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ON SPECTROSCOPIC RESOLVING POWER

By C. M. SPARROW

If a spectroscope is just able to separate two monochromatic lines of equal intensity and wave-lengths λ and $\lambda+\Delta\lambda$, the ratio $\frac{\lambda}{\Delta\lambda}$ is called the resolving power of the instrument for the wave-length λ. This is the *definition* of resolving power, and if we can determine by actual measurement the value of $\Delta\lambda$ for some particular instrument, we can obtain the resolving power of that instrument. If, however, our problem is to calculate the resolving power from the optical theory of the instrument, the *definition* must be supplemented by a *criterion* of some sort which will enable us to say when the two lines are to be considered as just resolved. In the case of a prism without absorption, or of a grating with many lines, the criterion proposed by Rayleigh[1] has hitherto been universally adopted. The intensity in a single line being given by

$$I=I_0\frac{\sin^2 x}{x^2}, \tag{1}$$

and that due to two lines by

$$I=I_0\left\{\frac{\sin^2(x-a)}{(x-a)^2}+\frac{\sin^2(x+a)}{(x+a)^2}\right\}, \tag{2}$$

the two lines are considered as just resolved when $a=\frac{\pi}{2}$, that is, when the maximum of one line coincides with the first minimum of the other. Under these conditions the composite diffraction pattern has a distribution of intensity given by the familiar curve 6 of Fig. 1. The ratio $\frac{I_{\text{min}}}{I_{\text{max}}}$ is in this case $\frac{8}{\pi^2}$ or about 0.81.

As originally proposed, the Rayleigh criterion was not intended as a measure of the actual limit of resolution, but rather as an index of the relative merit of different instruments. In the form in which it is stated above, the criterion is applicable only to instruments

[1] *Philosophical Magazine* (4), **47**, 193, 1874; (5), **9**, 266, 1879; also article on "Wave Theory" in the *Encyclopaedia Britannica*.

whose diffraction pattern is of the form (1). For such instruments it is as good an index as any other, and leads to simple formulae for the prism and grating. For instruments such as an absorbing prism or a Fabry and Perot interferometer it ceases to be immediately applicable. For such instruments we may, it is true, express the criterion in the form

$$\frac{I_{\min}}{I_{\max}}=0.81, \tag{3}$$

and this course has been generally adopted heretofore.[1] But now the criterion has lost its simple theoretical significance, and the choice of the value 0.81 for the right-hand side of (3) has become an arbitrary one. Moreover, the relative merit of different instruments will vary with our choice of the right-hand member of (3). Thus suppose that a grating and a Fabry and Perot interferometer have equal resolving power on the basis of (3). On a 90 per cent basis the interferometer would be superior, while on a 70 per cent basis the advantage would lie with the grating. If we should follow Schuster's proposal[2] and take complete separation as a basis, an infinitely thick prism with finite absorption would have zero resolving power.

It should be clear from the foregoing that the only fair basis on which such different instruments can be compared involves the adoption of a criterion which gives a measure of the actual limit of resolution. It has hitherto been assumed by many that the Rayleigh criterion does this, but the basis of fact on which this assumption rests is small and inconclusive; and, as we shall see, the true limit is quite different.

In the present paper we shall present the results of an empirical study of the actual appearance, visual or photographic, of different doublets. In this way the actual limits of resolution are determined. The results of these observations lead to the formulation of a criterion with a simple theoretical basis and applicable to a great variety of instruments. In addition the limits of resolution

[1] See, for example, Wadsworth, *Philosophical Magazine* (6), **5**, 355, 1903, where the effect on resolving power of absorption in a prism system is calculated.

[2] *Theory of Optics* (London, 1909), p. 158.

for two lines of different intensities have been determined—a case for which the theory gives no inkling as to what we may expect.

