presented by the Mount Wilson Observatory; photograph of the Kater tablet, Shetland Isles, presented by Major MacMahon; Astrographic Chart: 20 charts presented by the Tacubaya Observatory.

A Determination of the Frequency-law of Stellar Motions. (Plate 2.) By A. S. Eddington, M.A., M.Sc.

(*Communicated by the Astronomer Royal.*)

It is now well known that the motions of the stars, even after due allowance has been made for the effect of the solar motion, exhibit a strong polarity. There is a tendency to move parallel to a certain line, which is situated in the galactic plane; and both directions, towards either end of this line, are favoured directions of motion. This line cuts the celestial sphere near the point R.A. 94°, Dec. +12°, and the opposite point, called the vertex and antivertex; and the stream of stars moving towards the former point is called Stream I., and that towards the opposite point Stream II. The phenomenon is sometimes presented slightly differently by considering the motions of the streams relative to the Sun; from that point of view their directions are inclined at an angle of about 100°; but, when the solar motion is removed and the stream motions are referred to the centre of mass of the stars concerned, the directions must necessarily be opposite to one another.

In investigating quantitatively the amount and direction of this polarity, two principal forms of representation have been used. The two-drift theory expresses it as if it were due to two systems of stars moving in opposite directions along the line, the internal motions in each system being haphazard. The ellipsoidal hypothesis does not make any division into two systems, but expresses that the average motions of stars in different directions in space are related as the radii of a prolate spheroid, the major axis of the spheroid being in the direction of the polarity aforementioned. I have elsewhere * compared these two representations, showing that, although the expressions differ in mathematical form, it is possible for them to embody almost indistinguishable distributions of velocity.

In the present paper I have tried to determine what is the actual distribution or frequency-law of the linear velocities without assuming that it must be of either of these forms. This frequency-law is to be deduced from the observed distribution of the angular motions, not only as regards direction, but as regards both amount and direction. I had hoped that it might be possible to avoid making any assumption whatever as to the form of the frequency-law; but I was ultimately obliged to limit it to a form which, however, was sufficiently general to cover as particular cases the

* *Brit. Assoc. Report*, 1911.

ellipsoidal, two-drift, and Halm's three-drift theories. All these theories assume that the component motions at right angles to the line of polarity are distributed according to the law of accidental errors. I have used this property so far as concerns those stars whose whole motion (referred to the Sun) is at right angles to the line of polarity; in all other directions no limitation is imposed.

The result arrived at in this paper is opposed to the ellipsoidal and in favour of the two-drift (not excluding the three-drift) representation. The whole investigation, however, is somewhat tentative, and it is difficult to say how much reliance must be placed on any particular deduction. I feel sure that no minor difference in the mathematical treatment could make these particular data yield a result favourable to the ellipsoidal theory; for, before the solution now published was reached, three other nearly completed attempts were made, all of which indicated a two-drift distribution in the same way. But it may legitimately be urged that the data have been insufficient to give a great accuracy of detail in the resulting frequency-law, and I should rather hesitate to assert that the particular detail which distinguishes the rival mathematical hypotheses is entirely trustworthy. The conclusion on this particular point is perhaps disappointing; these results definitely favour the view that the stars are grouped in two (or three) distinct systems, but can hardly be considered to settle the question conclusively.

The principal results are given in a graphical form in the diagram (Plate 2). This gives the actual distribution of the linear motions determined by this method. The distribution of the stars as regards distance, which is incidentally determined (p. 386), is also of interest; but it should be received with even greater caution.

Mathematical Theory.

We assume that the law of distribution of the linear velocities of the stars does not depend on the distance of the stars considered from the solar system.

We have to consider the relation of three frequency-laws—

(1) The law of distribution of the linear motions of the stars.
(2) The law of distribution of the proper or angular motions of the stars.
(3) The distribution, as regards distance along the line of sight, of the stars of the catalogue.

These three laws are connected by an integral equation, from which, if any two of them are given, the third can be found. The catalogue of proper motions gives us (2); but to find (1) we must also know (3), or *vice versa.* My method was this:* as the

* I had expected to make use of a quite extraneous determination of law (3) by Professor Kapteyn, but that was found to be irreconcilable with my data and could not be used. It was therefore necessary to include the determination of (3) in my investigation.

linear motion relative to the Sun has the same direction as the proper motion, we can consider separately the proper motions in any direction. I chose the direction perpendicular to the line of polarity, in which the ellipsoidal and two-drift theories agreed that the law (1) was the law of errors (there is also a component of solar motion in that direction which must be taken into account). Knowing then (1) and (2) for this direction, I determined (3). But (3) applies to all the other directions, and therefore, knowing (2) and (3) for the other directions, I determined (1) for them. This completed the solution.

Confine attention to a limited area of the sky, which can be treated as plane, and to stars moving in directions within any given limits of position angle.

Let $g(u)\frac{du}{u}$ be the number of stars having linear motions between u and $u+du$.

Let $h(\alpha)\frac{d\alpha}{\alpha}$ be the number having proper motions between α and $\alpha+d\alpha$.

Let $f(r)\frac{dr}{r}$ be the proportion of stars (of the catalogue) comprised within the limits of distance from the Sun r and $r+dr$, so that

$$\int_0^\infty f(r)\frac{dr}{r}=1\,.$$

Then, expressing that $h(\alpha)\frac{d\alpha}{\alpha}$ is made up of stars at all possible distances r, and with linear motions u such that u is between $r\alpha$ and $r(\alpha+d\alpha)$, it is easily seen that

$$h(\alpha)\frac{d\alpha}{\alpha}=\int_0^\infty f(r)\frac{dr}{r}\cdot\frac{g(r\alpha)}{r\alpha}r\,d\alpha\,.$$

Hence

$$h(\alpha)=\int_0^\infty f(r)g(r\alpha)\frac{dr}{r} \quad . \quad . \quad . \quad . \quad (1)$$

For the solution of this integral equation, the method given by Prof. Schwarzschild (*Ast. Nach.*, No. 4422) for a somewhat similar problem can be used. Write

$$r=e^\lambda, \quad \alpha=e^\mu, \quad \text{and} \quad u=e^\gamma,$$
$$f(e^\lambda)=f_1(\lambda), \quad g(e^\gamma)=g_1(\gamma), \quad h(e^\mu)=h_1(\mu).$$

Equation (1) becomes

$$h_1(\mu)=\int_{-\infty}^{\infty} f_1(\lambda)g_1(\lambda+\mu)d\lambda \quad . \quad . \quad . \quad (2)$$

