most convenient to use the central values, T_0 and p_0, so that

$$k = \tfrac{4}{3}\frac{g_1}{p_0}\left(\frac{T_0}{T_1}\right)^4 \quad . \quad . \quad . \quad . \quad (19)$$

4. *Numerical Example.*—Before discussing possible modifications of the somewhat ideal conditions assumed, we shall give an example. The necessary tables and formulæ are given in Emden's *Gaskugeln* (pp. 85, 69). Emden actually works out this case (regarded as a case of adiabatic equilibrium) for the Sun, but we shall take a star in a more perfect gaseous condition. For the method of calculation reference may be made to Emden's examples (p. 96).*

Let the radius of the star be $r = 7 \times 10^{11}$ cm.
Let the mean density be $\rho_m = 0{\cdot}002$ gm. cm.$^{-3}$
Then the mass is $M = 2{\cdot}87 \times 10^{33}$ gm. (about $1{\cdot}5 \times$ Sun).

The central density in this form of equilibrium is *always* 54·2515 times the mean density. In this case

Central density $\rho_0 = 0{\cdot}1085$,

so that no serious deviation from Boyle's Law will occur.

To calculate the temperature and pressure, the only other quantity needed is the molecular weight. I have taken this to be 54 as a likely average (*e.g.* monatomic iron vapour). The corresponding value of R is $1{\cdot}536 \times 10^6$. I find at the centre

$$T_0 = 1{\cdot}52 \times 10^8 \text{ degrees Centigrade}$$
$$p_0 = 2{\cdot}535 \times 10^{13} \text{ dynes cm.}^{-2}$$

At the surface

$$g_1 = 390{\cdot}5 \text{ cm. sec.}^{-2}$$

Temperature gradient $= 6^\circ{\cdot}35$ per kilometre.

(The temperature gradient is nearly uniform for a great distance inwards; it increases slowly to a maximum of $6\frac{1}{2}$ times its surface value at a depth equal to $\frac{3}{4}$th of the radius.)

The following formulæ show how the conditions at the centre are altered by varying the mean density, mass, molecular weight (R^{-1}), or constant of gravitation:

$$\left.\begin{aligned} \rho_0 &\propto \rho_m \\ T_0 &\propto \rho_m^{\frac{1}{3}} M^{\frac{2}{3}} G R^{-1} \\ p_0 &\propto \rho_m^{\frac{4}{3}} M^{\frac{2}{3}} G \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (20)$$

To calculate k we must assume a value of the effective temperature. A star having the assumed mean density would probably be

* There is a misprint in the formula for Θ^3 on p. 97. The last bracket in the denominator should be squared.

of type F – G, and have an effective temperature of, say, 6500° absolute.

Substituting in (19) we find

$$k = 6 \cdot 2 \times 10^6.$$

This value is quite absurd, for it indicates that the radiation would be almost wholly absorbed in a path of 10^{-6} cm. at unit density, or 10^{-3} cm. at atmospheric density. Such a high opacity can scarcely be reconciled with the known properties of matter.

Inverting the argument and taking a more reasonable value of k, say 40, the effective temperature would have to be 130,000° C., which is, of course, out of the question.

A further indication that something is wrong is given by a calculation of the total amount of radiant energy contained in the star. The radiant energy per unit volume is $a\mathrm{T}^4$ ergs, where $a = 7 \cdot 1 \times 10^{-15}$ and T is reckoned in degrees. On integrating throughout the volume, we find the total imprisoned radiant energy is

$$\mathrm{H} = \tfrac{1}{2}\frac{a\mathrm{T}_0{}^4\mathrm{GM}^2}{p_0 r} = 5 \cdot 85 \times 10^{52} \text{ ergs},$$

whereas the whole energy so far generated by contraction is

$$\Omega = \tfrac{3}{2}\frac{\mathrm{GM}^2}{r} = 1 \cdot 18 \times 10^{48} \text{ ergs}.$$

One naturally asks, Where has all the radiant energy come from?

The radiant energy probably possesses electromagnetic mass; but this does not give rise to any difficulty. The mass is found by dividing H by the square of the velocity of light

$$\mathrm{H}/c^2 = 6 \cdot 5 \times 10^{31} \text{ grams},$$

which is less than $\frac{1}{40}$th of the whole mass of the star.

5. *Discussion of the Assumptions.*—The last paragraph shows that the theory has led to improbable results. We therefore examine the assumptions made.

