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AN ISOTHERMAL UNIVERSE

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ABSTRACT

We derive the metric for a simple class of isothermal inhomogeneous universes in which the nonzero pressure balances gravity. These models are spherical and static. There exists a family of such models in which the isothermal universe is a special case. These models may represent the ultimate state of an Einstein–de Sitter universe that undergoes a phase transition caused by gravitational clustering.

Subject headings: cosmology: theory — galaxies: clusters: general — gravitation — large-scale structure of the universe — methods: analytical

1. INTRODUCTION

Isothermal structures have a long history in astrophysics as an equilibrium approximation to more complicated systems that are close to a dynamically relaxed state. On the smallest scale, they were applied first to stars (Emden 1907), and then on increasing scales to globular clusters (Plummer 1911), galactic nuclei (e.g., Saslaw 1985), and clusters of galaxies (Zwicky 1957). In this paper we derive a model of an isothermal sphere consistent with general relativity on the largest possible scale, the universe itself.

The isothermal cosmological model turns out to have a remarkably simple metric. In addition to its intrinsic interest, it also has a possible (admittedly long-term) application. It may represent the asymptotic state of the standard Einstein–de Sitter ($\Omega_0 = 1$, $k = 0$) cosmological model after an infinite expansion where a hierarchial distribution of matter has clustered over the largest scales.

Although gravitational clustering of galaxies occurs in all expanding Einstein–Friedmann cosmological models, only in the Einstein–de Sitter model does it grow continuously. This is because the expansion timescale is essentially the same as the clustering timescale (e.g., Saslaw 1992). Moreover, there is no contracting phase to destroy the clustering. Essentially the Einstein–de Sitter model becomes static as $R \to \infty$. As this limit is approached, any local perturbation will cluster faster than the residual global expansion. This can lead to the growth of a centrally concentrated global distribution of matter. The universe would then undergo a phase transition from a statistically homogeneous state of translational and rotational symmetry around every point, to a state of rotational symmetry around one point only, and translational symmetry nowhere.

Since this phase transition is discontinuous and occurs only in the limit $R \to \infty$, we would not expect the isothermal model to match continuously onto the Robertson-Walker metric, and indeed we find that there is no hypersurface on which this matching holds. Another reason for this lack of matching is that at the transition, pressure becomes cosmologically significant.

Other models for spherical inhomogeneous cosmologies have been developed (see Kramer et al. 1980), particularly by Tolman (1934, 1939) and Bondi (1947). These models generally consist of pressure-free dust and are not static, or have complicated nonphysical equations of state.

In § 2 we derive the isothermal metric and then discuss its relation to a more general class of similar models and to the Robertson-Walker and Tolman-Bondi models. Section 3 discusses its applicability to the final state of a universe in the static limit as the Hubble parameter $H(t) = \dot{R}/R \to 0$.

2. THE ISOTHERMAL UNIVERSE

Isothermal metrics are characterized by a pressure gradient that balances the mutual self-gravity of its constituent particles (considered here to be idealized point galaxies). The dispersion of the particles' peculiar velocities is independent of position, with a simple equation of state

$$p = \alpha\rho , \quad (1)$$

for the pressure p and density ρ. This is independent of temperature, and α is a constant satisfying $0 < \alpha \leq 1$. The resulting configuration neither expands nor contracts, so the global solution is stationary. In the cosmological case, we take the particle motions to be nonrelativistic and ρ to be the total energy density including the rest mass energy. Thus we may be guided by the well-known Newtonian solution (e.g., Chandrasekhar 1939) for which ρ is finite in the core but decreases as r^{-2} throughout most of the configuration. The total mass and the extent of the isothermal sphere are infinite. To obtain the cosmological metric, we will make the simplifying approximation that $\rho \propto r^{-2}$ throughout the entire sphere. Although this implies a formal density singularity at the center, the mass interior to any radius r, which is the physically important quantity, remains finite, and $M \to 0$ as $r \to 0$.

Unlike the spherical, homogeneous Robertson-Walker models, which have no center (or equivalently where every spatial point can be considered to be the center), the isothermal model singles out the point with highest density at its

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center. Therefore, we start with the general static, spherically symmetric line element

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2 d\phi^2) , \quad (2)$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. The Einstein field equations are

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = -8\pi T_{ab} , \quad (3)$$

where

$$T_{ab} = (p + \rho)u_a u_b - pg_{ab} \quad \text{and} \quad u^a u_a = 1 , \quad (4)$$

is the usual energy-momentum tensor for a perfect fluid.