Let us form the Fourier integrals

$$\left.\begin{aligned} F(q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\lambda)e^{-\iota q\lambda}d\lambda \\ G(q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g_1(\lambda)e^{-\iota q\lambda}d\lambda \\ H(q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} h_1(\lambda)e^{-\iota q\lambda}d\lambda \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

Then we have the well-known reciprocal relation

$$\left.\begin{aligned} f_1(\lambda) &= \int_{-\infty}^{\infty} F(q)e^{\iota q\lambda}dq \\ g_1(\lambda) &= \int_{-\infty}^{\infty} G(q)e^{\iota q\lambda}dq \\ h_1(\lambda) &= \int_{-\infty}^{\infty} H(q)e^{\iota q\lambda}dq \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

From (2) we have

$$\begin{aligned} \int_{-\infty}^{\infty} h_1(\mu)e^{-\iota q\mu}d\mu &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\lambda)g_1(\lambda+\mu)e^{-\iota q\mu}d\lambda d\mu \\ &= \int_{-\infty}^{\infty} f_1(\lambda)e^{\iota q\lambda}d\lambda\int_{-\infty}^{\infty} g_1(\lambda+\mu)e^{-\iota q(\lambda+\mu)}d\mu \\ &= 2\pi F(-q) \,.\, 2\pi G(q). \end{aligned}$$

Hence $$H(q) = 2\pi F(-q)G(q) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This is the solution of the problem; for if the functions f_1 and h_1 are given, we can form the functions F and H, calculate G from (5), and deduce g_1 from G by means of (4).

The process may be summarised by saying that we form periodograms of the curves for f_1 and h_1, divide the second by the first, and form the periodogram of the result.*

There are several points of difficulty in the practical application of this solution.

We may first notice a result by Prof. Seeliger, which was kindly pointed out to me by Prof. Schwarzschild. If the density of distribution of the stars at a distance r from the solar system varies as $r^{-\kappa}$, so that

$$f(r) = Cr^{3-\kappa},$$

where C is constant, then by (1)

$$h(a) = C\int_0^{\infty} r^{2-\kappa}g(ra)dr;$$

or, writing $u = ra$,

$$h(a) = Ca^{\kappa-3}\int_0^{\infty} u^{2-\kappa}g(u)du.$$

* The periodogram must be understood to be a complex quantity expressing the phase as well as the amplitude.

But the integral on the right-hand side is independent of a, and hence

$$h(a) \propto a^{\kappa-3},$$

whatever may be the frequency-law of the linear motions. In such a case the solution for g clearly must be indeterminate.

It would, of course, be quite impossible for the distribution of the stars of the catalogue to obey the density law $r^{-\kappa}$ for all values of r, since that would involve an infinite number of stars close to the Sun if $\kappa > 3$, or an infinite number of very distant stars if $\kappa < 3$. But if such a law held for a large part of the range, we could expect only a weak solution of the equations. In the first place, therefore, it is desirable to see how nearly $h(a)$ approximates to the form $a^{\kappa-3}$.

In Table I. the material of this investigation is divided to show how many of the total proper motions are between $0''$ and $1''$, $1''$ and $2''$, etc., per century. In our notation this is $h(a)/a$. Inspection shows that no law of the form proposed fits the data for any considerable range; perhaps as good an approximation as possible is given by taking $\kappa = 2{\cdot}14$, so that $h(a)/a \propto a^{-1{\cdot}86}$. (This is determined by making the formula agree with observation near $a = 6''$ and $a = 22''$.) The calculated numbers are given in the third column. The failure of the formula for $a < 15''$ and $a > 40''$ is so considerable that we conclude there is no approach to the indeterminate case, and we may expect that the data will yield a fairly good determination of the function g.

TABLE I

Magnitudes of Proper Motions.

Limits of Centennial P.M.	No. of Stars.	Trial Formula.	Limits of Centennial P.M.	No. of Stars.	Trial Formula.
$''\ \ ''$			$''\ \ ''$		
0– 0·9	33	...	13–13·9	26	18
1– 1·9	95	...	14–14·9	19	16
2– 2·9	112	420	15–15·9	16	14
3– 3·9	112	223	16–16·9	15	12
4– 4·9	92	139	17–17·9	10	11
5– 5·9	84	95	18–18·9	8	10
6– 6·9	83	69	19–19·9	11	9
7– 7·9	70	53			
8– 8·9	56	42	20–25	38	34
9– 9·9	45	34	25–30	16	24
10–10·9	32	29	30–35	27	17
11–11·9	39	24	35–40	17	14
12–12·9	23	20	>40	54	108

Considerable difficulty is caused by the liability of the integrals to diverge. According to the equations (3), (4), and (5), the value of $g_1(\gamma)$, where $\gamma = \log_e u$, is

$$g_1(\gamma) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} e^{\iota q\gamma}\frac{dq}{\mathrm{F}(-q)}\int_{-\infty}^{\infty} h_1(\mu)e^{-\iota q\mu}d\mu \quad . \quad . \quad (6),$$

where the distribution of the stars as regards distance is assumed to be known, and therefore $\mathrm{F}(-q)$ is known.

Now the observed $h_1(\mu)$, as given directly by the proper motions of a finite number of stars, is a discontinuous function. Strictly speaking, if there are n stars, and we divide the range into a very large number of elements $\delta\mu$, $h_1(\mu)\delta\mu$ takes the value unity for n isolated elements, corresponding to the n proper motions, and is zero everywhere else. But if $h_1(\mu)$ is discontinuous in this manner, the Fourier Integral $\int_{-\infty}^{\infty} h_1(\mu)e^{-\iota q\mu}d\mu$ does not tend to zero for large values of q, but oscillates rapidly, and when divided by $\mathrm{F}(-q)$ becomes completely divergent; the solution thus breaks down. This is only to be expected; because there can be no exact solution of the problem in this case, and our formulæ are unfortunately not adapted to give an approximate solution when no exact one is possible.

We should naturally prefer to postpone any smoothing required until the values of $g_1(\gamma)$ had been arrived at; but we are compelled to start by smoothing $h_1(\mu)$, and making it into a continuous function. If this is done, no difficulty will arise in calculating the integral $\mathrm{H}(q)$; but even then it may, and often does, happen that the division by $\mathrm{F}(-q)$ causes divergence and makes the second integration impossible. To show under what conditions this happens, let us evaluate $g_1(\gamma)$ in a comparatively simple case.