(*a*) *The Radiative Hypotheses.*—It can scarcely be doubted that at the high temperatures which (on any theory) are held to prevail in the star, radiation is far more effective than any other cause in transferring heat. Its importance rapidly increases as the temperature rises, since it depends on the fourth power of the temperature. Convection currents would only arise if the transfer of heat by radiation were not sufficiently rapid to prevent the outer parts from cooling and sinking; our difficulty is that the transfer by radiation seems to be much too rapid.

(*b*) *The Boundary Condition.*—It will be found that the terms A, B, C become of the same order of magnitude near the boundary and our equations break down. The stellar atmosphere terminates in a way different from what we have supposed, and the correct equations are those given by Schwarzschild. The boundary temperature is, in fact, $T_1 / \sqrt[4]{2}$ instead of zero, and the pressure-temperature relation is more nearly

$$p \propto T^4(1 - T_1^4/2T^4);$$

that is to say, a small constant of integration is required in (16).

Until we are close to the boundary T_1^4 is very small compared with T^4, and it is easy to verify that as regards the interior of the star the correction is insignificant. Our calculation of k in § 3 is not affected by any failure of the equations at the boundary—we have not used T_1 as the temperature of a definite layer, but as measuring the total outflow of heat. The actual outflow must evidently be continuous with the outflow across a sphere contained a little within the boundary where our equations apply.

(*c*) *Variation of* k.—It has been assumed that k is constant, but it may vary with the temperature—either of the absorbing matter or of the radiation to be absorbed.

We can make calculations assuming that k varies as any power of the temperature, say T^δ. If $k = \kappa T^\delta$, then instead of (15) we have

$$\frac{dp}{dx} = \frac{4\pi G\mu}{3\kappa\epsilon} \frac{1}{T^\delta} \frac{dT^4}{dx} \quad . \quad . \quad . \quad . \quad (21)$$

whence $p \propto T^{4-\delta}$.

The equilibrium is still a form of adiabatic equilibrium, but with a different value of the ratio of the specific heats γ. Reference to the examples worked out by Emden shows that the central temperature is not very seriously altered by changing γ. We can show this best by proceeding at once to an extreme case. Let k vary as T^3, then $p \propto T$ and the star is a sphere of uniform density. I find in this case for the star discussed in § 4,

$$T_0 = 8{\cdot}9 \times 10^7 \text{ degrees Centigrade}$$
$$p_0 = 2{\cdot}7 \times 10^{11} \text{ dynes cm.}^{-2}$$

Temperature gradient near surface $= 25^\circ{\cdot}4$ per kilometre.

It is evident from the slight alteration of the central temperature that the outflow of radiation and store of radiant energy will again be excessively great. In fact, for an effective temperature of 6500° we find

$$k = 2{\cdot}7 \times 10^8 \text{ at the centre.}$$

(*d*) *Variation of* ϵ.—Similar small changes are produced if the radiated heat is not contributed by all parts equally, but comes more from the centre than the exterior layers, or vice versa. There are two cases in which we can give exact results:—

If $\epsilon \propto (1 - \xi^2/r^2)^3$, the density is constant.

If $\epsilon \propto (c^2 + \xi^2)$, where c is a certain constant, the case corresponds to adiabatic equilibrium with $\gamma = 1{\cdot}2$.*

* ϵ is defined by equation (13).

The two cases correspond to the heat being generated mainly in the interior and mainly in the exterior parts respectively, and give some idea of the range that is possible.

(*e*) *Effect of Scattering.*—In the introduction to his theory of radiative equilibrium Schwarzschild states that he has neglected *scattering*; but it appears that scattering can be included—at least in the simple theory given by him. He considers radiation of intensity A propagated along the positive direction of ξ, and radiation of intensity B along the negative direction. The equations, taking account of absorption and emission, are then

$$\left.\begin{aligned} \frac{dA}{d\xi} &= -k\rho A + k\rho\mu T^4 \\ -\frac{dB}{d\xi} &= -k\rho B + k\rho\mu T^4 \end{aligned}\right\} \quad . \quad . \quad . \quad (22)$$

If there is a coefficient of scattering s, so that, considering the radiation A, in a length $d\xi$, $\frac{1}{2}s\rho d\xi A$ is scattered forwards and the same amount backwards, the equations become

$$\left.\begin{aligned} \frac{dA}{d\xi} &= -k\rho A + k\rho\mu T^4 - \tfrac{1}{2}s\rho A + \tfrac{1}{2}s\rho B \\ -\frac{dB}{d\xi} &= -k\rho B + k\rho\mu T^4 - \tfrac{1}{2}s\rho B + \tfrac{1}{2}s\rho A \end{aligned}\right\} \quad . \quad . \quad (22a)$$

whence

$$\begin{aligned} \frac{d}{d\xi}(A - B) &= k\rho(2\mu T^4 - A - B) \\ &= 0 \text{ for the steady state} \\ \frac{d}{d\xi}(A + B) &= -(k+s)\rho(A - B). \end{aligned}$$

These differ only from the corresponding equations derived from (22) by the replacement of k by $(k+s)$. Thus scattering should be included in the absorption coefficient, and the theory then applies without modification.