From equations (2)–(4), the field equations take the general form

$$8\pi\rho = \frac{\lambda'}{r} e^{-\lambda} + \frac{1}{r^2}(1 - e^{-\lambda}) , \quad (5)$$

$$8\pi p = \frac{\nu'}{r} e^{-\lambda} + \frac{1}{r^2}(e^{-\lambda} - 1) , \quad (6)$$

$$\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' - \frac{\nu'}{r} - \frac{\lambda'}{r} + \frac{2}{r^2}(e^{\lambda} - 1) = 0 . \quad (7)$$

The energy density and pressure are measured relative to the comoving 4-velocity

$$u^a = e^{-\nu/2}\delta^a_t . \quad (8)$$

Substituting the isothermal equation of state and the density distribution $\rho \propto r^{-2}$ into the field equations, we obtain the exact solution

$$e^{\nu} = Ar^{4\alpha/(1+\alpha)} , \quad (9)$$

$$e^{\lambda} = 1 + \frac{4\alpha}{(1+\alpha)^2} , \quad (10)$$

where A is an arbitrary constant, for the metric. The corresponding solutions for the energy density and pressure are

$$8\pi\rho = \frac{4\alpha}{4\alpha + (1+\alpha)^2}\frac{1}{r^2} , \quad (11)$$

and

$$8\pi p = \frac{4\alpha^2}{4\alpha + (1+\alpha)^2}\frac{1}{r^2} . \quad (12)$$

This is a remarkably simple result and is related to the solutions found by Tolman (1939) and Chandrasekhar (1972). More complicated isothermal solutions for different energy density distributions $\rho(r)$ can also be found, although they involve integrating Abel equations of the second kind, which are highly nonlinear. Thus, an isothermal sphere has a consistent interpretation as a cosmological model.

The spherically symmetric static field equations (5)–(7) also have the more general class of exact solutions with α now an arbitrary positive nonzero constant (so that $p/\rho > 0$ as $r \to 0$):

$$e^{\nu} = Ar^{4\alpha/(1+\alpha)} , \quad (13)$$

$$e^{\lambda} = a(1 - cr^b)^{-1} , \quad (14)$$

$$8\pi\rho = \frac{1}{ar^2}[a - 1 + (b+1)cr^b] , \quad (15)$$

$$8\pi p = \frac{1}{ar^2}\left[\frac{4\alpha}{(1+\alpha)}(1 - cr^b) - (cr^b + a - 1)\right] , \quad (16)$$

where

$$a = 1 + \frac{4\alpha}{(1+\alpha)^2} , \quad (17)$$

$$b = \frac{2[(1+\alpha)^2 + 4\alpha]}{(1+\alpha)(1+3\alpha)} , \quad (18)$$

and c is the arbitrary constant. These models reduce to the previous isothermal model in the special case $c = 0$. However, their equations of state are more complicated than the isothermal case. They make the important point that departures from isothermality also lead to self-consistent static cosmological models. It is interesting to observe that solutions of the type (13)–(18) also arise in the analysis of fluid spheres in spacetime dimensions greater than 4, which are important in superstring theory and supergravity (Maharaj & Patel 1996).

The question of the stability of these models now arises, and we consider several possibilities. First, in an isothermal sphere of galaxies the pressure gradient adjusts itself to hydrostatic equilibrium with the gravitational force, and so any Jeans type of instability caused by local linear perturbations would damp. The critical Jeans length is essentially the size of the sphere—in this case infinite. If local instabilities were to develop, their nonlinear interactions with surrounding galaxies would ultimately thermalize. This could change the value of α in equation (1), but the result would still be an isothermal sphere. The density distribution $\rho \propto r^2$ need not change significantly if the temperature changes.

Second, consider whether the isothermal models are stable relative to the more general class of models in equations (15)–(18). For this more general class of models to be mechanically stable requires $(\partial p/\partial \rho)_T > 0$ at all radii, r. In equations (15) and (16) we may eliminate cr^b to obtain $p \propto -\rho$ for large r, and therefore these models are not mechanically stable. In the isothermal limit $c \to 0$, or equivalently $8\pi a\rho r^2 \to 4\alpha/(1+\alpha)^2$, these models become mechanically stable everywhere. Therefore, the isothermal models are the mechanically stable subset of this more general class.

Third, consider a detailed dynamical instability that has long been known to apply to an isothermal sphere. This is its tendency to form a core halo type of structure (reviewed, e.g., Saslaw 1985). In a finite system such as a globular cluster containing N stars, the timescale for this density redistribution is approximately of order $N(G\bar{\rho})^{-1/2}$, where $\bar{\rho}$ is the average mass density. In an infinite universe, for any reasonable value of N, this global timescale is much longer than the Hubble expansion timescale $\sim(G\bar{\rho})^{-1/2}$, which itself becomes infinite for the Einstein–de Sitter universe as $\bar{\rho} \to 0$. (However, smaller clusters of galaxies will tend to evaporate.) In the central part of the isothermal universe, galaxies may redistribute energy and alter the density by forming binaries that exchange energy with their neighbors. This process is self-limiting for two reasons. First, as the binaries become smaller and more deeply bound, their effective interaction cross section with their neighbors decreases. Second, after the separation of the binary decreases to the radius of its larger member, the binary merges and ceases to transfer energy to its neighbors. This creates some very massive galaxies at the center, perhaps forming black holes. The resulting departures from an r^{-2} density profile, however, would be relatively local.

Fourth, even though detailed galaxy interactions may alter the density profile, the result may still belong to the class of isothermal static solutions, albeit without $\rho \propto r^{-2}$. It is reasonable to assume that changes in the density profile will not change the form of the equation of state of the universe, $p = \alpha\rho$. As mentioned earlier, one can, however, find an even more general and complicated class of solutions than those given by equations (13) and (14) by solving the (formidable) nonlinear Abel equations of the second kind. As long as equation (1) holds, these solutions will remain in the isothermal class of models.