Suppose

$$\left.\begin{aligned} f_1(\lambda) &= \sqrt{\frac{\xi}{\pi}}e^{-\xi(\lambda-\sigma)^2} \\ h_1(\mu) &= \mathrm{N}\frac{\kappa}{\sqrt{\pi}}e^{-\kappa^2(\mu-\mu_0)^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (7)$$

The constants σ and μ_0 could be made zero by suitably choosing the units of distance and proper motion, but it is convenient to retain them. N is the number of stars considered.

The expressions for h_1 and f_1 actually used later in this paper are more complicated; but the error-curves given by (7) are sufficiently good approximations to give a valuable insight into the conditions of solution.

We have from (3)

$$2\pi\mathrm{F}(q) = \int_{-\infty}^{\infty}\sqrt{\frac{\xi}{\pi}}e^{-\xi(\lambda-\sigma)^2-\iota q\lambda}d\lambda,$$

which reduces to *

$$2\pi F(q) = \exp.\left\{ -\frac{q^2}{4\xi} - \iota q\sigma \right\} \quad . \quad . \quad . \quad (8)$$

Similarly

$$2\pi H(q) = N \exp.\left\{ -\frac{q^2}{4\kappa^2} - \iota q\mu_0 \right\} \quad . \quad . \quad (9)$$

Therefore

$$2\pi G(q) = \frac{H(q)}{F(-q)} = N \exp.\left\{ -q^2\left(\frac{1}{4\kappa^2} - \frac{1}{4\xi}\right) - \iota q(\mu_0 + \sigma) \right\}.$$

And hence by (4)

$$g_1(\gamma) = \frac{N}{2\pi}\int_{-\infty}^{\infty} \exp.\left\{ -q^2\left(\frac{1}{4\kappa^2} - \frac{1}{4\xi}\right) - \iota q(\mu_0 + \sigma - \gamma) \right\} dq.$$

This reduces to

$$g_1(\gamma) = \frac{N}{\sqrt{\pi\left(\frac{1}{\kappa^2} - \frac{1}{\xi}\right)}} \exp. -\frac{(\gamma - \mu_0 - \sigma)^2}{\frac{1}{\kappa^2} - \frac{1}{\xi}} \quad . \quad . \quad (10)$$

provided

$$\frac{1}{\kappa^2} > \frac{1}{\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If the condition $\frac{1}{\kappa^2} > \frac{1}{\xi}$ is not fulfilled, the solution is divergent. This is a useful guide, by which we may avoid divergent solutions in the general case. Equation (7) gives to $h_1(\mu)$ the form of an error-curve, and the smaller the value of $\frac{1}{\kappa^2}$ the more sharply peaked will this curve be. In order that a solution may be possible, the peak must not be too sharp, as is shown by (11). When $h_1(\mu)$ is not assumed to be an error-curve, we must still take care that there are no peaks or roughnesses which are sharper than a certain limit; if the data of observation appear to indicate too abrupt a peak, it shows that either the assumed law for $f_1(\lambda)$ is wrong, or the peak is a mere accident and must be smoothed down to come within the necessary limit.

In finding $g_1(\gamma)$ from equation (6) solutions are additive. Apparently we may arbitrarily divide the given distribution of proper motions, expressing it as the sum of two or more distributions, determine the distribution of linear motions corresponding to each, and add together the results. But the necessity of performing the division so that all the integrals are convergent imposes a very strict limitation in practice.

* The general formula for reducing these expressions is

$$\int_{-\infty}^{\infty} e^{-b_1 x - b_2 x^2} dx = \sqrt{\frac{\pi}{b_2}} \exp. \frac{b_1^2}{4b_2},$$

subject to obvious conditions of convergency.

When $f_1(\lambda)$ is an error-function as in (7), a complete algebraic solution can be found when $h_1(\mu)$ is of the form

$$(a_0 + a_1(\mu - \mu_0) + a_2(\mu - \mu_0)^2 + \ . \ . \ . \ .)e^{-\kappa^2(\mu - \mu_0)^2}.$$

It happens that the data of observation can usually be expressed very closely in the form

$$h_1(\mu) = (a_0 + a_4(\mu - \mu_0)^4)e^{-\kappa^2(\mu - \mu_0)^2},$$

which is of the above type; but the solution turns out to be useless, as the value found for $g(u)$ makes wide oscillations. The fact is that we have no means of imposing the condition that $g(u)$ is to be always positive and that it is to be a smooth curve, and we are always liable to obtain solutions which disregard these conditions. Thus, in place of finding 10, 12, and 14 stars with velocities of 1, 2, and 3 km./sec. respectively, our solution may give 30, −28, and 34 stars with those velocities. It is too scrupulously exact, and, in place of disregarding trivial roughnesses in the observed data, makes the most astounding oscillations in the attempt to represent them exactly.

As the result of a great deal of experiment, I find that the safest course is to express $h_1(\mu)$ as the sum of a number of error-curves: generally three are found necessary. If care is taken that for each of them $\frac{1}{\kappa^2}$ is above a certain limit,* the solution for $g_1(\gamma)$ will be everywhere positive and free from oscillations.

Application to the observed Data.

The material consists of the stars of Professor Boss's Preliminary General Catalogue (excluding those of the Orion type) contained within the two opposite areas

R.A. $21^h\ 36^m$ to $2^h\ 24^m$, Dec. $-36°$ to $+36°$
and R.A. $9^h\ 36^m$ to $14^h\ 24^m$, " " "

This corresponds to Regions VIII., XII., XIII., and XVII. of my previous investigation.

The whole number of stars is 1129.

The centres of the areas are on the equator at 0^h and 12^h, so that we are really investigating the distribution of linear motions in planes parallel to that which cuts the sphere in the meridians of 6^h and 18^h, neglecting altogether the components perpendicular to this plane. This area of the sky might appear to be unduly large to treat as plane; but, owing to the particular choice of plane, I think no error will arise from this cause. Both the solar apex and the vertex are approximately 90° from the centre of the region, and therefore, in projecting on the tangent plane, the lines joining all the different points of the region to either

* This limit is gradually found by practice. It is somewhat greater than $\frac{1}{\xi}$, because, when $\frac{1}{\kappa^2}$ is only just greater than the critical value, the solution, though convergent, is unsuitable.

apex or vertex project into nearly parallel lines. This would not be true in any other part of the sky; the directions towards the vertex or apex, or both, would converge, and in so large an area the inclination would be very considerable. It follows that near this particular point it is permissible to use a much larger area of the sky than could be admitted elsewhere. Whilst this consideration necessarily determined the choice, the region was especially suitable for another reason. Being near the galactic poles, the proper motions are on the average large, and therefore proportionately more accurately determined than in lower galactic latitudes. Further, in this part of the sky the full effect of the star-streaming is seen without foreshortening.