6. *Radiation-Pressure.*—It appears then that no modification of the assumptions mentioned in § 5 could affect the main result, viz. that the interior temperature is so high as to cause much too great an outflow of radiation, unless we imagine the material to be almost perfectly opaque. The high temperature in the interior is inevitable if the gas—necessarily of moderate density in a giant star—is to support the weight of the enormous column of material above.

This points to a way out of the discrepancy—the weight of the column may be partly supported by radiation-pressure. In introducing radiation-pressure at this stage we do so not as a *hypothesis* to explain the discrepancy, but because in the conditions we have found radiation-pressure would be extremely powerful. Whether radiation-pressure is important or not depends on the value of k. If we had arrived at a low value of k, our neglect of radiation-pressure would have been justified. With the high value found radiation-pressure would far outbalance gravity, and our neglect of it is clearly illegitimate.

As there seems to be a rather widespread impression that gases are not subject to radiation-pressure, it may be advisable to state the theory briefly. The pressure is simply a consequence of absorption or scattering. A beam of radiation carries a certain forward-momentum proportional to its intensity; after passing through a sheet of absorbing medium a weaker beam emerges carrying proportionately less momentum; the difference of incident and emergent momentum is retained by the medium and constitutes the pressure. The medium, in fact, absorbs the momentum of the beam in the same proportion as it absorbs the energy. The calculations of radiation-pressure on small solid particles are simply calculations of absorption and scattering by these particles; it is not possible to apply such methods to atoms and molecules, which absorb by some internal mechanism. But the relation between absorption and pressure is a perfectly general one, depending only on the conservation of momentum.

Consider a small disc of thickness $d\xi$, and let radiation carrying momentum h fall on it, travelling at an angle θ to the normal. The length of path is $d\xi \sec\theta$. Hence the momentum absorbed is $k\rho h d\xi \sec\theta$. Resolving along the normal, the normal outward momentum absorbed is $k\rho h d\xi$, which is independent of θ.

In the present problem we have energy $\pi\mathrm{A} + \frac{2}{3}\pi\mathrm{B}$ flowing outwards and $\pi\mathrm{A} - \frac{2}{3}\pi\mathrm{B}$ inwards across unit surface; hence the net outward momentum absorbed is $c \,.\, \frac{4}{3}\pi\mathrm{B}k\rho d\xi$, where c is the factor relating the momentum and energy of a beam. Now the pressure on a black body is numerically equal to the density of the energy (assumed isotropic). The density of the energy at temperature T is $a\mathrm{T}^4$, where a is the universal constant $7{\cdot}06 \times 10^{-15}$. The outward flow of isotropic energy across unit surface is $\pi\mathrm{A} = \pi\mu\mathrm{T}^4$, and the outward flow of momentum is $c \,.\, \pi\mu\mathrm{T}^4$. Hence

$$a\mathrm{T}^4 = c\pi\mu\mathrm{T}^4,$$

so that

$$c = a/\pi\mu.$$

It follows that the force of radiation-pressure on an element $d\xi$ is

$$\frac{4}{3}\frac{a}{\mu}\mathrm{B}k\rho d\xi \qquad . \quad . \quad . \quad . \quad (23)$$

$$= -\frac{4}{3}\frac{a}{\mu}d\mathrm{A} \quad \text{by (6)}$$

$$= -\frac{4}{3}a d(\mathrm{T}^4) \,. \qquad . \quad . \quad . \quad . \quad (24)$$

Equation (24) is quite general. Assuming now the constancy of $k\epsilon$ so that $\mathrm{T}^4 \propto p$, the radiation-force is

$$-\frac{4}{3}a\frac{\mathrm{T_0}^4}{p_0}dp.$$

For the star discussed the values of T_0 and p_0 in § 4 give

$$\tfrac{4}{3}a\frac{T_0^4}{p_0} = 2\cdot0 \times 10^5,$$

so that radiation-pressure would be 200,000 times stronger than gravitation. Of course, the result does not mean that the existing radiation-pressure is so strong as this ; it shows that our conclusions contradict our premises, viz. that radiation-pressure was negligible.