Fifth, since the particle horizon of the Einstein–de Sitter universe expands as $3ct$ (here c is the velocity of light, not to be confused with c in eqs. [14]–[16]), and the universe itself expands as $t^{2/3}$, all the galaxies will have been able to interact in the limit $t \to \infty$ (and $\bar{\rho} \to 0$), so new instabilities cannot enter the horizons.

Sixth, could perturbations in the density or equation of state cause the type of instability found in the Einstein or the Lemaitre models? The important point to appreciate is that in both these other models equilibrium is attained because the cosmological constant provides a repulsive force. This equilibrium depends crucially on a constant that cannot respond to perturbations. This is the cause of their instability. That is, however, not the case for the isothermal models we consider here. Because they are in hydrostatic equilibrium (independent of a global property like the cosmological constant), any perturbations in density will produce corresponding changes in pressure so as to maintain the equilibrium. Thus, the isothermal models will be stable in this respect as well.

Since the isothermal equilibrium state forms only asymptotically, even if these instabilities were important they would need an infinite time to develop. However, the stability of the isothermal state strengthens the tendency towards its establishment.

3. DISCUSSION

We have shown that there is a remarkably simple consistent solution of the Einstein field equations that represents a spherical static isothermal cosmological model in which pressure is important. We have also found a larger family of models with different equations of state or density distributions to which the isothermal model belongs. Thus, changes in the model's density or equation of state also give consistent solutions. Isothermal spherically symmetric models can also be obtained (Dadhich 1996) by considering a metric that is conformal to a "minimally" curved spacetime rather than to flat spacetime.

As our universe expands, higher amplitude structures can form gravitationally on larger and larger scales. It is interesting to ask what the long-term outcome of this growth may be. If the universe is closed and recollapses ($\Omega_0 > 1$, $k = +1$), any large-scale structure will eventually be destroyed in the big crunch. If the universe is open and expands forever ($\Omega_0 < 1$, $k = -1$), it will expand so rapidly after redshifts $z \lesssim \Omega_0^{-1}$ that significantly larger structures will generally cease to form and the structure at $z \approx \Omega_0^{-1}$ will be frozen in (e.g., Saslaw 1992).

The flat universe ($\Omega_0 = 1$, $k = 0$), which has zero total energy and reaches $R \to \infty$ asymptotically with $\dot{R} = 0$, is the most interesting in this regard. Its gravitational clustering can continue forever. The largest linear perturbation or inhomogeneity at any time begins to grow on a timescale slightly less than the Hubble timescale $\tau_H = 2R(t)/3\dot{R}(t)$ at that time. Given enough time, all the inhomogeneities are accreted around the region of maximum density in the universe. Then discrete and collective dynamical relaxation processes among the galaxies redistribute their orbital energy until the system becomes approximately isothermal. (Here we do not consider even more speculative possibilities concerning the final end state of matter itself.)

One may consider whether the Einstein–de Sitter universe can evolve into the isothermal end state continuously. We have the following three stages: first, the Einstein–de Sitter expansion, which tends asymptotically to stationarity; second, the condensing state that leads to the development of cosmologically significant pressure; and finally the isothermal sphere with $p = \alpha\rho \propto r^{-2}$. Note that the first two stages are taken to be pressure free, while the third has a nonzero cosmological pressure. For matching of spacetimes, continuity of pressure across a specific hypersurface $r =$ constant is required by the junction conditions. This obviously cannot be achieved in our case. Rather, we should treat the discontinuous changes as phase transitions, first from the uniform expanding phase to the centrally condensed inhomogeneous state, and later from the pressure-free condensation to isothermal equilibrium with nonzero pressure.

Bonnor & Vickers (1981) have discussed the different sets of junction conditions in general relativity. They considered matching of a condensing region to a Friedmann Robertson-Walker model with vanishing pressure. Bonnor & Chamorro (1990) have shown that Tolman-Bondi models tend asymptotically to a Friedmann universe. However, these are all situations of vanishing pressure. In our situation, it is envisaged that the Einstein–de Sitter model at time $t \to \infty$ tends to an expansion-free state, and then the universe condenses into an isothermal static sphere which is described by the metric of equations (2), (9), and (10). It is clear that isothermal universes with nonzero pressure cannot be matched to the pressure-free Einstein–de Sitter model across a specific $r =$ constant hypersurface. However, this is not inconsistent with our model because we require a phase transition for condensation via galaxy clustering in the late universe that leads to the isothermal model. This phase transition is marked by developing a nonzero cosmologically significant pressure from an earlier model in which the cosmological importance of galaxy motions (pressure) could be neglected.

In conclusion, it appears that an isothermal universe is likely to be a good approximation to the distribution of galaxies in an Einstein–de Sitter universe after an infinite time has elapsed. The application of the isothermal model to our own universe is admittedly speculative. If our universe were to evolve into this isothermal cosmology, it would represent the ultimate astrophysical prediction.

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