Having calculated the magnitude and direction of each proper motion, I grouped them according to direction in 30° sectors, *i.e.* stars moving in the directions 335°–5°, 5°–35°, 35°–65°, etc. The direction θ is measured so that $\theta = 0$ is towards the point R.A. 6^h Dec. 0°, and $\theta = 90°$ is towards the South Pole. Table II. shows, for each sector, the number of stars for which the decimal logarithm of the proper motion (unit $= 0''\cdot001$ per annum) lies between the limits given in the first column. This is practically the function $h_1(\mu)$, only that the decimal logarithm μ' is used instead of the natural logarithm μ.

TABLE II.

Number of Proper Motions within given Limits.

μ' (=$\log_{10}$ P.M.)	Sectors. 5° to 35°	35° to 65°	65° to 95°	95° to 125°	125° to 155°	155° to 185°	185° to 215°	215° to 245°	245° to 275°	275° to 305°	305° to 335°	335° to 5°
<1·0	2	5	3	4	5	1	2	2	2	4	4	1
1·0–1·1	1	4	4	1	1	1	1	0	1	1	0	2
1·1–1·2	8	4	5	3	4	2	1	1	0	1	0	3
1·2–1·3	8	8	4	1	3	4	1	0	4	2	5	4
1·3–1·4	10	6	5	5	6	6	1	0	5	1	3	9
1·4–1·5	15	13	6	5	15	4	0	1	4	5	4	9
1·5–1·6	7	14	13	11	5	6	4	1	2	4	3	15
1·6–1·7	28	12	8	12	5	4	3	0	1	4	8	13
1·7–1·8	22	22	8	7	15	9	0	2	0	2	5	18
1·8–1·9	21	28	10	8	12	4	0	0	3	0	2	33
1·9–2·0	32	14	7	3	13	3	2	2	1	0	1	25
2·0–2·1	20	12	7	8	3	5	1	0	0	1	3	25
2·1–2·2	19	9	1	4	6	4	1	1	1	1	3	15
2·2–2·3	11	4	4	4	3	1	0	0	0	1	4	11
2·3–2·4	13	4	3	5	6	0	0	0	2	1	2	8
2·4–2·5	11	0	1	4	0	0	0	0	0	2	0	8
2·5–2·6	5	7	3	2	6	0	0	1	0	0	1	7
2·6–2·7	10	0	4	2	0	0	0	0	0	0	0	7
>2·7	1	6	2	2	2	2	1	0	1	0	4	7
Total	244	172	98	91	110	56	18	11	27	30	52	220

If we smooth the columns and plot the results, we obtain curves which at once suggest the error-curve. The curves are almost symmetrical about the maximum ordinate. This could hardly have been anticipated *a priori*, and is a very fortunate circumstance from the analytical standpoint. On attempting to fit an error-curve to the observations, we find that, although it is a very fair first approximation, there is a quite definite difference, If the whole error-curve is divided by ordinates into six parts of equal area, the observed curve falls below it in the 2nd and 5th segments, and rises above it in the two extreme and two middle segments. This was found to be true in all cases.

The most natural modification is to compare the observed curve with $y=(a_0+a_4x^4)e^{-\kappa^2x^2}$ instead of the simple error-curve $y=e^{-\kappa^2x^2}$; but, as already explained, the solution arrived at in this way turns out to be an oscillating one. It is not possible to represent the observed distribution by the sum or difference of two error-curves, whether coaxial or not. The method I finally adopted was to use three error-curves, of which one principal curve represents the majority of the motions, another provides for the excess of very small proper motions, and a third provides for the excess of very large motions. A very good approximation can be obtained in this way. In the sectors, which contain less than 50 stars, a single error-curve was deemed sufficient.

Representing, then, $h_1(\mu)$ as the sum of 3 terms, thus—

$$h_1(\mu)=\sum_{r=1,2,3} \mathrm{N}_r \frac{\kappa_r}{\sqrt{\pi}} e^{-\kappa_r{}^2(\mu-\mu_r)^2},$$

or, using the corresponding decimal logarithms μ' and μ_r',

$$h_1(\mu)=\sum_{r=1,2,3} \cdot 434\ \mathrm{N}_r \frac{\kappa'_r}{\sqrt{\pi}} e^{-\kappa'_r{}^2(\mu'-\mu_r')^2};$$

the coefficients in the latter expression are given in Table III. Here $\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3$ are the number of stars in the three components into which the distribution is divided (so that $\mathrm{N}_1+\mathrm{N}_2+\mathrm{N}_3$ is approximately equal to the observed number of stars moving in the sector); μ_1', μ_2', μ_3' mark the positions of the axes of the three component error-curves; and $\kappa_1', \kappa_2', \kappa_3'$ are their moduli. Also $\kappa_r=\cdot 434\kappa_r'$ and $\mu_r=\mu_r'/\cdot 434$.

From these $\mathrm{H}(q)$ can be determined by (9), viz.

$$2\pi\mathrm{H}(q)=\sum_{r=1,2,3} \mathrm{N}_r \exp.\left(-\frac{q^2}{4\kappa_r}-\iota q\mu_r\right)$$

An example of the function $\mathrm{H}(q)$ thus calculated is given later (Table IV.); a certain constant multiplier has been ignored. q is given in degrees, but the corresponding value in radians must be used in the formula.

Table III.

Coefficients of the Formula used to represent the observed P.M.'s.

Sector.	Coefficient.	$r=1$.	$r=2$.	$r=3$.
335°–5°	N_r	171·5	18·4	29·8
	μ_r'	1·90	1·30	2·575
	$1/\kappa_r'$	·40	·40	·40
5°–35°	N_r	207·0	19·1	19·3
	μ_r'	1·875	1·40	2·525
	$1/\kappa_r'$	·47	·45	·34
35°–65°	N_r	134·6	26·2	10·0
	μ_r'	1·80	1·25	2·65
	$1/\kappa_r'$	·40	·40	·40
65°–95°	N_r	83·0	4·8	9·9
	μ_r'	1·675	0·975	2·55
	$1/\kappa_r'$	·50	·34	·57
95°–125°	N_r	65·2	5·7	16·5
	μ_r'	1·725	0·10	2·40
	$1/\kappa_r'$	·40	·40	·42
125°–155°	N_r	92·5	7·9	12·1
	μ_r'	1·75	1·05	2·425
	$1/\kappa_r'$	·45	·45	·38
155°–185°	N_r	48·0	4·2	6·0
	μ_r'	1·675	1·175	2·20
	$1/\kappa_r'$	·41	·34	·34

The remaining sectors (containing very few stars) were represented by simple error-curves.