We must now form the modified equations in which radiation-pressure is included. The weight of an element $g\rho d\xi$ will be partly counterbalanced by the radiation-force (23)

$$\tfrac{4}{3}\frac{a}{\mu}Bk\rho d\xi.$$

Hence in the pressure-equation we replace g by

$$g - \tfrac{4}{3}\frac{a}{\mu}Bk.$$

Using (13), this gives

$$g - \tfrac{4}{3}\tau\frac{x^3}{\xi^2}\cdot\frac{ak\epsilon}{\mu}$$

$$= g\left(1 - \frac{ak\epsilon}{\pi G\mu}\right) = \beta g, \quad \text{say} \quad . \quad . \quad . \quad (25)$$

We shall suppose as before that $k\epsilon$ is constant, and accordingly β is constant. Evidently the effect of radiation-pressure is exactly equivalent to an alteration of the constant of gravitation to βG. We see by (20) that the density distribution will be unaltered, but the pressures and temperatures previously calculated must be multiplied by β. Instead of (16) we shall have

$$p = \frac{4\pi\mu\beta G}{3k\epsilon}T^4 \quad . \quad . \quad . \quad . \quad (26)$$

To determine β we have from (25)

$$1 - \beta = \frac{ak\epsilon}{\pi G\mu}$$

$$= \tfrac{4}{3}a\beta\frac{T^4}{p} \quad \text{by (26),} \quad . \quad . \quad . \quad (26a)$$

so that

$$1 - \beta = \tfrac{4}{3}a\beta^4\left(\frac{T_0^4}{p_0}\right)_c \quad . \quad . \quad . \quad . \quad (27)$$

where $(T_0^4/p_0)_c$ refers to the values calculated in § 4 without taking account of radiation-pressure.

Also k is given by changing g_1 to βg_1 in (19)

$$\tfrac{3}{4}k = \frac{\beta g_1}{T_1^4}\frac{T^4}{p},$$

or

$$\tfrac{3}{4}k = \beta^4\frac{g_1}{T_1^4}\left(\frac{T_0^4}{p_0}\right)_c \quad . \quad . \quad . \quad . \quad (28)$$

Moreover, from (27) and (28),

$$1 - \beta = ka\mathrm{T}_1{}^4/g_1 \quad . \quad . \quad . \quad . \quad (29)$$

7. *Results for a Giant Star.*—Substituting in (27) the values calculated in § 4, we have

$$1 - \beta = [5{\cdot}2980]\beta^4.$$

I have given the logarithm of the coefficient. Solving this equation

$$\beta = 0{\cdot}0468,$$

so that less than one-twentieth of the weight is left uncompensated. It will be seen from (20) that $(\mathrm{T}_0{}^4/p_0)_c$ does not depend on the density; thus β is the same for all giant stars of the same mass.

Using this value of β in (29) we obtain

$$k = 29{\cdot}5.$$

This value should presumably be the same for all stars. It indicates that the radiation would be reduced in the ratio $1/e$ after passing through a column containing $\frac{1}{30}$ gm. per sq. cm. section (about 25 cm. of air). This seems possible, since it must be remembered that at the high temperatures concerned the bulk of the radiation is of very short wave-length.

The new values of the central temperature and pressure are

$$\mathrm{T}_0 = 7{\cdot}12 \times 10^6 \text{ degrees}$$
$$p_0 = 1{\cdot}19 \times 10^{12} \text{ dynes cm.}^{-2}$$

Surface temperature gradient $= 0^\circ{\cdot}30$ per kilometre.

The total radiant energy imprisoned is

$$\mathrm{H} = 2{\cdot}81 \times 10^{47} \text{ ergs}$$
$$= 0{\cdot}238\ \Omega,$$

so that it is no longer an excessive amount.

8. *The Relation between Density and Effective Temperature.*—It appears that a gaseous star must settle itself into a state in which radiation-pressure almost balances gravitation. If the outward flow of radiation is too strong, so that gravitation is overbalanced, the star will blow out, the temperature will fall, and radiation will be reduced. On the other hand, if radiation-pressure falls much below gravitation the star will contract and the outflow of radiation will increase. We can make use of the approximate relation radiation-pressure = gravitation (or the more rigorous equation (29)) to determine the variation of effective temperature with mean density and mass.