The experimental method is simple, and by no means new, being the same as that used by Langley for the conversion of bolographs into spectrographs. The form of a diffraction pattern was calculated and the intensity-curve, in rectangular co-ordinates, was drawn carefully on black paper. The area between the curve and the x-axis was then cut out, making a screen with an aperture of the required form. This screen was placed against a uniformly illuminated background and viewed or photographed with a cylindric lens having its axis of curvature parallel to the y-axis of the curve. In this way "artificial doublets" of any form and separation could be produced. The screens were about the size of a lantern slide, and the lens, of about 15 cm focus, was about 10 m from the screen. The camera was fitted with a multiplying back, so that six exposures could be taken on one plate. In order to test the focus, one exposure on each plate was made of a screen with a pair of narrow parallel slits. Hammer lantern plates (white label) were used; they were developed with hydrochinon. Visual observations on the lines led in all cases to the same results as the photographs; hence they are not specially mentioned in what follows.

Grating with infinitely narrow slit.—The actual appearance of a doublet whose separation is that given by the Rayleigh criterion is shown in the first row (1–5) of photographs in Plate II for different relative intensities of the two components. Considering for the moment only the case of equal intensity, it is obvious that the lines are quite distinctly resolved. On the plates the effect is so much more pronounced that most spectroscopists would call the separation measurable. The numbers which give the separation are the values of $2a$ in (2). They are thus half the phase difference in radians between the maximum of either line and the position on its diffraction pattern where the other maximum falls. The corresponding intensity-curves for each line are given in the first row of Fig. 1. In the second row of photographs (6–10) the components are closer, but are still clearly resolved. (The intensity-curves bear the corresponding numbers in Fig. 1.) As the lines are brought still closer, the central minimum becomes shallower, until it finally disappears. To find the value of $2a$ corresponding to this condition

