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Analytical properties of the R$^{1/m}$ law

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Abstract. In this paper we describe some analytical properties of the $R^{1/m}$ law proposed by Sersic (1968) to categorize the photometric profiles of elliptical galaxies. In particular, we present the full asymptotic expansion for the dimensionless scale factor $b(m)$ that is introduced when referring the profile to the standard effective radius. Surprisingly, our asymptotic analysis turns out to be useful even for values of m as low as unity, thus providing a unified analytical tool for observational and theoretical investigations based on the $R^{1/m}$ law for the entire range of interesting photometric profiles, from spiral to elliptical galaxies.

Key words: galaxies: elliptical and lenticular, cD – galaxies: fundamental parameters – galaxies: kinematics and dynamics – galaxies: photometry

1. Introduction

After its introduction as a generalization of the $R^{1/4}$ law (de Vaucouleurs 1948), the so–called Sersic law (Sersic 1968) has found a variety of applications. On the observational side, it has been used as a tool to quantify the non-homology of elliptical galaxies (see, e.g., Davies et al. 1988; Capaccioli 1989, hereafter C89; Caon et al. 1993; Young & Currie 1994; D'Onofrio et al. 1994; Prugniel & Simien 1997, hereafter PS97; Wadadekar et al. 1999). In addition, it has been applied to the description of the surface brightness profiles of galaxy bulges (see Andredakis et al. 1995; Courteau et al. 1996). One research area where the usefulness of the Sersic law as a statistically convenient description has been exploited is that of the Fundamental Plane of elliptical galaxies (Graham et al. 1996; Ciotti et al. 1996; Graham & Colless 1997; Ciotti & Lanzoni 1997; Graham 1998). On the theoretical side, it has been the focus of several general investigations (see, e.g., Makino et al. 1990; Ciotti 1991, hereafter C91; Gerbal et al. 1997; Andredakis 1998).

According to this law, the surface brightness profile is given by

$$I(R) = I_0 e^{-b\eta^{1/m}}, \tag{1}$$

where $\eta = R/R_{\rm e}$, m is a positive real number, and b a dimensionless constant such that $R_{\rm e}$ is the effective radius, i.e., the projected radius encircling half of the total luminosity associated with $I(R)$. In a broad statistical sense, it is found that bright ellipticals are well fitted by the Sersic law with m around 4, dwarf ellipticals and galaxy disks with m around 1, and finally bulges and intermediate luminosity ellipticals with $1 \leq m \leq 4$. For some galaxies, a value of m even higher than 10 has been found (e.g., see NGC 4552, Caon et al. 1993).

The projected luminosity inside the projected radius R is given by

$$L(R) = 2\pi \int_0^R I(R')R'dR' = I_0 R_{\rm e}^2 \frac{2\pi m}{b^{2m}} \gamma(2m, b\eta^{1/m}), \tag{2}$$

where (for $\alpha > 0$)

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt \tag{3}$$

is the *(left) incomplete gamma function*. The total luminosity is then given by

$$L = I_0 R_{\rm e}^2 \frac{2\pi m}{b^{2m}} \Gamma(2m), \tag{4}$$

where $\Gamma(\alpha) = \gamma(\alpha, \infty)$ is the *complete gamma function*. From the definition of $R_{\rm e}$ it follows that $b(m)$ is the solution of the following equation:

$$\gamma(2m, b) = \frac{\Gamma(2m)}{2}. \tag{5}$$

2. Asymptotic expansion

Unfortunately, Eq. (5) cannot be solved in explicit, closed form, and so it is usually solved numerically[1]. This is inconvenient for a number of observational and theoretical applications. The exact values of $b(m)$ are recorded in Table 1 for $1 \leq m \leq 10$. For the de Vaucouleurs law, $m = 4$ and $b(4) \approx 7.66924944$. Interpolation formulae for $b(m)$ have been given in the literature, namely $b \simeq 1.9992m - 0.3271$ by C89 (as reported by Graham

[1] For $m = 1$, i.e., the exponential profile, the solution can be formally expressed using the *Lambert* W function, as $b(1) = -1 - W(-1, -1/2e) = 1.678346990...$

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& Colless 1997), $b \simeq 2m - 0.324$ by C91, $b \simeq 2m - 1/3$ (for m integer) by Moriondo et al. (1998), and the "numerical solution" $b(m) \simeq 2m - 1/3 + 0.009876/m$ by PS97. These expressions provide an accurate fit in the range $0.5 \leq m \leq 10$; curiously, their leading term is *linear* in m, with a slope very close to 2. In the following we show that this behavior results from a general property of the gamma function.