Since g_1 (gravity at the surface) is proportional to $\mathrm{M}^{\frac{1}{3}}\rho^{\frac{2}{3}}$, we have from (29)

$$\mathrm{T}_1{}^4 \propto \mathrm{M}^{\frac{1}{3}}\rho^{\frac{2}{3}}(1 - \beta),$$

whence, approximately

$$\mathrm{T}_1 \propto \mathrm{M}^{\frac{1}{12}}\rho^{\frac{1}{6}} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The relation is rigorous as regards ρ; and as regards M the correction due to the dependence of β on M is usually insignificant. For example, if M is increased eightfold, formula (30) gives an increase of T_1 in the ratio 1·189, whereas the exact ratio is 1·198.

The range of mass in stars is believed to be small, and will affect T_1 only slightly. The following table shows how T_1 varies with the density for a star of mass $1·5 \times$ Sun.

Effective Temperature.	Mean Density (Water = 1).
13 000°	0·128
10 000	·026
8 000	·007
6 000	·0012
4 500	·00022
3 000	·00002

This seems to correspond fairly well with what we know of the densities and temperatures of the giant stars of different types.*

At 10 000° the central density should be $54·25 \times ·026 = 1·4$, so that deviations from Boyle's Law will occur near the centre. We may, however, anticipate the results of § 10, and state that it is not until temperatures above 13 000° are reached that the deviations from the gas-law begin to influence the effective temperature.

The total radiation varies as $r^2 T_1^4$, which by (29) varies as $g_1 r^2(1-\beta)$, that is, as $M(1-\beta)$, and is independent of the density. We thus obtain the result that

The bolometric magnitude of a gaseous star is independent of its stage of evolution, and depends only on its mass.

This is in accordance with H. N. Russell's conclusion that the absolute magnitudes of giant stars of different types are nearly the same. Differences of one or two magnitudes may occur, owing partly to differences of mass and partly to differences of the ratio of the luminous to the total radiation.

From the table the density of a giant star having the same effective temperature as the Sun (6000°) is ·0012. The Sun's density is 1·38. Hence by comparing their superficies I find that a giant star is $5^{m}·4$ brighter than the Sun, which agrees well with observation. (According to Russell, the average magnitude of a giant star at 10 parsecs distance is probably about zero; that of the Sun is usually given as $5^{m}·5$, but some recent determinations make it about $0^{m}·5$ brighter.)

* Russell has given the following estimates of the average densities of giant stars of different types in terms of the Sun's density (*Nature*, vol. xciii. pp. 282–3):—Type A, 1/10; G, 1/350; K, 1/2800; M, 1/25000. Assuming that type G has the same effective temperature as the Sun (6000°), the law $T_1 \propto \sqrt[6]{\rho}$ gives the following effective temperatures:—Type A, 10800°; G, 6000°; K, 4250°; M, 2950°. This agrees almost exactly with the temperatures usually assigned.

Comparison with the Lane-Ritter Theory.—According to Ritter * the effective temperature of a star varies as the square-root of the mass. Our result is the twelfth root, which seems to accord better with the comparatively small range of surface temperatures observed in stars of similar density. Ritter further states that the temperature is nearly independent of the density, whereas according to the present view density is the main determining factor.

Had Ritter taken $\gamma=\frac{4}{3}$, his formula would have led to $T_1 \propto \rho^{\frac{1}{6}}$, agreeing in this respect with (30).

Lane's law, strictly speaking, is not concerned with the effective temperature, but with the temperature at any definite point in the interior; this is proportional to the cube root of the density on Lane's theory as on the present theory. It has often been recognised that the effective temperature will follow a different law; but I do not think the sixth root has been suggested before.

Range of Stellar Masses.—We have found that the total radiation of a giant star varies as $M(1-\beta)$; or, since β is small and varies only slightly, the radiation is practically proportional to the mass. On this view we must attribute the differences in luminosity of individual giant stars of the same spectral type to differences in the masses; and if we knew the range of variation of luminosity we should be able to deduce the range of variation of the masses.

H. N. Russell has given provisionally the necessary data for giant stars of type M.† He has found that these are distributed about a mean absolute magnitude approximately according to the law of errors, the "probable error" being $\pm 0^{m}\!\cdot\!6$. By the law of errors more than four-fifths of the stars will be included within a total range of $2^{m}\!\cdot\!4$ (*i.e.* twice the probable error on either side). The corresponding range of luminosity is 1 : 9; and the range of the masses must be the same.

We conclude that the masses of the stars are dispersed about a median mass M, so that four-fifths of the whole number are between the limits $\frac{1}{3}$M and 3M. This result applies primarily to the giant M stars; but Russell's investigation indicates that the range is very similar in the other types. This remarkably small range of mass is in general agreement with our knowledge derived from observation of binary stars.

