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QUASI-MODES AS DISSIPATIVE MAGNETOHYDRODYNAMIC EIGENMODES: RESULTS FOR ONE-DIMENSIONAL EQUILIBRIUM STATES

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ABSTRACT

Quasi-modes, which are important for understanding the MHD wave behavior of solar and astrophysical magnetic plasmas, are computed as eigenmodes of the linear dissipative MHD equations. This eigenmode computation is carried out with a simple numerical scheme, which is based on analytical solutions to the dissipative MHD equations in the quasi-singular resonance layer. Nonuniformity in magnetic field and plasma density gives rise to a continuous spectrum of resonant frequencies. Global discrete eigenmodes with characteristic frequencies lying within the range of the continuous spectrum may couple to localized resonant Alfvén waves. In ideal MHD, these modes are not eigenmodes of the Hermitian ideal MHD operator, but are found as a temporal dominant, global, exponentially decaying response to an initial perturbation. In dissipative MHD, they are really eigenmodes with damping becoming independent of the dissipation mechanism in the limit of vanishing dissipation. An analytical solution of these global modes is found in the dissipative layer around the resonant Alfvénic position. Using the analytical solution to cross the quasi-singular resonance layer, the required numerical effort of the eigenvalue scheme is limited to the integration of the ideal MHD equations in regions away from any singularity. The presented scheme allows for a straightforward parametric study. The method is checked with known ideal quasi-mode frequencies found for a one-dimensional box model for the Earth's magnetosphere (Zhu & Kivelson). The agreement is excellent. The dependence of the oscillation frequency on the wavenumbers for a one-dimensional slab model for coronal loops found by Ofman, Davila, & Steinolfson is also easily recovered.

Subject headings: MHD — Sun: corona — Sun: magnetic fields — waves

1. INTRODUCTION

In ideal MHD, the quasi-mode (global mode, collective mode, virtual eigenmode, etc.,) corresponds to a singularity of the Green's function that has to be constructed during the solution of the initial-value problem with the Laplace-transform approach (Sedláček 1971; Goedbloed 1983; Zhu & Kivelson 1988). This singularity gives rise to a pole in the complex frequency plane that is independent of the spatial coordinate. Therefore, it represents an orderly "collective" mode. The collective mode is exponentially damped in time and therefore cannot correspond to a eigenmode of the Hermitian differential operator of ideal MHD. The damping of these modes is due to the resonant coupling to localized Alfvén waves. This phenomenon is analogous to the Landau damping in the Vlasov description of plasmas, where the wave energy goes into the acceleration of resonant particles. Laplace-transform approach has the advantage that causality is naturally built in, but is very complicated mathematically, and physically not always transparent.

Knowing that there is a collective response exponentially damped because of resonant absorption in ideal MHD, one can expect the existence of a resistive eigenmode that has quasi-singular behavior (resonance) around the point at which the oscillating part of the eigenfrequency matches to the local Alfvén frequency. This is shown numerically by Poedts & Kerner (1991). In the limit of vanishing plasma resistivity, the eigenfrequency converges to the ideal quasi-mode frequency, but the eigenfunction does not.

The coupling of different MHD wave modes in nonuniform media is a fundamental problem that is of interest to all plasma physicists. We discuss briefly the role of the quasi-mode in four main areas of research: CTFR, solar physics, astrophysics, and magnetospheric physics. The objective of the thermonuclear fusion research program is to confine a sufficiently hot and dense plasma for a long enough time so that the power produced by the fusion reactions exceeds the power necessary to heat the plasma. Since the Joule heating resulting from the toroidal plasma current in a tokamak is insufficient to bring the plasma into the ignition regime, supplementary heating is necessary. Resonant excitation of shear Alfvén waves has been proposed as a possible candidate. (Currently, it is not believed to be a viable mechanism.) Numerical results (Poedts, Kerner, & Goossens 1989) show that resonant absorption is extremely efficient when the plasma is excited with a frequency near that of a quasi-mode. In this case, all the energy supplied by the external source is dissipated and there is no energy circulating in the system.