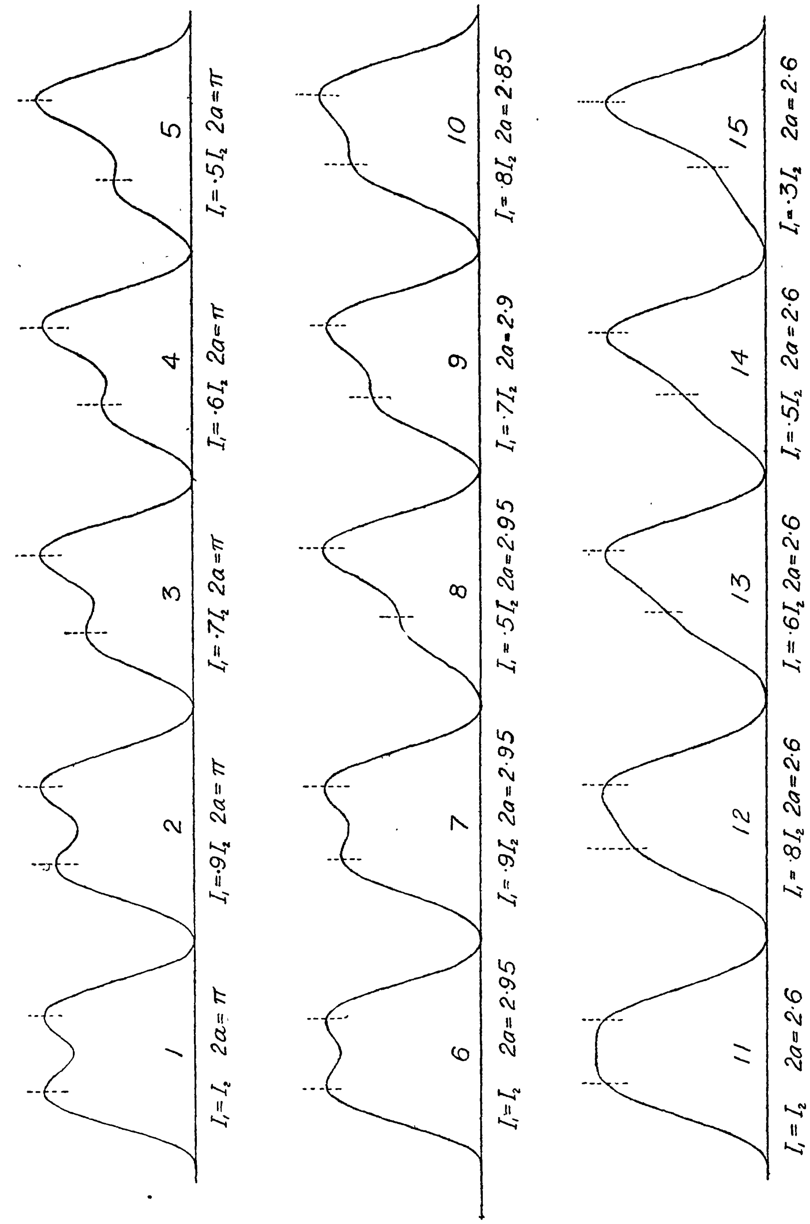

FIG. 1

(for equal intensities) we may differentiate (2) twice with respect to x and put

$$\frac{\delta^2 I}{\delta x^2}=0 \text{ when } x=0, \tag{4}$$

and solve for a. Since the two curves are symmetrical, the odd derivatives necessarily vanish at the origin, and thus (4) gives the composite curve an "undulation" at the origin. I shall refer to (4) hereafter as the "undulation condition."

It is obvious that the undulation condition should set an upper limit to the resolving power. The surprising fact is that this limit is *apparently actually attained*, and that the doublet still appears resolved, the effect of contrast so intensifying the edges that the eye supplies a minimum where none exists. The effect is observable both in positives and in negatives, as well as by direct vision. It cannot be seen in prints because of insufficient illumination. I have therefore not attempted to reproduce it here, but have given only the forms of the intensity-curves (11–15, Fig. 1). My own observations on this point have been checked by a number of my friends and colleagues. The same phenomenon has been noted by Wood in connection with the apparent reversal of a broad spectral line. A very slight further diminution of the separation rounds the top of the intensity-curve so that there is no resolution; the undulation condition thus defines the limit of resolution quite sharply.

A solution of (4) by successive approximations leads to the value $2a=2.606$. Thus the actual resolving power of a perfect grating of n lines in the Nth order is

$$\frac{\pi}{2.606}\cdot Nn=1.26\,Nn, \tag{5}$$

and similarly for a prism of thickness t and refractive index μ,

$$1.26\,t\frac{d\mu}{d\lambda}. \tag{6}$$

When the intensities are unequal, the form of the intensity-curve is of course completely altered. Nevertheless the actual observations show the remarkable fact that the *limit of resolution remains about the same*. In this case it is of course impossible to say definitely where resolution stops: a line which one observer would call resolved would perhaps be regarded by another observer as a single line shaded on one side. Nevertheless the form of the intensity-curve is quite sensitive to small changes in separation, the result being that the actual limits given by different observers who have examined the plates vary by only a small percentage. When the ratio of intensities is greater than 10:2 the appearance of the doublet is complicated by the greater relative prominence of the secondary maxima; it has therefore been found difficult to draw definite conclusions for intensity-ratios greater than this. The form of the intensity-curves is rather noteworthy; the curves, 8, 9, for example, show no actual minimum, but slope away continuously from the vertex, while those (11–15) which correspond to the limiting case hardly suggest a doublet by their shape.

Instruments with diffraction patterns other than the normal one.—Besides imperfections in the optical surfaces (which we shall not consider here), there are two principal causes for the deviation of the diffraction pattern from the form (1); namely, the use of a slit of finite width, and absorption or loss of light by reflection. The previous work with the normal pattern having shown that the resolving power varies at most only slightly with the relative intensity, it was found possible in subsequent work to simplify the experimental method. Instead of making a separate screen for each doublet (about a hundred such were used for the observations described above), one screen was made having the form of the diffraction pattern of a single line. This screen was mounted so that it could be displaced parallel to the x-axis through any required distance, and the doublets were made by superposing two exposures of the screen in different positions on the same plate, intensities being regulated by the time of exposure.