Coefficients.

	185°–215°.	215°–245°.	245°–275°.	275°–305°.	305°–335°.
N.	18·1	11·0	27·0	30·1	52·1
μ_1'	1·55	1·625	1·45	1·55	1·70
$1/\kappa'$	·64	·76	·60	·62	·71

As explained in the introductory paragraphs, I assumed that in the direction at right angles to the line joining the vertex and antivertex, the linear motions were distributed according to Maxwell's law (after abstracting the component of solar motion in that direction). The ellipsoidal, two-drift, and three-drift theories are all in agreement with this assumption. The direction of the vertex is near $\theta = 348°$, so that the sector $\theta = 65° - 95°$ and the opposite one are those to which this assumption is applicable. If τ be the component of the solar motion in the direction $\theta = 80°$ (expressed in terms of the usual unit $\frac{1}{h}$, which depends on the mean peculiar velocities), we have

$$\frac{\text{No. of stars in the sector } 65°\text{–}95°}{\text{No. of stars in the sector } 245°\text{–}275°} = \frac{f(\tau)}{f(-\tau)}$$

where $f(\tau)$ is the quantity tabulated in my paper "Systematic Motions of the Stars" (*M. N.* lxvii. p. 37).

We find that the ratio is $\frac{98}{27}$, which leads to $\tau = {\cdot}36$. The determination is, of course, a very weak one, and there is reason to think it is a rather small value. Using the constants found from an analysis of the whole catalogue, $\tau = {\cdot}65$; but the higher proportion of Type II. stars in this particular region, which have large peculiar motions, would probably alter the unit $\frac{1}{h}$, and thus reduce the appropriate value of τ. Ultimately I started two solutions, one with the value $\tau = {\cdot}43$ and the other for $\tau = {\cdot}56$. After some trial the former was thought to be most satisfactory, and was completed; the latter was carried far enough to test the effect on the general character of the results.

Assuming then $\tau = {\cdot}43$, we have (omitting a constant multiplier)

$$\frac{g(u)}{u}du = ue^{-(u-{\cdot}43)^2}du$$

for the sector 65°–95°; and therefore putting $u = e^{\gamma}$,

$$g_1(\gamma) = e^{2\gamma}e^{-(e^{\gamma}-{\cdot}43)^2}.$$

This was tabulated and the Fourier Integral G(q) calculated (partly by formulæ and the residuum by quadratures).

We have now H(q) and G(q) for this sector, and F($-q$) is obtained by division. These are given in Table IV.; in all cases constant multipliers are disregarded.

TABLE IV.

Sector 65° *to* 95°.

q.	H (q).			G (q).			F ($-q$).		
0°	55·2	expi *	0°	9·72	expi	0°	5·68	expi	0°
25	49·4	,,	− 99·0	9·41	,,	+ 1·2	5·25	,,	− 100·2
50	36·2	,,	− 195·4	8·57	,,	+ 1·2	4·22	,,	− 196·6
75	23·2	,,	− 288·1	7·45	,,	− 0·4	3·11	,,	− 287·7
100	14·1	,,	− 381·3	6·26	,,	− 4·0	2·25	,,	− 377·3
125	8·29	,,	− 480·6	5·15	,,	− 9·4	1·61	,,	− 471·2
150	4·60	,,	− 585·9	4·16	,,	− 16·6	1·105	,,	− 569·3
175	2·48	,,	− 689·5	3·31	,,	− 25·0	·750	,,	− 664·5
200	1·24	,,	− 785·0	2·60	,,	− 34·9	478	,,	− 750
225	·519	,,	− 866·5	2·06	,,	− 46·2	·252	,,	− 820
250	·219	,,	− 940·8	1·59	,,	− 58·5	·138	,,	− 882
275	·094	,,	− 1001	1·20	,,	− 72·3	·078	,,	− 929
300	·040	,,	− 1041	·892	,,	− 87·5	·045	,,	− 953

* expi $\theta \equiv e^{i\theta}$.

Since $F(-q)$ expresses the distribution of the stars as regards distance, the same value is applicable to all the sectors. We therefore work out $2\pi G(q) = \dfrac{H(q)}{F(-q)}$ for all the other sectors and obtain by quadratures (or Fourier Analysis) for each of them

$$g_1(\gamma) = \int_{-\infty}^{\infty} G(q) e^{iq\gamma} dq.$$

The results are given in Table V. This is a table analogous to Table II. showing for each sector the number of stars for which the logarithm of the *linear* motion (in an arbitrary unit) lies between the limits γ and $\gamma + d\gamma$ corresponding to the value of γ given in the first column. The logarithms are to the base e.

TABLE V.

Values of $g_1(\gamma)$ *in the twelve sectors.*

γ.	5° to 35°.	35° to 65°.	65° to 95°.	95° to 125°.	125° to 155°.	155° to 185°.	185° to 215°.	215° to 245°.	245° to 275°.	275° to 305°.	305° to 335°.	335° to 5°.
−1·2	...	...	6·8	...	...	...	...	2·6	7·3	...	...	...
−1·0	...	...	10·2	...	2·9	2·9	4·6	2·9	9·2	7·7	10·0	...
− ·8	5·3	12·8	15·2	−4·9	4·5	8·0	5·6	3·2	10·9	9·7	12·4	3·4
− ·6	15·9	16·5	22·4	5·0	11·1	14·4	6·5	3·3	12·0	11·3	14·4	4·5
− ·4	33·8	28·6	31·9	19·8	26·1	21·5	7·0	3·3	12·1	12·4	16·0	8·2
− ·2	58·9	47·7	43·4	36·7	43·0	30·3	7·2	3·1	11·4	12·7	16·9	19·6
·0	89·1	74·6	54·4	52·7	60·0	38·3	6·9	2·9	9·9	12·1	16·9	45·0
·2	117·9	105·3	60·1	63·9	71·3	42·9	6·2	2·6	7·8	10·8	16·1	84·9
·4	138·3	129·2	54·3	65·1	71·8	40·2	5·1	2·2	5·5	8·8	14·6	131·7
·6	143·0	132·5	36·1	52·5	59·5	28·7	4·0	...	...	6·5	12·6	168·2
·8	128·6	104·3	14·9	28·0	37·6	11·7	...	...	...	...	10·3	175·4
1·0	96·0	47·4	2·9	0·9	13·5	...	...	...	...	...	7·9	140·4
1·2	33·7	−17·2	...	...	−4·3	...	...	...	...	...	...	69·6

The result of omitting constant multipliers at various stages of the work is that the figures in this table are 3·6 times too great, compared with the actual number of stars observed. The multiplier is of no theoretical importance; but in estimating the significance of any discrepancy, it may be useful to remember that 3·6 in any of the numbers entered in Table V. is equivalent to a single star in the observed data.