Prompted by Eq. (5) we now address the following:

Problem *Solve for* x

$$\gamma(\alpha, x) = \frac{\Gamma(\alpha)}{2}, \tag{6}$$

for given $\alpha > 0$.

Because no explicit solution in closed form is available, we will focus on the asymptotic expansion of $x(\alpha)$ for $\alpha >> 1$. In fact, it is well known that in many cases asymptotic expansions turn out to give excellent approximations of the true function even for relatively small values of the expansion parameter. The starting point of our study is the asymptotic relation (see Abramowitz & Stegun 1965)

$$\Gamma(\alpha) \sim e^{-\alpha}\alpha^{\alpha}\sqrt{\frac{2\pi}{\alpha}}\left[1 + \frac{1}{12\alpha} + \frac{1}{288\alpha^2} - \frac{139}{51840\alpha^3} - \frac{571}{2488320\alpha^4} + \frac{163879}{209018880\alpha^5} + \mathrm{O}(\alpha^{-6})\right]. \tag{7}$$

This is the *Stirling formula*[2], which is known to be associated with a relative error smaller than 3×10^{-6} already for $\alpha = 2$.

Let us now introduce the sequence

$$x_n = \alpha + \sum_{k=0}^{n-1} \frac{c_k}{\alpha^k} \tag{8}$$

with $x_0 = \alpha$, so that $x_{n+1} = x_n + c_n/\alpha^n$. Here c_k are coefficients (to be determined at a later stage), independent of α. Then we start by proving the following asymptotic results, applicable for $\alpha >> 1$.

Lemma 1 *The following asymptotic relation*

$$\gamma(\alpha, x_0) \sim \frac{\Gamma(\alpha)}{2} + e^{-\alpha}\alpha^{\alpha}\sum_{k=0}^{\infty}\frac{P_k^{(0)}}{\alpha^{k+1}}, \tag{9}$$

holds, where $P_k^{(0)}$ *are rational numbers.*

The validity of Eq. (9) can be established by means of a standard asymptotic expansion (e.g., see Bender & Orszag 1978, Bleinstein & Handelsman 1986) of the integral

$$\gamma(\alpha, \alpha) = \alpha^{\alpha}e^{-\alpha}\int_{-1}^{0}\frac{\exp[-\alpha s + \alpha\ln(1+s)]}{1+s}ds. \tag{10}$$

In fact, the argument of the integral is the same as that of the integral representation of $\Gamma(\alpha)$. In both cases the stationary point for the exponent occurs at $s = 0$, but for $\Gamma(\alpha)$ the stationary point is in the middle of the domain of integration, because the integral extends to ∞ (instead, for the integral in Eq. [10] the upper limit is precisely $s = 0$). Thus, when we consider the power series expansion (in s) of the argument of the integral around the stationary point, for $\gamma(\alpha, \alpha)$ the *even* powers of s contribute exactly *one half* of their contribution to $\Gamma(\alpha)$, while the *odd* powers determine the terms in Eq. (9) associated with the coefficients $P_k^{(0)}$ (in contrast, the odd powers do not contribute to $\Gamma(\alpha)$, by symmetry). The calculation of $P_k^{(0)}$ is tedious, but straightforward.

Note that there is a "shift" of powers, by $\alpha^{1/2}$, between the two terms on the right hand side of Eq. (9). In particular, the second term is smaller by a factor $\mathrm{O}(\alpha^{-1/2})$. This already shows that $x_0 = \alpha$ is a first approximate solution to the problem set by Eq. (6).

Lemma 2 *The following asymptotic relation*

$$\gamma(\alpha, x_{n+1}) \sim \gamma(\alpha, x_n) + e^{-\alpha}\alpha^{\alpha}f(\alpha) \tag{11}$$

holds, with $f(\alpha) = \mathrm{O}(\alpha^{-n-1})$. *To leading order,* $f(\alpha) \sim c_n/\alpha^{n+1}$.

This result easily follows from the definitions of the quantities involved (Eqs. [3] and [8]), which give

$$\gamma(\alpha, x_{n+1}) = \gamma(\alpha, x_n) + e^{-x_n}\int_0^{c_n/\alpha^n} e^{-t}(t + x_n)^{\alpha-1}dt. \tag{12}$$

At this point we can proceed to prove the following theorem:

Theorem *For large (real) values of* α, *the full asymptotic expansion of the solution to the problem posed by Eq. (6) can be expressed as*

$$x(\alpha) = \alpha + \sum_{n=0}^{\infty}\frac{c_n}{\alpha^n}, \tag{13}$$

where

$$c_n = -P_n^{(n)}, \tag{14}$$

and the coefficients $P_k^{(n)}$ *can be calculated by* iteration *on the relation*

$$\gamma(\alpha, x_n) \sim \Gamma(\alpha)/2 + e^{-\alpha}\alpha^{\alpha}\sum_{k=n}^{\infty}\frac{P_k^{(n)}}{\alpha^{k+1}}. \tag{15}$$