Two cases were studied in detail: that of a grating with "4-normal" slit-width, and that of an infinitely thick prism with finite absorption. Detached instances were studied for other cases. The general results may be summed up by the same criterion as that found for the grating with a narrow slit; namely, that *the limit of resolution is given by the undulation condition.* Since this was found to hold for a narrow slit and a wide slit, it seemed safe to assume that it would hold for intermediate slit-widths. Since it holds for an infinitely thick absorbing prism and for a perfectly transparent prism, it may be assumed to hold for all cases which are intermediate between these two, or which approximate them very closely.

As will be seen below, these cases include a finite absorbing prism, an echelon grating, a Lummer-Gehrcke plate, and a Fabry and Perot interferometer. As this list includes most of the important forms of spectroscopic apparatus, it may be concluded that the undulation condition furnishes a criterion of very general applicability. There thus remains only the task of formulating this criterion for the different types of instrument. This formulation is best expressed by the use of factors which indicate the relative resolving power of such instruments with respect to the more perfect instruments. We thus have two sets of factors, slit-width factors and absorption factors. In order to calculate, for instance, the resolving power of an echelon grating, taking account of absorption and slit-width, we have only to multiply the resolving power for the ideal instrument by a suitable factor which depends only on the absorption and slit-width, not on the type of instrument.

The slit-width factors.—The intensity pattern of a doublet may be written in the form

$$I=I_0\left\{\int_{x-d}^{x+d}\frac{\sin^2(x+a)}{(x+a)^2}\,dx+\int_{x-d}^{x+d}\frac{\sin^2(x-a)}{(x-a)^2}\,dx\right\}. \tag{7}$$

The analytic expression for the undulation condition is not in this case easy to apply. The values of $2a$ were therefore obtained by a

TABLE I

Slit-Width $\times f/d$	$2a$	Slit-Width Factor (C.M.S.)	Purity Factor (Schuster)
0	2.606	1.00	1.00
0.25	2.64	0.99	0.986
0.5	2.72	0.96	0.943
0.75	2.91	0.90	....
1.00	3.14	0.83	0.780
1.25	3.77	0.69	....
1.50	4.75	0.55	0.579
1.75	5.61	0.46	....
2.00	6.30	0.41	0.450

combination of graphical and numerical methods. The results are given in Table I. The "purity factors" of Schuster,[1] which

[1] *Astrophysical Journal*, **21**, 197, 1905; *Theory of Optics* (London, 1909), p. 163.

were calculated for the same purpose, but with a different theoretical basis, are given for comparison in the fourth column. The difference in the two sets of factors is not great, and either would probably prove sufficiently accurate for most practical purposes.

Absorption factors.—The problem of finding the diffraction pattern is here one of combining n disturbances with amplitudes in geometric progression and phases in arithmetic progression. The summation leads to the well-known formula of Airy which we may write in the form

$$I=s_0^2\,\frac{1-2r^n\cos n\phi+r^{2n}}{1-2r\cos\phi+r^2}\,, \tag{8}$$

where r is the ratio of the $(p+1)$th to the pth amplitude, s_0 the initial amplitude, and ϕ the phase difference between successive disturbances. The equation which expresses the undulation condition is here quite complicated unless n is infinite. For most practical purposes an approximate formula will do as well. We may obtain such a formula by making n infinite, while the total phase change $n\phi$ and the total absorption r^n, as well as ns_0, approach finite limits. Writing $r=e^{-k}$ and multiplying and dividing numerator and denominator in (8) by n^2,

$$I=\frac{n^2s_0^2(1-2e^{-nk}\cos n\phi+e^{-2nk})}{n^2(1-e^{-k})^2+4e^{-k}\cdot n^2\sin^2\frac{\phi}{2}} \tag{9}$$

$$=I_0\frac{(1-2e^{-k}\cos\Phi+e^{-2k})}{k^2+\Phi^2}\,. \tag{10}$$

Here I_0 is the maximum intensity which we should have without absorption, Φ is the total phase difference between the two extreme disturbances, and k is the logarithm of the ratio of the final to the initial disturbance. The approximation amounts to this: if we represent the disturbances by vectors, the vector sum (9) is a polygon inscribed in a logarithmic spiral; in (10) we pass from the polygon to the limiting spiral. Equation (10) is a rigorous expression for an absorbing prism[1] and an approximate one for the

