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Measuring the main sequence spread: a new mathematical approach

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Abstract. In this paper a new mathematical approach for measuring the main sequence spread based on the interval analysis is presented. This new approach takes into account the error bars on the observations.

1. The Astrophysical problem

It has long been suggested that the analysis of the main sequence spread, in the HR diagram, for low mass stars can give indications about the value of $\Delta Y/\Delta Z$, the relative helium to heavier elements enrichment.

Perrin et al. (1977) was the first to perform a quantitative determination of $\Delta Y/\Delta Z$ based in the assumption that this spread is mainly due to the chemical compositon variations. They found $\Delta Y/\Delta Z = 5$ in the solar neighbourhood and show that a single theoretical main sequence could fit the best observational sample.

Twenty years later Fernandes et al. (1996), using the same method of analysis of main sequence spread and a more accurate observational sample of nearby Population I stars, showed that a single theoretical ZAMS could not explain the observed spread. Nevertheless they used ground base parallaxes. So it still remains a considerably large error on M_{bol}. This result was confirmed later using HIPPARCOS data (Pagel & Portinary 1998).

The determination of $\Delta Y/\Delta Z$ are performed with the only preoccupation to constraint the observational points between the two ZAMS or isochrones, representing the extreme values on metallicity, $[Fe/H]$. No detailed discussion is currently done about the error on the position of the ZAMS or isochrones induced by observational error on T_{eff} or M_{bol}.

The aim of this paper is to present a new mathematical approach for measuring the main sequence spread based on the interval analysis, which takes into account the error bars on the observations. Attending to our main goal, in section 2 we consider several concepts from the interval analysis and also the interval Legendre polynomial introduced in Patrício et al. (2001). The numerical results are presented in section 3 using data from Lebreton et al. (1999). The diagrams are computed using the interval polynomials studied in Patrício et al. (2001) but also piecewise version of those polynomials. In order to obtain a robust envelop of all data an extension of the mentioned polynomials is introduced. Finally an analysis of the results is presented in section 4.

2. The Mathematical Solution

2.1. Basic definitions

Let us start by introducing some basic definitions.

Definition 1 *A real interval polynomial with degree n is defined by* $P_n(x) = \sum_{j=0}^{n} A_j x^{n-j}$ *with* $A_0 = 1$, $A_j = [a_j^{(1)}, a_j^{(2)}] \subset \mathbb{R}$, $j = 1, \ldots, n$.

Definition 2 *Let $P_n(x)$ be a real interval polynomial. The graph of $P_n(x)$ is given by* $G(P_n) = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : \exists p_n(x) \in P_n(x), \tilde{y} = p_n(\tilde{x})\}$

The graph of an interval polynomial can be given using the following real polynomials:

$$\begin{aligned} q_+(x) &= \sum_{j=0}^{n} q_{+,j} x^{n-j},\ r_+(x) = \sum_{j=0}^{n} r_{+,j} x^{n-j},\ q_-(x) \\ &= \sum_{j=0}^{n} q_{-,j} x^{n-j}, r_-(x) = \sum_{j=0}^{n} r_{-,j} x^{n-j} \end{aligned}$$

with $q_{+,0} = r_{+,0} = q_{-,0} = r_{-,0} = 1$, $q_{+,j} = a_j^{(2)}$, $r_{+,j} = a_j^{(1)}, j = 1, \ldots, n$, and

$$q_{-,j} = \begin{cases} a_j^{(2)}, & \text{if } n-j \text{ is even} \\ a_j^{(1)}, & \text{if } n-j \text{ is odd} \end{cases}, \qquad r_{-,j} = \begin{cases} a_j^{(1)}, & \text{if } n-j \text{ is even} \\ a_j^{(2)}, & \text{if } n-j \text{ is odd.} \end{cases}$$

Lemma 1 *Let $P_n(x)$ be a interval polynomial. The graph of P_n verifies*

$$G(P_n) = \{(x, y) \in \mathbb{R}^2 : (r_+(x) \le y \le q_+(x)\ if x \ge 0)\ or\ (r_-(x) \le y \le q_-(x)\ if x < 0)\}.$$

2.2. The interval Legendre polynomials

Let us recall the interval Legendre polynomials and their properties studied in Patrício et al. (2001).

Definition 3 *For $k \in \mathbb{N}$, let $\mathbb{L}_{n,k}(x)$ for $n \in \mathbb{N}$, be defined by*

$$\mathbb{L}_{n+1,k}(x) = \frac{2n+1}{n+1} x \mathbb{L}_{n,k}(x) - \frac{n}{n+1} \mathbb{L}_{n-1,k}(x)$$

with $\mathbb{L}_{0,k}(x) = [1 - \frac{1}{k}, 1 + \frac{1}{k}]$ *and* $\mathbb{L}_{1,k}(x) = [1 - \frac{1}{k}, 1 + \frac{1}{k}]x$. *We call $\mathbb{L}_{n,k}(x)$ the interval Legendre polynomial.*

In the next result we establish some properties of the introduced interval polynomials.