9. *Energy of a Star.*—Former investigators have found a difficulty that the heat energy of a star amounts to a large proportion of the energy acquired by contraction, leaving only a small fraction to account for the radiation during past history. Thus Perry showed that if $\gamma<\frac{4}{3}$ the heat (molecular) energy alone is greater than could be acquired by contraction; he neglected the imprisoned radiant energy, which would have made the deficit very much worse. On the present theory this difficulty is avoided, and only about a quarter of the energy of contraction is retained within the star.

* *Astrophysical Journal*, vol. viii. p. 307.
† *Observatory*, vol. xxxvii. p. 170; *Nature*, vol. xciii. p. 255.

Let c_v be the specific heat at constant volume,
γ the ratio of specific heats,
$d\mathrm{V}$ an element of volume,
K the molecular energy of the star,
H the imprisoned radiant energy,
Ω the energy acquired by contraction from infinite diffusion.

Then
$$\mathrm{K} = \int c_v \rho \mathrm{T} d\mathrm{V} = \frac{c_v}{\mathrm{R}} \int p d\mathrm{V} = \frac{1}{\gamma - 1} \int p d\mathrm{V},$$

since
$$c_v + \mathrm{R} = \gamma c_v.$$

Also
$$\mathrm{H} = \int a\mathrm{T}^4 d\mathrm{V} = a \frac{\mathrm{T}_0{}^4}{p_0} \int p d\mathrm{V} = \frac{3(1-\beta)}{4\beta} \int p d\mathrm{V} \quad \text{by } (26a).$$

Now if there were no radiation-pressure we should have
$$\int p d\mathrm{V} = 4\pi \int p \xi^2 d\xi = -\tfrac{4}{3}\pi \int \xi^3 dp = \tfrac{1}{3}\Omega. \quad (\textit{Gaskugeln}, \text{ p. } 124).$$

But radiation-pressure multiplies the values of p in the ratio β, so that
$$\int p d\mathrm{V} = \tfrac{1}{3}\beta\Omega.$$

Thus
$$\mathrm{K} = \frac{\beta}{3(\gamma - 1)}\Omega \qquad \mathrm{H} = \frac{1-\beta}{4}\Omega.$$

We cannot estimate K accurately, since we do not know γ. The material is probably monatomic at the high temperature, but it seems unlikely that the usual value $\gamma = \frac{5}{3}$ is valid for radiating atoms. The factor β in the numerator will make this energy small in almost any case.

Taking Perry's case, $\gamma = \frac{4}{3}$, we have (for mass $= 1{\cdot}5 \times \odot$)
$$\mathrm{K} = {\cdot}047\Omega \qquad \mathrm{H} = {\cdot}238\Omega.$$

Altogether 0·285 of the energy of contraction is retained within the star as heat, leaving plenty of margin for radiation or for the formation of radioactive elements.

It is interesting to note that the greater part (H) of the energy resides in the æther. Only a small part of the star's store of heat is represented by the molecular movements; the bulk of it consists of æther-waves travelling in all directions but unable to escape, except very slowly, through the meshes of matter which imprison them. This condition applies to the gaseous stars only, and not to dwarf stars such as the Sun.

As the star contracts Ω varies as $\rho^{\frac{1}{3}}$, or as $\mathrm{T}_1{}^2$. Hence, since the total rate of radiation remains constant, we ought to have—

The age of a gaseous star increases as the square of its effective temperature.

I do not place much reliance on this, because it is well known that a star cannot have a reasonably long life if its supply of energy is due solely to contraction.

10. *Dwarf Stars.*—If we were to extrapolate the law $T_1 \propto \rho^{\frac{1}{3}}$ for a star having the density of the Sun, 1·38, we should find an effective temperature of 19 300,° whereas the Sun's effective temperature is about 6000°. The difference is due to the failure of the gas-law at this high density, which renders the calculated values of $(T_0{}^4/p_0)_c$ erroneous. It should be noted that we have not used the gas-law in deriving equations (27), (28), (29), or indeed in any of the numbered equations except (20). Thus the whole theory holds for the dwarf stars, except that the calculations of § 4 break down.

The effect of a failure of the gas-law is to alter the value of $(T_0{}^4/p_0)_c$, which by (27) alters β. β begins to increase. By (29) T_1 varies as $\sqrt[4]{(1-\beta)}$. Since β is at first about 0·05, the increase of β two- or three-fold makes at first but little difference in this factor. It is not until long after the internal distribution of temperature and density has deviated strongly from that given by the gas-law that T_1 begins to be affected.