It now remains to reduce the results to a convenient form for illustrating the law of distribution of the linear motions. The *frequency* $D(\xi, \eta)$ of a linear motion whose rectangular components are (ξ, η) is defined by the relation :—

Number of stars with linear motion between ξ and $\xi + d\xi$, and η and $\eta + d\eta = D(\xi, \eta) d\xi d\eta$.

Turning to the definition of g, we see that the frequency is $\frac{g(u)}{u^2}$, or $g_1(\gamma)e^{-2\gamma}$, allowing for the increase in width of the sector as u increases.

To exhibit the frequency of different values of the linear motion we draw the curves of equal frequency and obtain the figure (Plate 2). The curves are drawn through points determined from Table V. on the twelve equidistant radii; all the irregularities have been allowed to remain. But no attempt has been made to determine points within a certain distance of the origin ($\gamma < -\cdot 35$), because, owing to the increase of the factor $e^{-2\gamma}$, the frequency depends on more and more slender material as we approach the origin. Within the circle the equifrequentals have been joined up by the dotted lines in the way which seemed best to complete the diagram.

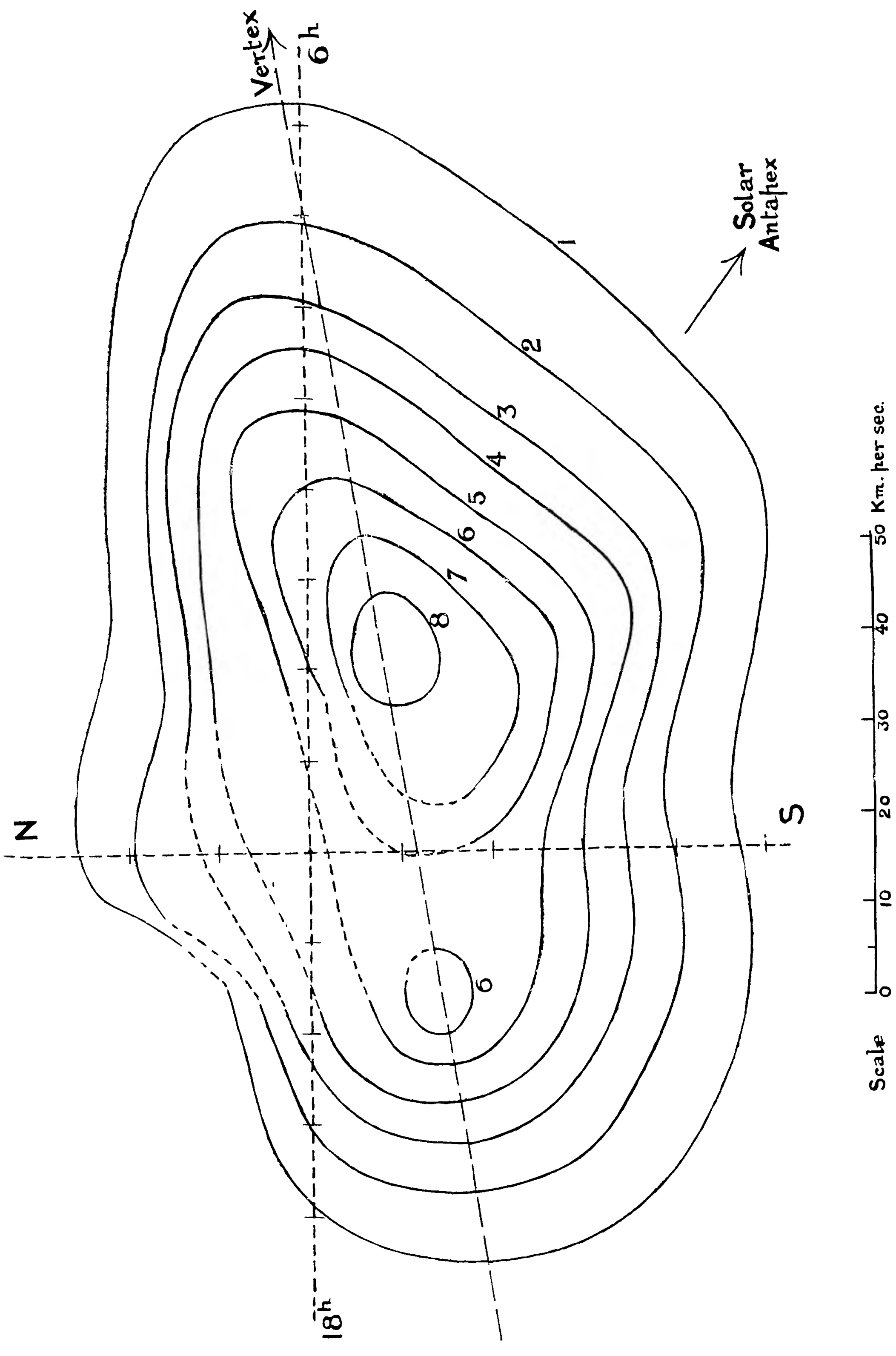
Explanation of the Diagram.

The diagram refers to the plane through the North and South Poles and the meridians of 6^h and 18^h R.A., and represents the distribution of stellar velocities in that plane, the components perpendicular to it being ignored. Any particular speed and direction is represented in the usual way by a point on the diagram, and the frequency of that motion (*i.e.* the proportionate number of stars having that motion) is indicated by contour-lines (or equifrequentals) drawn on the diagram. Starting from the outside of the figure, the successive curves correspond to the relative frequencies 1, 2, 3, 4, 5, 6, 7, 8. The most favoured values of stellar velocities are those which correspond to the points within the equifrequental line 8. The scale of velocity is given below the diagram, and, in addition, intervals corresponding to 10 km./sec. are set off along the axes of coordinates.

The curves are like the contour-lines of a region with two peaks; the height of the peak on the right would be almost 9·0, that on the left 6·4, and the pass between them 5·4. The meaning of these curves may be made clearer by some examples. Suppose it is required to compare the number of stars moving in each case with a velocity of 25 km./sec. towards (1) North Pole, (2) South Pole, (3) R.A. 6^h on the equator, (4) R.A. 18^h on the equator. Estimating from the curves near the corresponding points, just as we should estimate heights from the contours on a map, we find the relative numbers are

1·0, 5·1, 6·3, 2·7.