In the context of solar physics, resonant absorption could be an important heating mechanism for coronal loops. Again, the fractional absorption coefficient reaches a maximum at the frequency of the global mode (Poedts, Goossens, & Kerner 1989; Steinolfson & Davila 1993). Recently, Ofman, Davila, & Steinolfson (1995) investigated the parametric dependence of the global mode frequency and the corresponding heating rate in coronal loops with the numerical solution of the linearized time-dependent MHD equations for a full compressible, low-β, resistive plasma using an implicit integration scheme. Mullan & Johnson (1995) have put these theoretical results in an astrophysical context. They argue that the occurrence of periodicity in coronal emission is consistent with the heating process, where global MHD waves are absorbed resonantly in a coronal loop in M dwarfs. Hence, resonant absorption models would be subject to an important test if periodicities could be identified, since other proposed heating mechanisms (like MHD turbulence, magnetic reconnection, etc.,) are not expected to manifest any obviously periodic properties (on short timescales). Resonance on the contrary implies the existence of a preferred frequency.

The work of Mullan & Johnson is an attempt to use theoretical results on MHD waves to study the outer atmospheres of stars other than the Sun. In essence, it is the first application of MHD wave spectroscopy to stellar atmospheres with the quasi-modes as the key tool. The solar atmosphere is the ideal place to use MHD wave spectroscopy. The various magnetic structures can support MHD waves. The observed frequencies can be used to obtain information on the distributions of density and magnetic fields inside these magnetic structures. This requires a good theoretical understanding of the relationship between the frequencies and the physical quantities. Since quasi-modes turn out to be global natural oscillations of the magnetic structures, they will probably be most easily observed. Therefore, it is important to know how the frequencies of the quasi-modes are related to the distribution of the physical quantities and the geometry of the structure.

Numerical simulations in a coronal loop model by Ofman & Davila (1996) show that quasi-modes are excited by a broadband driver in the nonlinear regime and resonantly heat the loop with a time-varying heating rate.

In the context of magnetospheric physics, ultra-low frequency waves are believed to be standing Alfvén waves on dipolar field lines, coupled to a global compressional eigenmode (Kivelson & Southwood 1985). These modes could be excited throughout the entire magnetospheric cavity in response to sudden impulses in the solar wind.

Thus, in the four areas, the quasi-mode is of fundamental importance. The quasi-mode is nothing more than a global mode (natural coherent oscillation of the system; discrete eigenmode), in which frequency lies in the continuous spectrum and therefore damped because of resonant absorption. In ideal MHD, these natural coherent oscillations can be found by solving the initial value problem with the Laplace transform approach. In resistive MHD, the quasi-modes are eigenmodes. Therefore, in this paper, the terms global mode and quasi-mode refer to the same object. Since the global mode is a natural oscillation of the inhomogeneous plasma, it is easily understood why maximum absorption occurs when driving with a global mode frequency (Balet, Appert, & Vaclavik 1982; Poedts & Kerner 1992; Steinolfson & Davila 1993), as well as why driving with a broadband spectrum gives rise to discrete Alfvén resonances in the magnetospheric cavity (Wright & Rickard 1995). It should also be noted that because of the origin of the global modes, these modes are not essentially resistive eigenmodes. The mode should also be recovered in the spectrum of other dissipative MHD operators.

In view of the obvious importance of quasi-modes for solar physics, astrophysics, and terrestrial magnetophysics, it would be very helpful to have a simple and easy to use numerical scheme for their computation. Such a scheme would make it possible to relate the frequency of the quasi-mode to the distributions of the physical quantities and the geometry of the equilibrium state. In addition, it will help to demystify the quasi-mode by identifying it as a eigenmode of the linear dissipative MHD operator. In this paper, we present a numerically easy and physically straightforward method to determine the complex frequencies of the global modes. Since the global modes are irreversibly coupled to localized Alfvén waves, dissipation (in this paper, resistivity) is taken into account to remove the singularity in ideal MHD associated with the resonance. First, the discrete eigenmode frequencies are calculated, solving an ordinary eigenvalue problem with parameter values so that there is no resonance. When a frequency is lying in the Alfvén continuum, a change in the parameters will cause resonance. This resonance gives rise to damping and a change in oscillation frequency. Assuming the imaginary part of the frequency (damping) to be small, the solution in the dissipative layer can be found analytically. This solution is used to cross the quasi-singular layer around the resonant position.

In § 2, the solution of the quasi-mode in the dissipative layer is derived. Section 3 presents a simple numerical scheme using the solutions of § 2 to determine the complex eigenfrequencies of the global modes. In § 4, the current of thought and the numerical scheme is checked with known ideal quasi-mode frequencies of a one-dimensional box model for the Earth's magnetosphere, employed by Zhu & Kivelson (1988). The dependence of the oscillation frequency on the wavenumbers in a one-dimensional slab model for coronal loops found by Ofman et al. (1995) is recovered in § 5. Section 6 gives a summary.