The proof is obtained by induction. In fact, Eq. (9) shows that the statement is true for $n = 0$, with the coefficients $P_k^{(0)}$ available from the asymptotic analysis outlined in the proof of Lemma 1. If we now refer to the result of Lemma 2, with the leading order expression for $f(\alpha)$, and assume the statement (related to Eq. [15]) to hold true for x_n, we find

$$\gamma(\alpha, x_{n+1}) \sim \frac{\Gamma(\alpha)}{2} + e^{-\alpha}\alpha^{\alpha}\left[\sum_{k=n}^{\infty}\frac{P_k^{(n)}}{\alpha^{k+1}} + \frac{c_n}{\alpha^{n+1}} + ...\right]. \tag{16}$$

In other words, the statement is found to hold true also for x_{n+1}, provided $c_n = -P_n^{(n)}$, as required by Eq. (14). The method thus

[2] The derivation of this formula can be found in standard textbooks. The coefficients appearing in the asymptotic expansion of $\ln\Gamma(\alpha)$ for $\alpha \to \infty$ can be expressed in terms of the so–called *Bernoulli numbers*; see, e.g., Arfken & Weber 1995, Chapts. 5 and 10.

provides a way to systematically improve the approximation to $x(\alpha)$ by means of n steps, leading to an estimate x_n; step by step the possible presence of undesired "shifted" (odd) terms is eliminated and the process leads to the complete determination of the coefficients defining the asymptotic series (13). At a given level n of desired accuracy, the coefficients $P_k^{(n)}$ depend on the values $P_k^{(i)}$ for $i = 0, ..., n-1$.

The explicit computation yields:

$$x(\alpha) \sim \alpha - \frac{1}{3} + \frac{8}{405\alpha} + \frac{184}{25515\alpha^2} + \frac{1048}{1148175\alpha^3} - \frac{17557576}{15345358875\alpha^4} + \mathrm{O}(\alpha^{-5}). \quad (17)$$

The first two terms can be easily checked using standard general formulae for the leading terms of the relevant steepest descent asymptotic expansion.

3. Analytical properties of the Sersic law

Therefore, the first terms of the asymptotic expansion of $b(m)$ (for real m) are

$$b(m) \sim 2m - \frac{1}{3} + \frac{4}{405m} + \frac{46}{25515m^2} + \frac{131}{1148175m^3} - \frac{2194697}{30690717750m^4} + \mathrm{O}(m^{-5}). \quad (18)$$

Eq. (18) now clearly explains the value of the interpolation formulae found earlier (C89, C91, PS97); note that $4/405 = 0.0098765...$. How many terms in the asymptotic expansion are required to obtain a better representation of $b(m)$ when compared to the previously introduced interpolations?

We have computed the relative errors of the various expressions with respect to the true value of $b(m)$ (obtained by solving numerically Eq. [5] with a precision of 20 significant digits) for integer values of m in the range $1 \le m \le 10$, and the results are reported in Table 1. The first result is that using the first four terms of the expansion the true value of $b(m)$ is recovered with a relative error of 6×10^{-7} for $m = 1$, and 4×10^{-9} for $m = 10$, i.e., the asymptotic expansion so truncated performs much better than the formulae cited previously. Obviously, for larger values of m the error becomes correspondingly smaller. The second somewhat surprising result is the fact that Eq. (18) is already very accurate for m as small as 1. This allows us to include, within the reach of the present analysis, the case of exponential profiles. A third point that we have noted is that, for *fixed* m, there is an *optimal truncation* of the asymptotic expansion, beyond which, as is well known in the general context of asymptotic analysis, increasing the number of terms in the expansion does not improve the accuracy of the estimate. For example, for $m = 1$, the optimal truncation occurs at the fourth term, for which the relative error is 6×10^{-7}. For simplicity, in the following part of this Section we will record a number of interesting analytical expressions restricted to their leading order. Of course, the asymptotic analysis provided here would allow us to give explicitly any higher order term, not shown below, if so desired.