Again, the equifrequental line 3 passes through points corresponding to velocities 60 km./sec. towards R.A. 6^h, and 23 km./sec. towards R.A. 18^h; these values of the velocity accordingly occur with equal abundance. Similarly, a little measurement shows that a velocity 57 km./sec. towards R.A. 6^h Dec. $-45°$ should occur just as often as a velocity of 15 km./sec. in the opposite direction. Again, for every star moving towards the South Pole with velocity



DISTRIBUTION OF LINEAR VELOCITIES OF STARS.—A. S. EDDINGTON.

47 km./sec., there are five moving in the same direction with velocity 26 km./sec.

In all these examples it is understood that components of motion perpendicular to the plane of the diagram are ignored, and by the number of stars having a given motion is meant the number whose motions differ from it (geometrically) by less than a fixed small amount. The motions are taken relatively to the Sun.

It will be noticed that the origin of coordinates bears no obvious relation to the arrangement of the curves. This was to be expected, since the origin is simply the point which represents the motion of the Sun, and has no cosmical importance. If we leave out the origin, the diagram becomes perfectly general and represents the true cosmical arrangement of the motions of stars. The origin, however, has enabled us to determine the scale of the figure, since the distance of the origin from the axis of the figure is a component of the solar motion whose value in kilometres per second is known approximately.

The very noticeable elongation of the distribution represents the now well-known polarity of the motions. Its direction, indicated by the broken line, is within about 4° of that assigned in my general investigation of Boss's catalogue. Other determinations generally agree closely. As regards the cause of this polarity, the diagram clearly suggests that it is due to two distinct systems of stars, whose motions overlap to some extent, but are, nevertheless definitely distinguished by the two peaks. I have given elsewhere * diagrams of the theoretical appearance of the equifrequental lines for the ellipsoidal and the two-drift theories. The lines actually found appear to be due to a two-drift distribution; the left-hand drift contains fewer stars than the right-hand one, as former results have indicated. On a unitary theory such as the ellipsoidal hypothesis, there should be no separation into two peaks.

Exactly the same type of figure was obtained in three other investigations of these data, viz.—

1. A first approximation assuming f_1 and h_1 to be simple error-curves

2. Taking f_1 an error-curve and h_1 of the form

$$\{a_0 + a_4(\mu - \mu_0)^4\}e^{-\kappa^2(\mu-\mu_0)^2},$$

and smoothing out rather arbitrarily the wildly oscillating functions which result.

3. As in this investigation, but taking the component of solar motion ·56 instead of ·43.

The same two peaks and even the same minor irregularities showed themselves in all the investigations.

The third stream or drift makes itself evident in the bulge of the figure towards the solar antapex, and produces the principal disturbance of the symmetry of the figure; this is the more

* "Stellar Distribution and Movements," *Brit. Assoc. Report*, 1911.

remarkable because the third stream is less fully represented in this part of the sky than nearer the galactic plane, and the Orion stars which form an important part of it have been excluded. When I first became aware of this stream, I regarded it as a small additional stream, and anticipated that it would be found to consist of stars more remote than those belonging to the two star-streams. The latter suggestion has been negatived by the work of Professor Kapteyn. According to Dr. Halm's views the bulge is not so much caused by a small extra stream, but is an indication that the right-hand drift in my diagram is itself made up of two systems of about equal importance. This is quite consistent with the appearance of the diagram.

Distribution of the Stars along the Line of Sight.

I had originally intended to derive the function $f(r)$ from Professor Kapteyn's work on the distribution and luminosity of the stars. In *Astron. Jour.*, No. 566 (1904), Table VI., Kapteyn gives provisionally the information that we require; this work has since been amended by him in some points (*Amsterdam Proceedings*, x. p. 626 (1908)), but the results of this later work are not given in a form suitable for our purpose. The emendations affect chiefly the fainter and more distant stars, and do not appear to affect vitally the part of Table VI. which I am about to use. Using the 1904 paper, I find that for stars brighter than $6^m{\cdot}6$, Kapteyn's figures correspond closely to the formula

$$f_1(\lambda) = \text{const.} \times e^{-\xi(\lambda-\sigma)^2},$$

where

$$\xi = {\cdot}45$$

and σ corresponds to a parallax of $0''{\cdot}008$.

But if

$$h_1(\mu) \propto e^{-\kappa^2(\mu-\mu_0)^2},$$

we have seen that

$$\frac{1}{\kappa^2} \text{ must be greater than } \frac{1}{\xi}.$$

Hence $\frac{1}{\kappa} > 1{\cdot}49$, or, changing to decimal logarithms,

$$\frac{1}{\kappa'} > {\cdot}65.$$

(If we had taken the lower limit of magnitude to be $7^m{\cdot}6$, we should have found $\frac{1}{\kappa'} > {\cdot}62$; thus the indefiniteness of the lower limit of Boss's stars makes little difference.)

Now, the observed values of κ' in Table III. generally fail to satisfy the inequality, and it would be impossible to obtain a solution with this value of $f_1(\lambda)$. Even if, instead of splitting $h_1(\mu)$ into three error-curves, we are content with an approximate

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representation by a single error-curve, the corresponding values of $1/\kappa'$ are higher, but still fall below the limit; the values (for sectors containing more than 60 stars) are in fact—

Sector.	335°–5°	5°–35°	35°–65°	65°–95°	95°–125°	125°–155°
$1/\kappa'$	·53	·56	·56	·62	·63	·59

The physical meaning appears to be this. The logarithms of the individual proper motions deviate from the mean value for two causes: (1) because the logarithms of the individual linear motions deviate from the mean, (2) because the logarithms of the distances of the stars deviate from the mean; these two sources of deviation compound like probable errors. Now, the deviations in distance given by $\xi = \cdot 45$ are already more than sufficient to explain the observed differences of proper motion, leaving less than nothing to be attributed to cause (1), although we cannot doubt that that ought to make a large contribution. There is thus a contradiction between our observed data and the Table VI. of *Astron. Jour.* 566.