2. ANALYTICAL SOLUTION IN THE DISSIPATIVE LAYER

The present numerical scheme for determining the complex eigenfrequencies, $\omega = \omega_R + i\omega_I$, of the global eigenmodes (quasi-modes) is inspired by the analysis of Sakurai, Goossens, & Hollweg (1991) and Goossens, Ruderman, & Hollweg (1995) of driven resonant MHD waves in nonuniform plasmas. The analysis of Goossens et al. is based on the observation that the large values of the viscous and magnetic Reynolds numbers in solar and astrophysical plasmas imply that dissipation is very weak and can be neglected altogether except in narrow layers of steep gradients. In the case of the global modes, the dissipative terms in the MHD equations are only important in a narrow layer around the resonance position, where the real part of the eigenfrequency equals the local Alfvén frequency. Outside this narrow layer, the eigenmode is accurately described by the equations of ideal MHD.

The numerical method we use for obtaining the global eigenmodes does not require the numerical solution of the dissipative MHD equations. For our purposes, it suffices to obtain numerical (or, for a simple equilibrium state, analytical) solutions to the eigenmode equations of ideal MHD to the left and the right of the dissipative (resonance) layer. Hence, the equations of the dissipative MHD have to be solved analytically in the dissipative layer and in two overlap regions (to the left and the right of the dissipative layer), where ideal MHD is valid too. The combination of the dissipative layer and the overlap region is only a tiny fraction of the equilibrium state; hence, it is possible to use simplified versions of the dissipative MHD equations that allow analytical solutions. These analytical solutions to the dissipative MHD equations, combined with the numerical solutions to the ideal MHD, can be used to design a simple eigenvalue code for the quasi-modes.

The method can even be taken a step further. Asymptotic analysis of these analytic solutions inside the dissipative layer give connection formulae that can be used to connect the ideal solutions over the dissipative layer. This allows us to forget about the dissipative layer. The object of this section is to obtain the analytical solution in the dissipative layer and in the two overlap regions to cross the quasi-singular layer, while integrating the linearized resistive MHD equations:

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla p' + \frac{1}{\mu}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}' + \frac{1}{\mu}(\nabla \times \boldsymbol{B}') \times \boldsymbol{B} , \quad (1)$$

$$p' = -\boldsymbol{\xi} \cdot \nabla p - \gamma p \nabla \cdot \boldsymbol{\xi} , \quad (2)$$

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \nabla \times \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{B}\right) + \eta \nabla^2 \boldsymbol{B}' , \quad (3)$$

where ξ is the displacement vector, and p' and $\boldsymbol{B}'$ are the Eulerian variations of the pressure and the magnetic field, respectively. The coefficient of magnetic diffusivity is η, and γ is the ratio of specific heats. Both are assumed to be uniform. Note that the ohmic heating term in equation (2) is neglected. The approximation by the adiabatic equation is justified by numerical results of Poedts, Beliën, & Goedbloed (1994).

We consider plasmas of which the equilibrium quantities vary in only one direction. For such a kind of configurations, jump conditions are already derived for the driven problem in the asymptotic state (Sakurai et al. 1991; Goossens et al. 1995). For the eigenvalue problem for global modes, the frequencies to be determined are complex. Therefore, the dissipative solutions these authors found are no longer valid and their derivation has to be redone in order to see the effect of nonstationarity ($\omega_I \neq 0$). To illustrate the analysis, we take a cylindrical configuration with the equilibrium quantities varying only in radial direction.

As noted, we assume the resistivity to be small so that the resistive terms are only important in a narrow layer around the resonance point, r_A [the position at which the oscillating part, ω_R, of the eigenfrequency is equal to the local Alfvén frequency $\omega_A(r_A)$]. Then in the vicinity of the resonance point, equations (1), (2), and (3) reduce (after Fourier analyzing with respect to the two homogeneous directions) to two differential equations of the third order. In cylindrical geometry, they are

$$D_\eta \frac{d(r\xi_r)}{dr} = C_1 r\xi_r - C_2 rP' ,$$

$$D_\eta \frac{dP'}{dr} = C_3 \xi_r - C_1 P' , \quad (4)$$

where $D_\eta = \rho(c^2 + v_A^2)(\omega^2 - \omega_C^2)[\omega^2 - \omega_A^2 - i\eta\omega(d^2/dr^2)]$. The coefficients C_1, C_2, and C_3 are functions of the equilibrium quantities and ω, and they can be found in the paper by Sakurai et al. (1991). Series expansions around $s = r - r_A = 0$ of D_η, C_1, C_2, and C_3 give simplified versions of equation (4) that are valid in the interval $[-s_A, s_A]$, where the linear Taylor polynomial is a valid approximation of $\omega^2 - \omega_A^2$.