[1] See Wadsworth, *op. cit.*, ... , where essentially the same formula is derived.

case of an echelon grating, or a Lummer-Gehrcke plate. To form some idea of the degree of approximation the value of $2a$ for $n=5$, $k=1$ was computed by both (8) and (10). The value from the exact formula was about 2 per cent greater than that from the approximate formula. As this value of n is very small for an actual instrument, and as the accuracy of (10) increases very rapidly with increasing n, we may consider (10) a sufficient approximation for most purposes.

The undulation condition for equation (10) was solved for different values of k by successive approximations, giving the absorption factors listed in Table II. The first column gives the values of k, the second the corresponding values of $e^{-k}(=r^n$, see (8)), the third gives the values of $2a$, and the fourth the absorption factors.

TABLE II

k	e^{-k}	$2a$	Absorption Factor
0	1.0000	2.606	1.00
0.5	0.6065	2.611	0.998
1.0	0.3679	2.637	0.988
1.5	0.2231	2.662	0.979
2.0	0.1353	2.710	0.962
4.0	0.0183	3.041	0.857

For infinite values of k the expression (10) becomes indeterminate, since I_0 also becomes infinite. By returning to (9) we may obtain an expression for the intensity in this case, which is the case of an infinitely thick prism with finite absorption. The expression here reduces to the simple form

$$I=\frac{I_1}{\Phi_1^2+k_1^2}, \tag{11}$$

where I_1, k_1, and Φ_1 have the same meaning as the corresponding quantities in (10) except that they refer to any finite portion of the prism. If we make I_1 the intensity of the incident light, k_1 the logarithmic decrement due to loss by reflection and absorption, and Φ_1 the phase difference between two successive interfering beams, the expression (11) is an approximate expression for the intensity-curve of the Fabry and Perot interferometer. The undulation condition obtained from (11) leads to an extremely simple formula for the resolving power of this instrument. Writing

$$I=\frac{s_1^2}{k_1^2+(\Phi_1+\Phi_0)^2}+\frac{s_1^2}{k_1^2+(\Phi_1-\Phi_0)^2},$$

differentiating twice as to Φ_1 and putting $\Phi_1=0$, we obtain

$$4\Phi_0^2=k_1^2+\Phi_0^2 \text{ or } \Phi_0=\frac{k_1}{\sqrt{3}}. \tag{12}$$

If D is the distance between the plates, this gives for the resolving power

$$\frac{2\pi D}{\lambda}\frac{\sqrt{3}}{k_1}=\frac{10.9\,D}{\lambda k_1}. \tag{13}$$

It is worth while to compare this result with that obtained from the exact formula, which may be obtained from (8) by making n infinite. The undulation condition leads to a quadratic in $\cos\Phi$, the solution of which gives

$$\cos\Phi_0=-\frac{1+r^2}{4r}+\sqrt{\frac{(1+r^2)^2}{16r^2}+2}\quad (r=e^{-k_1}). \tag{14}$$

For $k_1=0.1$ we obtain from (12) $\Phi_0=0.1155$, and from (14) $\Phi_0=0.1158$, thus showing that (11) represents the form of the Fabry and Perot fringes in the neighborhood of a maximum with a high degree of approximation.

There is one further advantage of the criterion furnished by the undulation condition, namely, that it is independent of any particular photographic process; for contrast can be enhanced by photography only where it exists, so that we should expect the appearance of a pair of lines at the limiting separation to undergo little change with any variation of the photographic process.