It is of great interest to determine the turning-point at which the effective temperature ceases to increase with the density.

For high densities and pressures the necessary modification of Boyle's Law is given approximately by Van der Waal's equation, and the gas-equation becomes

$$p = R\rho T\left(1 - \frac{\rho}{\rho_0}\right)^{-1} \quad . \quad . \quad . \quad . \quad (31)$$

where ρ_0 is the maximum possible density of the material. This equation must be combined with (26*a*)

$$p = \tfrac{4}{3}a\frac{\beta}{1-\beta}T^4 \quad . \quad . \quad . \quad . \quad (32)$$

and

$$\frac{1}{\rho}\frac{dp}{d\xi} = -\beta g \quad . \quad . \quad . \quad . \quad . \quad (33)$$

to determine the interior conditions. An exact solution is, of course, impossible; but we can without much difficulty attain our object by quadratures.

Since it is clear that on this theory the Sun must be compressed to something near the maximum density throughout most of its mass, I take ρ_0 to be 1·5.* Our object is—given the mass and mean density, to find β. We have to proceed by guessing β and then working from the outside inwards with equations (31), (32), and (33) to test whether the value fits.

For a star of the same mass as before, and density 0·7, I find that β must be 0·18. The following table gives the radii of the surfaces of equal density, the mass contained within them, the temperature, and the acceleration of gravity there.

* N. Ekholm gives 1·486. His neglect of radiation-pressure and other divergences will make no serious difference in the case of the Sun. See *Gaskugeln*, p. 467.

Dense Star.

(Mass = 1·5 ☉ ; Density = 0·7.)

Density.	Radius.	Interior Mass.	Temperature.	Gravity.
0	$9{\cdot}92 \times 10^{10}$	$2{\cdot}87 \times 10^{33}$	...	$1{\cdot}94 \times 10^4$
0·05	9·32	2·87	$3{\cdot}4 \times 10^6$	2·20
0·15	9·05	2·85	5·0	2·31
0·25	8·87	2·81	6·1	2·38
0·35	8·70	2·76	7·0	2·43
0·45	8·54	2·70	7·8	2·46
0·55	8·37	2·62	8·6	2·49
0·65	8·18	2·52	9·5	2·51
0·75	7·95	2·39	10·4	2·52
0·85	7·67	2·22	11·3	2·51
0·95	7·29	1·98	12·5	2·48
1·05	6·76	1·65	13·8	2·41
1·15	5·86	1·16	15·4	2·25

The mass remaining within the last sphere must correspond to a mean density greater than 1·15, and less than the limiting density 1·5. The actual value deduced from the table is 1·37, and this constitutes our test that the value of β is correct—we have stripped off just the right amount. Had we chosen $\beta = 0{\cdot}20$, the interior density would have been less than 1·15; similarly a smaller β would have given a density greater than 1·5.

According to formula (30) this star would have an effective temperature of 17 300°; the factor $(1-\beta)^{\frac{1}{4}}$ reduces this to 16 600°. It is remarkable that such an extreme deviation from the conditions of a perfect gas should lead to so small a correction. It is clear that 16 600° must be about the turning-point of the temperature, because when once the term in β has begun to take effect it will soon counterbalance any increase of density.

By (30) the maximum effective temperature will be proportional approximately to the twelfth root of the mass. The correction depending on β can be determined from (27), since *for the same mean density* $(T_0{}^4/p_0)_c$ varies as M^2, whether the star be gaseous or otherwise (*cf.* (20)). Thus for the same density

$$\frac{1-\beta}{\beta^4} \propto M^2.$$

Introducing this correction, I find the following results:—

Mass = 4·5 × Sun.	Maximum temperature = 18 600°
,, 1·5 ,,	,, ,, 16 600°
,, 0·5 ,,	,, ,, 14 600°

These values are perhaps too high. The observational evidence of Nordmann and of Wilsing and Scheiner seems to point to a maximum of about 14 000°. But the agreement seems sufficiently close to be of interest. Lower maxima, in the ratio five-sixths, would have been obtained if Russell's densities (*footnote*, p. 29) had been used instead of the datum density 0·002 corresponding to temperature 6500°.

The dwarf-stars must be comprised within the range of density 0·7 to 1·5. The importance of radiation-pressure compared with gravitation decreases rapidly as we descend along the dwarf series. For the Sun, radiation-pressure is practically negligible. The differences of luminosity of the dwarf types must be due mainly to differences of effective temperature, since there is not much scope for variation of superficial area.