Theorem 1 *1. The interval Legendre polynomial $\mathbb{L}_{n,k}(x)$ is equal to the interval polynomial obtained from the Legendre polynomial $\ell_n(x)$ considering their coefficients multiplied by $[1-\frac{1}{k}, 1+\frac{1}{k}]$.*

2. The interval Legendre polynomials $\mathbb{L}_{n,k}(x), n \in \mathbb{N}$, satisfy:

(a) If n is even then

$$q_+(x) = \sum_{j=0}^{\frac{n}{2}} a_j(1+\frac{(-1)^j}{k})(-1)^j x^{2j}\,,\ q_-(x) = \sum_{j=0}^{\frac{n}{2}} a_j(1+\frac{(-1)^{j+1}}{k})(-1)^j x^{2j},$$

and $r_+(x) = q_+(x)$, $r_-(x) = q_-(x)$, $a_j = \dfrac{(n+2j)!}{2^n(2j)!(\frac{n}{2}+j)!(\frac{n}{2}-j)!}$;

(b) If n is odd then

$$q_+(x) = \sum_{j=0}^{\frac{n-1}{2}} a_j(1+\frac{(-1)^j}{k})(-1)^{\frac{n-1}{2}-j} x^{2j+1}$$

$r_+(x) = \sum_{j=0}^{\frac{n-1}{2}} a_j(1+\frac{(-1)^{j+1}}{k})(-1)^{\frac{n-1}{2}-j}x^{2j+1}$ *and* $r_-(x) = q_+(x)$, $q_-(x) = r_+(x)$, $a_j = \dfrac{(n+1+2j)!}{2^n(2j-1)!(\frac{n+1}{2}+j)!(\frac{n+1}{2}-j)!}$.

2.3. The minimum square approximations

Let us consider the discrete set of data $\{(x_i, Y_i), i = 1, \ldots, n\}$ where $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, is increasing and $Y_i = [y_i^{(1)}, y_i^{(2)}]$, $i = 1, \ldots, n$, are compact real intervals. In the sequel we introduced an interval minimum square approximation of the last set.

We start by introducing a new set of discrete data $\{(\hat{x}_i, \hat{y}_i), i = 1, \ldots, 2n\}$ defined by $\hat{x}_{2j} = \hat{x}_{2j-1} = x_j$, $\hat{y}_{2j} = \hat{y}_{2j-1} = y_j$, for $j = 1, \ldots, n$.

Let $\hat{\ell}(x) = \sum_{j=0}^{m} a_j^* \ell_j(x)$ be computed with the solution a_j^*, $j = 0, \ldots, m$, of the minimization problem

$$(P) \quad \min_{a_0,\ldots,a_m \in \mathbb{R}} \sum_{j=1}^{2n} (\hat{y}_j - \ell(x_j))^2 = \sum_{j=1}^{2n} (\hat{y}_j - \hat{\ell}(x_j))^2 \ \text{ with } \ \ell(x) = \sum_{j=0}^{m} a_j \ell_j(x),$$

that is, $\hat{\ell}(x)$ is the minimum square approximation for the set $\{(\hat{x}_i, \hat{y}_i), i = 1, \ldots, 2n\}$.

Definition 4 *Let $\hat{\mathbb{L}}_{n,k}(x)$ be defined by $\hat{\mathbb{L}}_{n,k}(x) = \sum_{j=0}^{m} a_j^* \mathbb{L}_{j,k}(x)$ with $a_j^*, j = 0, \ldots, m$, the solution of the minimization problem (P). We call $\hat{\mathbb{L}}_{n,k}(x)$ the minimum square interval Legendre approximation of degree n for the discrete set $\{(x_i, Y_i), i = 1, \ldots, n\}$.*

If in $\mathbb{L}_{n,k}(x)$ we replace $[1-\frac{1}{k}, 1+\frac{1}{k}]$ by $[1-\frac{1}{k_1}, 1+\frac{1}{k_2}]$ then a new interval Legendre polynomial is obtained which we denote by $\mathbb{L}_{n,k_1,k_2}(x)$. This new family of interval polynomials allow us to introduce $\hat{\mathbb{L}}_{n,k_1,k_2}(x)$ as in definition 4.

Let us assume now that the nature of the set $\{(x_i, Y_i), i = 1, \ldots, n\}$ induces a division of the initial set into several subsets and for each subset the minimum square interval Legendre polynomial is computed. Collecting all the minimum square interval Legendre polynomials, a piecewise version of the minimum square interval Legendre polynomial is obtained. We remark that the piecewise minimum square interval Legendre polynomial can be discontinuous and a continuous approximation for the initial set can be computed using for instance interpolation.

3. Numerical Results

3.1. Observational sample

Following the *sample 1* of Lebreton et al. (1999) we consider stars with metallicity ranging from -0.97 to 0.10, where individual values were based on detailed spectra analysis with a typical error of 0.1 dex. The T_{eff} is from Alonso et al. (1996) and the parallaxe is from HIPPARCOS. According to these criteria we have 33 stars, certainly one of the best accurate sample ever plotted in a HR diagram.