The contradiction may be partially explained by the fact that we are considering a limited region of the sky of high galactic latitude, whereas Professor Kapteyn's table refers to an average over the whole sky; the number of very distant stars in this region will be very much smaller than the average. It is difficult to make a precise calculation of how this will affect ξ; but it is hard to believe that this will account for the whole of the discrepancy.*

As I was thus unable to rely on an extraneous determination of distribution of the stars as regards distance, I calculated the functions in the way already described. The function $F(-q)$ has been given in Table IV.; and we can by equation (4) calculate the corresponding distribution of the stars as regards distance from the Sun. The result will refer to the stars of the catalogue, *i.e.* to stars brighter than a limiting magnitude which may be taken to be about $6^{m}\cdot 6$; this must not be confused with the totally different problem of the absolute density of distribution of stars in different parts of space. The latter problem has a more immediate application to the structure of the stellar universe; but there is a practical interest and importance in learning the distribution of those stars which appear bright to us. The present result is provisional only, because it is a bye-product in an investigation made for a different purpose; and it would have been possible to obtain a superior result, if this had been made my main object.†

* [Added March 9.]—It is interesting to note that in a paper just received Professor Charlier (*Lund Observatory*, "*Meddelanden*" Series 2, No. 8, p. 48) similarly finds himself unable to reconcile Professor Kapteyn's parallaxes with the proper motions of Boss's Catalogue.

† I have now in hand a direct investigation of this kind, strictly limited to stars brighter than $6^{m}\cdot 0$ (for which Boss's Catalogue is complete) and divided according to spectral type.

The result of the calculation is shown in Table VI. The values of $f_1(\lambda)$ for $\lambda = -6{\cdot}0$, $-5{\cdot}6$, etc., are set down as giving the number of stars between distances for which $\lambda = -6{\cdot}2$ and $-5{\cdot}8$, $-5{\cdot}8$ and $-5{\cdot}4$, etc. This is correct, neglecting second differences. To obtain the parallaxes in the second column, I proceeded thus:—The unit of distance is that at which the adopted unit of linear velocity will subtend the adopted unit of angular velocity. As regards the former, the component of solar motion in the direction $\theta = 80°$ was taken to be 0·43 unit. But from Campbell's value L.O.B., No. 196, this component $= 19{\cdot}5 \cos 55° = 11{\cdot}2$ km./sec. Hence our unit of linear motion $= 26{\cdot}0$ km./sec. The unit of angular motion was taken to be 0″·001 per annum. From these it follows that the unit of distance corresponds to a parallax 0″·000182, and therefore

$$\text{parallax} = 0''{\cdot}000182\ e^{-\lambda}.$$

TABLE VI.

Number of Boss's Stars within given Limits of Distance (Provisional Result) for a Region of high Galactic Latitude.

Limits of λ.	Limits of Parallax.	No. of Stars per 1000.
−6·2	″·0897	
		20·6
−5·8	·0601	
		28·7
−5·4	·0403	
		42·9
−5·0	·0270	
		81·1
−4·6	·0181	
		144·3
−4·2	·01213	
		201·4
−3·8	·00814	
		206·0
−3·4	·00545	
		144·2
−3·0	·00365	
		64·7
−2·6	·00245	
		23·3
−2·2	·00164	
		14·1
−1·8	·00110	

For an equal density of distribution of the stars the numbers in the third column of the table would form a geometrical progression increasing in the ratio 1 : 3·32. The much slower rate of increase is mainly due to the increasing proportion of more distant stars falling below the limit of magnitude of the catalogue; there is believed to be also an actual falling off of density.

By interpolation I obtain the following Table:—

TABLE VII.

Parallax.	Percentages of Stars.
$''$·10 – $''$·08	1·0
·08 – ·06	1·5
·06 – ·04	2·9
·04 – ·02	9·5
·02 – ·015	8·6
·015 – ·010	17·5
·010 – ·008	11·7
·008 – ·006	14·9
·006 – ·004	16·6
·004 – ·002	10·0
·002 – ·001	2·4

Owing to a possible cumulative error, it would hardly be fair to deduce that the 3·4 per cent. unaccounted for above is at all closely equal to the number of stars with parallax greater than 0$''$·10 or less than 0$''$·001.

It is interesting to note that 70 per cent. of the stars have parallaxes between the rather narrow limits 0$''$·020 and 0$''$·004.

Tables VI. and VII. rest on rather slender material, and, moreover, depend on the assumption $g(u) = ue^{-(u-\cdot43)^2}$ for the sector 65°–95°, which was made somewhat reluctantly. But notwithstanding these drawbacks, they should give a rough idea of the true distribution; and in the present very limited state of our knowledge even a rough approximation may be useful. Further, it seems well to exhibit the result, because it isolates, as it were, all the more doubtful part of our argument. If Table VII. can be shown to be seriously in error, the other results of the paper will need to be modified accordingly; but if this table can be accepted as giving a probable distribution of the stars, then the rest of the results, including the diagram of velocity distribution, follow quite rigorously on the sole assumption that the law of velocities is the same at all distances from the Sun.

A Tentative Explanation of the "Two Star Streams" in Terms of Gravitation. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. The following speculations started from reading the admirable summary of our modern knowledge of the stellar universe drawn up by Mr. Eddington for the last meeting of the British Association at Portsmouth. It would perhaps be difficult to point to anything quite new in the summary, but its clear presentation of the situation is essentially suggestive. I quote two important passages.

"It is believed that the great mass of the stars, excluding the Milky Way, are arranged in the form of a lens or bun-shaped system. Our Sun occupies a nearly central position, or at least a position midway between the two flattened surfaces."

"It is convenient for the purposes of explanation and for mathematical analysis to represent this bipolarity in the stars' motions as being the result of two systems of stars having become intermingled."

2. These passages will serve to illustrate an inconsistency to which we have been led. Briefly, our stellar universe is thought of sometimes as one, sometimes as two. Is there any way of removing this discordance? Let us take the first attitude and think of our system as a whole, but as finite in extent. We are led to inquire whether this great mass of stars exerts any attraction on the individuals composing it. If we regard it as a homogeneous sphere, the attraction would be towards the centre, and varying directly as the distance, according to a well-known proposition. It is probably neither homogeneous nor spherical, but as a first rude approximation let us consider the results of this central attraction.

3. The kind of result with which we are most familiar is the description of open orbits round the centre. An extreme form of such motion is a perfectly circular orbit, such as is described by a particle of a solid body. The existence of a definite axis characterising our system, viz., that perpendicular to the plane of the Milky Way, naturally suggests preferential motions round this axis.

4. But there is a wholly different extreme type of motion, in which every star would describe a rectilinear orbit through the centre, with harmonic motion. Any motion intermediate between these extreme types, the circular and the rectilinear, can be regarded as made by combining them, and will have the characteristics of both to a partial extent. Let us therefore consider the result of this second type of motion alone.

5. We could divide all the stars into two groups:

(A) Those approaching the centre,
(B) Those leaving the centre.