Hence, s_A has to satisfy

$$s_A \ll \left|\frac{2\omega_A^{2\prime}}{\omega_A^{2\prime\prime}}\right| ,$$

where the prime denotes the derivative with respect to r. Hence, close to the Alfvén resonance point, equation (4) reduces to

$$\left(2i\omega_A \omega_I + s\Delta - i\omega_A \eta \frac{d^2}{ds^2}\right)\frac{d\xi_r}{ds} = \frac{g_B}{\rho B^2} C_A(s) \quad (5)$$

$$\left(2i\omega_A \omega_I + s\Delta - i\omega_A \eta \frac{d^2}{ds^2}\right)\frac{dP'}{ds} = \frac{2f_B B_\phi B_z}{\mu\rho r_A B^2} C_A(s) \quad (6)$$

$$\left(2i\omega_A \omega_I + s\Delta - i\omega_A \eta \frac{d^2}{ds^2}\right)\frac{dC_A}{ds} = 0 , \quad (7)$$

where the equilibrium quantities are evaluated at the ideal resonance position, and

$$C_A = g_B P' - \frac{2f_B B_\phi B_z}{\mu r_A} \xi_r , \quad \Delta = \left[\frac{d}{dr}(\omega^2 - \omega_A^2)\right]_{r=r_A} ,$$

$$f_B = \frac{mB_\phi}{r} + kB_z , \quad g_B = \frac{mB_z}{r} - kB_\phi .$$

It is important to note that we assume that the damping caused by resonant absorption is weak: $|\omega_I| \ll |\omega_A|$, what has to be checked a posteriori.

The highest derivative terms are multiplied with the electrical resistivity. Thus, for very high Reynolds numbers, equations (5), (6), and (7) represent a singular perturbation problem. Resistivity is only important in a narrow layer that is of the order

$$\delta_A = \left(\frac{\omega\eta}{|\Delta|}\right)^{1/3} ,$$

where $\delta_A \ll s_A$ because of the high Reynolds number considered here.

Now it is convenient to introduce a new scaled variable, $\tau = s/\delta_A$, which is of the order 1 in the dissipative layer. With this new variable, equations (5), (6), and (7) for ξ_r, P', and C_A, respectively, take the following form:

$$\left[\frac{d^2}{d\tau^2} + i\,\mathrm{sign}\,(\Delta)\tau - \Lambda\right]\frac{d\xi_r}{d\tau} = i\frac{g_B}{\rho B^2|\Delta|} C_A , \quad (8)$$

$$\left[\frac{d^2}{d\tau^2} + i\,\mathrm{sign}\,(\Delta)\tau - \Lambda\right]\frac{dP'}{d\tau} = i\frac{2f_B B_\phi B_z}{\rho B^2 \mu r_A |\Delta|} C_A , \quad (9)$$

$$\left[\frac{d^2}{d\tau^2} + i\,\mathrm{sign}\,(\Delta)\tau - \Lambda\right]\frac{dC_A}{d\tau} = 0 , \quad (10)$$

where $\Lambda = 2\omega_I \omega_A/\delta_A|\Delta|$. In the Appendix, analytical solutions to these equations are derived using a Fourier transform technique. C_A is a conserved quantity across the resonant layer. The solutions for the global modes in the dissipative layer are then given by

$$\xi_r(\tau) = -\frac{g_B C_A}{\rho B^2 \Delta} G(\tau) + cte_{\xi_r} ,$$

$$P'(\tau) = -\frac{2f_B B_\phi B_z C_A}{\rho B^2 \mu r_A \Delta} G(\tau) + cte_{P'} , \quad (11)$$

where

$$G(\tau) = \int_0^\infty \frac{e^{iu\,\mathrm{sign}\,(\Delta)\tau - \Lambda u} - 1}{u} e^{-u^3/3}\,du .$$

The same kind of solution has been found by Ruderman, Tirry, & Goossens (1995) for resonant damped Alfvén