Visual resolving power.—It is obvious that the undulation criterion should apply equally to the calculation of the visual (telescopic) resolving power of a rectangular aperture. For apertures of other shapes we should not a priori expect it to apply. The problems presented are of far less practical importance than those furnished by the spectroscopic case, and it has not seemed worth while to carry the investigation farther in this direction.

SUMMARY

1. The actual limit of the resolving power of a perfect grating or prism has been determined experimentally. It is found that this limit is given, for equal intensities of the two lines, by the "undulation condition," that is, by the condition that the central minimum shall just disappear. This gives a theoretical resolving power about 26 per cent greater than that obtained by the Rayleigh criterion.

2. The limit given by the undulation condition has been found to hold for unsymmetrical doublets when the ratio of intensities of the two components is less than 10:3.

3. The undulation condition gives the limit for all cases in which the diffraction pattern is modified by finite slit-width, or by a decrease in geometric progression of the intensities of the interfering beams, whether this is due to absorption or to loss of light by reflection. These cases include most of the important forms of spectroscopic apparatus.

4. The effect of slit-width and absorption can be introduced by the use of suitable factors. These factors have been calculated for various values of the slit-width and absorption.

5. A simple approximate formula has been given for the resolving power of the Fabry and Perot interferometer.

The foregoing work was begun during the last Christmas vacation in the Physical Laboratory of the Johns Hopkins University. I am indebted to the Department of Physics there for the facilities so freely placed at my disposal during the beginning of the work, and for the loan of the cylindric lens with which I have continued the work here. I am also especially indebted to Dr. J. A. Anderson for his valuable advice and assistance.

Rouss Physical Laboratory
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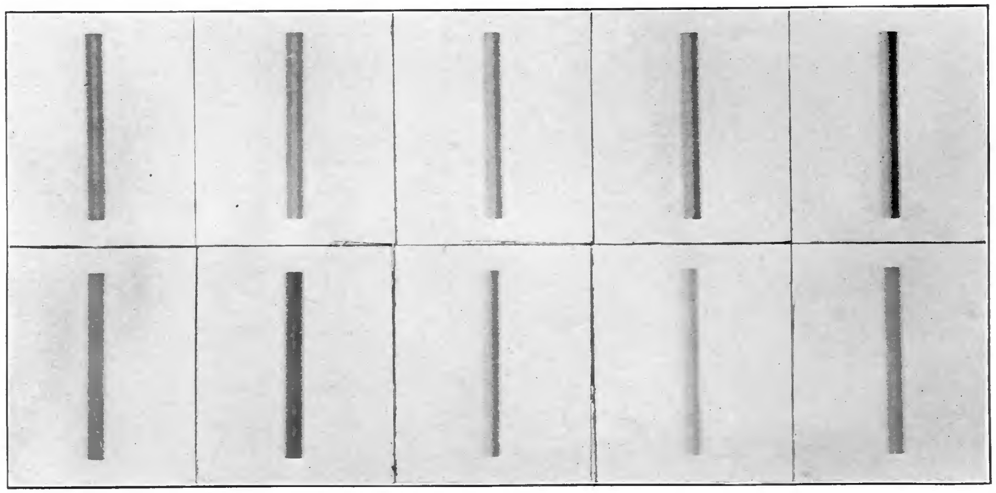
PLATE II

1	2	3	4	5
$I_1=I_2$	$I_1=0.9I_2$	$I_1=0.7I_2$	$I_1=0.6I_2$	$I_1=0.5I_2$
$2a=\pi$	$2a=\pi$	$2a=\pi$	$2a=\pi$	$2a=\pi$



6	7	8	9	10
$I_1=I_2$	$I_1=0.9I_2$	$I_1=0.5I_2$	$I_1=0.7I_2$	$I_1=0.8I_2$
$2a=2.95$	$2a=2.95$	$2a=2.95$	$2a=2.9$	$2a=2.85$

SPECTROSCOPIC RESOLVING POWER