3.2. Results

The figures represents the observational HR Diagram plotted not using the classical astrophysics representation but the mathematical one: horizontal axis, increasing from left to right; vertical axis, increasing from bottom to top.

In Figure 1a is plotted the minimum square interval Legendre polynomial with degree 1 and $k = 2 \times 10^2$, that is the graph of $\hat{\mathbb{L}}_{1,2\times 10^2}(x)$. The graph of the minimum square interval Legendre polynomial of degree two with $k_1 = 4.5 \times 10^3$, $k_2 = 10^4$, that is the graph of $\hat{\mathbb{L}}_{1,4.5\times 10^3,10^4}(x)$ is plotted in Figure 1b. In Figure 2a is plotted the graphs of the piecewise linear minimum square interval Legendre polynomial computed with $k = 4.5 \times 10^3$ (for the first branch) $k = 1.6 \times 10^2$(for the second branch), $k = 1.7 \times 10^2$ (for the third branch). Finally the piecewise linear minimum square interval Legendre polynomial with $k_1 = 5 \times 10^3, k_2 = 6.5 \times 10^2$(for the first branch), $k_1 = 6 \times 10^2, k_2 = 8.5 \times 10^2$(for the second branch), $k_1 = 1.7 \times 10^2, k_2 = 4.5 \times 10^2$(for the third branch) is plotted in Figure 2b.

4. Discussion

The astrophysical interpretation of the results is non trivial. The higher part of the diagram (M_{bol} higher than 6.2) shows quite a narrow band because the sample includes only 4 stars in this region. On the other hand for the hot region of the diagram (M_{bol} lower than 5.2), clearly other effects than chemical

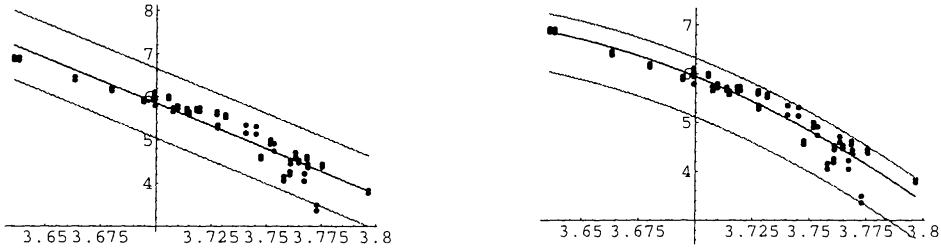

Figure 1. (a) Left: Degree 1 (see text). (b) Right: Degree 2 (see text)

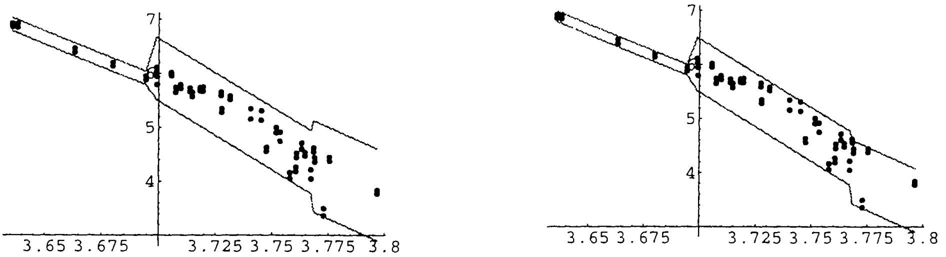

Figure 2. (a) Left: Degree 1 (see text). (b) Right: Degree 2 (see text)

composition as the evolution are already present. So the measurement of the main sequence spread can only be done in the middle part of the diagram.

The best measurement is obtained between blue and superior red line in Figure 6. For a fixed value of T_{eff} we have at least $\Delta M_{bol} \approx 0.40$. This is typically **0.15 higher** than the value found by the simply adjustment of the observational sample using ZAMS or isochrones (see Lebreton et al. 1999).

A finer analysis must be performed and take into account important effects in the hottest part of the HR diagram (evolution, rotation, overshooting, ...) in order to obtain a correct value of the spread for most of the part of the diagram.

References

Alonso, A., Arribas, S., Martínez-Roger, C. 1996 A&AS, 117, 227

Fernandes, J., Lebreton, Y., & Baglin, A. 1996 A&A, 311, 127

Ferreira, J.A., Patrício, P., & Oliveira F. 2001 J. Comp. App. Math. 136, 271

Lebreton, Y., Perrin, M.N., Cayrel, R., Baglin A., & Fernandes, J. 1999 A&A, 350, 587

Pagel, B., & Portinari L. 1998 MNRAS, 298, 747,

Patrício, F., Ferreira, J.A., & Oliveira, F. 2001 On the interval Legendre polynomils, submitted.

Perrin, M.N., Cayrel de Strobel, G., Cayrel, R., & Hejlesen, P.M. 1977 A&A, 54, 779