Table 1. True values and relative errors on $b(m)$ for integer values of m, using the C89, C91, and PS97 formulae. As(4) is the relative error for the asymptotic expansion given in Eq. (18), truncated to the first four terms.

m	$b(m)$	C89	C91	PS97	As(4)
1	1.67834699	4×10^{-3}	10^{-3}	10^{-3}	6×10^{-7}
2	3.67206075	2×10^{-4}	10^{-3}	10^{-4}	10^{-6}
3	5.67016119	6×10^{-5}	10^{-3}	4×10^{-5}	4×10^{-7}
4	7.66924944	6×10^{-5}	9×10^{-4}	10^{-5}	10^{-7}
5	9.66871461	2×10^{-5}	8×10^{-4}	8×10^{-6}	5×10^{-8}
6	11.6683632	2×10^{-5}	7×10^{-4}	4×10^{-6}	3×10^{-8}
7	13.6681146	6×10^{-5}	6×10^{-4}	3×10^{-6}	2×10^{-8}
8	15.6679295	9×10^{-5}	5×10^{-4}	2×10^{-6}	9×10^{-9}
9	17.6677864	10^{-4}	5×10^{-4}	10^{-6}	6×10^{-9}
10	19.6676724	10^{-4}	4×10^{-4}	9×10^{-7}	4×10^{-9}

3.1. *Total luminosity and central potential for a spherical system with an R$^{1/m}$ projected luminosity profile*

From Eqs. (4) and (18) the total luminosity is found to be

$$L = I_0 R_e^2 \frac{2\pi m}{b^{2m}} \Gamma(2m) \sim I_0 R_e^2 2\pi^{3/2} e^{1/3} e^{-2m} \sqrt{m} [1 + \mathrm{O}(m^{-1})], \quad (19)$$

where we have used the fact that $b^{2m} \sim e^{-1/3}(2m)^{2m}[1 + \mathrm{O}(m^{-1})]$. Following C91, the central potential of the spherically symmetric density distribution associated with the Sersic law is given by:

$$\Phi_0 = -G\frac{M}{L} I_0 R_e \frac{4\Gamma(1+m)}{b^m}. \quad (20)$$

Here M is the total mass of the system, and the mass–to–light ratio is taken to be constant. Thus, an asymptotic estimate at given I_0, R_e is

$$\Phi_0 \sim -\frac{GM}{L} I_0 R_e 2^{5/2} \pi^{1/2} e^{1/6} \frac{\sqrt{m}}{(2e)^m} [1 + \mathrm{O}(m^{-1})]. \quad (21)$$

We note that, for $m = 1$ and $m = 4$, a truncation of Eq. (19) to the leading term is characterized by a relative error of 5.7 per cent and of 1.5 per cent, respectively. The corresponding truncation on Eq. (21) is associated with a relative error of 8.6 per cent and 2.3 per cent.

The proper normalization of the R$^{1/m}$ profile, to be considered for a case with given scales L and R_e, is

$$I(R) = \frac{L}{R_e^2} \frac{b^{2m} e^{-b\eta^{1/m}}}{2\pi m \Gamma(2m)}, \quad (22)$$

which thus provides the useful quantity

$$I_e = I(\eta = 1) \sim \frac{L}{R_e^2} \frac{1}{2\pi^{3/2}\sqrt{m}} [1 + \mathrm{O}(m^{-1})], \quad (23)$$

so that

$$\Phi_0 \sim -\frac{GM}{R_e} \frac{2^{3/2}}{\pi e^{1/6}} \left(\frac{e}{2}\right)^m [1 + \mathrm{O}(m^{-1})]. \quad (24)$$

For $m = 1$ and $m = 4$, a truncation of Eq. (23) to the leading term is characterized by a relative error of 7.3 per cent and of 1.8 per cent, respectively. The corresponding truncation on Eq. (24) is associated with a relative error of 3.1 per cent and 0.8 per cent.

4. Conclusions

In this paper, the full asymptotic expansion for the dimensionless scale factor $b(m)$ appearing in the Sersic profile has been constructed. It is shown that this expansion, even when truncated to the first four terms as

$$b(m) = 2m - \frac{1}{3} + \frac{4}{405m} + \frac{46}{25515m^2} \tag{25}$$

performs much better than the formulae given by C89, C91 and PS97, even for m values as low as unity, with relative errors smaller than $\simeq 10^{-6}$. The use of this simple formula is thus recommended both in theoretical and observational investigations based on the Sersic law. With the aid of this formula, we have been able to clarify a number of interesting properties associated with the Sersic profile. The additional material presented in Appendix A can be compared to the simple power law R^{-2}, often used in the past to fit the photometric profiles of elliptical galaxies.