11. *Molecular Weight.*—The calculations have been based throughout on a molecular weight 54. To illustrate the effect of modifying this, I give some results for a molecular weight 18. The gas-constant R becomes multiplied by 3. The central pressure becomes multiplied by 3 very approximately, the central temperature being almost unaltered. The new value of β is ·137, so that about $\frac{6}{7}$ of gravity is compensated by radiation-pressure, instead of $\frac{19}{20}$. The constant of absorption k is slightly reduced to 26·7. The results of § 8 are based on the general relation $T_1 \propto M^{\frac{1}{12}} \rho^{\frac{1}{6}}$, and are not altered. The calculations of the internal energy of the star in § 9 involve β, and the numerical results are altered, but they are of the same general character; the molecular energy is, however, increased and the radiant energy slightly diminished.

The most important effect of diminishing the molecular weight to 18 is to reduce the maximum possible effective temperature and to cause it to occur at a lower mean density. It is not possible to give accurate figures without a great deal of labour, but it appears that the maximum effective temperature would be decreased by about 1200°.

The importance of radiation-pressure in determining stellar equilibrium would be reduced if the molecular weight were still smaller; but even for a star composed of helium, it would still be intense enough to counterbalance half the gravitation.

12. *The Constant of Absorption.*—We have found that k has a value about 30 in C.G.S. units, and this should be the constant of absorption for stellar material at temperatures chiefly between 10^6 and 10^7 degrees prevailing within stars. Unfortunately, there seems no other evidence to indicate whether this value is at all near the truth. At temperatures of 10^6 and 10^7 degrees the radiation of maximum intensity is of wave-length 30 and 3 tenth-metres respectively—shorter than ordinary spectral radiation, but longer than X-rays and γ-rays. It has one merit for our purpose—it cannot very well experience selective absorption, since no spectrum (X-ray or ordinary) contains lines within this range.

It happens that the experimental values of k for hard X-rays absorbed by solid material are usually about equal to the value here found.* In the absence of a theory to guide us, we cannot

* Bragg, *X-Rays and Crystal Structure*, p. 177. Table A gives k, defined as in this paper.

infer that the same values would hold at a temperature of 10^6 degrees and for considerably longer waves. The agreement is probably accidental. The work which seems to bear most nearly on the question is J. J. Thomson's theory of the scattering of X-rays.* For very short waves the scattering depends only on the number of electrons present—for a gas, practically on the mass. The theory, confirmed by the experiments of Barkla, gives a scattering of ·00025 by a cubic centimetre of air. This is equivalent to $k=0{\cdot}2$ only; but this is only the part of the coefficient due to scattering, and the full absorption coefficient must be larger.

Experimental results for the extreme ultra-violet have, I think, no application to our problem, for the absorption may be molecular and is not likely to persist at high temperatures.

The Stellar Magnitude Scales of the Astrographic Catalogue. Tenth Note. The Melbourne Magnitudes. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. During the visit of the British Association to Australia in 1914 we were glad to learn that the measurement and reduction of the Melbourne zones ($-65°$ to $-90°$, the portion corresponding to that undertaken by Greenwich in the northern hemisphere) were nearing completion, and that funds had been provided for the printing. It is earnestly to be hoped that neither the war nor the retirement of Mr. Baracchi from the Directorship will affect the availability of these funds, and that the printing may go forward rapidly. The Council of the British Association, after consultation with the Council of the Royal Astronomical Society, drew the attention of the Victorian Government to the importance of this printing, expressing great satisfaction that the means had been promised.

2. Meanwhile Mr. Baracchi, in kind response to a request, sent me the counts of the plates in zone $-65°$; material of great value, seeing that nothing similar has yet been published south of $-42°$ (Cape). Experience of the Vatican plates in similar latitudes and of the Oxford plates indicates that the results of a single zone are liable to be sensibly affected by accidental errors (*Mon. Not.*, lxxv. pp. 468 and 603). Hence the following results may possibly be modified when other zones are available.

3. We will depart a little from previous procedure by considering first the magnitude scale as a whole. In Table I. are given, first, d, the diameter as measured at Melbourne in units of $0''{\cdot}25$; next, N′, the total number of stars on all the plates in the zone with a diameter greater than d; and next, log N, the average *per plate*, *i.e.* log N′ − log 80, there being 80 plates in the zone.

* Richardson, *Electron Theory of Matter*, p. 485.