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Appendix A: remarks on a power law approximation

The surface brightness profile given in Eq. (1) is sometimes expressed as (see Eqs. [22]-[23])

$$I(R) = I_{\rm e} e^{b(1-\eta^{1/m})}. \tag{A1}$$

In this case, for a *fixed* location $\eta > 0$, we may consider an asymptotic expansion of the surface brightness profile at constant $I_{\rm e}$ for $m >> 1$. This is obtained by introducing a stretched radial coordinate $\xi = \ln \eta$ and by noting that

$$\begin{aligned} b(1-\eta^{1/m}) &= b(1-e^{\xi/m}) \\ &= -2\xi\left(1 - \frac{1}{6m} + \frac{2}{405m^2}\ \ldots\right) \\ &\quad \times \left(1 + \frac{\xi}{2m} + \frac{\xi^2}{6m^2}\ \ldots\right). \end{aligned} \tag{A2}$$

Thus, we find

$$I(R) \sim \frac{I_{\rm e}}{\eta^2}[1 + {\rm O}(m^{-1})]. \tag{A3}$$

We may recall here that the photometric profiles of elliptical galaxies have often been described in the past in terms of power laws (see, e.g., Hubble 1930). A naive inspection of the first term omitted in the expansion (A2) suggests that Eq. (A3) is adequate provided $|\xi(\xi - 1/3)/m| << 1$. Note that the term involved vanishes at $\xi = 0$ and at $\xi = 1/3$. This is an indication

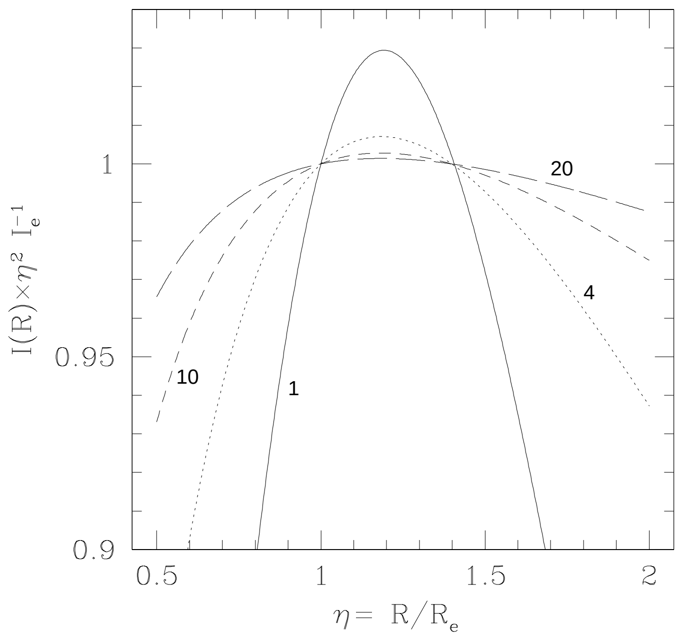

Fig. A1. Ratio between the $R^{1/m}$ profile (normalized at $I_{\rm e}$) and a properly scaled R^{-2} profile, for $m = 1, 4, 10, 20$.

that the quality of the power law approximation is asymmetric, with a modest bias to the outer region.

Consider the function $f(\xi) = b(1-e^{\xi/m})+2\xi$. The quantity $\exp(f)$ gives the ratio between the $R^{1/m}$ profile and its R^{-2} approximation. The function f diverges to $-\infty$ both for $\xi \to -\infty$ (i.e., $\eta \to 0$) and for $\xi \to +\infty$ (i.e., $\eta \to +\infty$), and has a single maximum f_M at ξ_M, defined by the relation $e^{\xi_M/m} = 2m/b$, where

$$f_M = b - 2m + 2m \ln\left(\frac{2m}{b}\right) \sim \frac{1}{36m} + {\rm O}(m^{-2}). \tag{A4}$$

Note that $2m/b$ is close to unity (in fact, $\xi_M \sim 1/6$), i.e., that ξ_M/m is small. Therefore, the power law profile is slightly underluminous with respect to the $R^{1/m}$ profile in the radial range between the effective radius and an outer radius $\xi_r \sim 2\xi_M \sim 1/3$, where $f(\xi_r) = 0$, which coincides with the outer location identified by a previous naive inspection (see comment after Eq. [A3]). Outside such radial range the power law profile is brighter than the $R^{1/m}$ profile. In such a region, where $f < 0$, we may ask how far (in radial range) Eq. (A3) applies, by studying the condition $1 - e^f \leq \epsilon$, i.e., $|f| \leq \epsilon$. We expand f for negative values of ξ around $\xi = 0$, and around ξ_r for $\xi > \xi_r$. The range of applicability of Eq. (A3) is thus constrained by the condition $(1–3\epsilon)^m \lesssim \eta \lesssim (1 + 3\epsilon)^m e^{1/3}$. This situation is illustrated in Fig. A1.

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