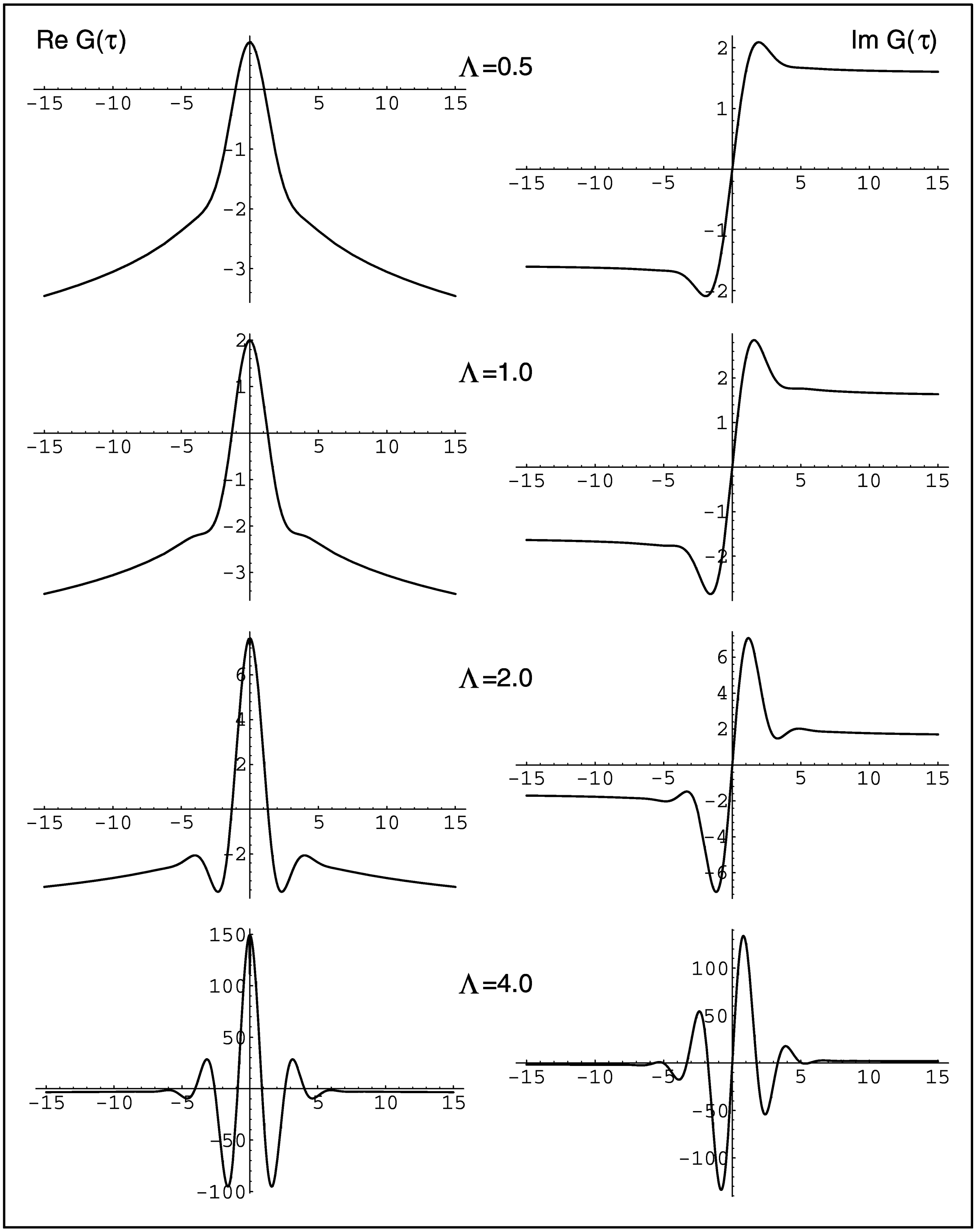

FIG. 1.—Real (*left-hand column*) and imaginary (*right-hand column*) part of the $G(\tau)$ function for different values of Λ (0.5, 1, 2, 4)

surface waves in a resistive and viscous plasma with incompressible motions. The $\nabla \cdot v = 0$ assumption implies that the Alfvén and cusp singularities in ideal MHD coincide. Therefore, the surface wave is damped becaue of Alfvén and cusp resonances. In this paper, we will consider the coupling between global compressional modes and localized Alfvén waves.

Putting $\Lambda = 0$ gives the solution in the vicinity of the resonance point for the driven problem in the asymptotic state. $\Lambda \neq 0$ shows us the effect of nonstationarity ($\omega_I \neq 0$) on the decaying normal modes. In order to explain what we mean by nonstationarity, let us assume that we are in a coordinate system moving with the wave-phase velocity. Then, in the driven problem, the solution that describes resonant absorption of MHD waves is independent of time. Consequently, the dissipative layer in the driven problem can be called a stationary dissipative layer. In studying solutions of the MHD equations in the form of normal decaying modes, we deal with a nonstationary situation, even in a coordinate system moving with the wave-phase velocity. The reason is, of course, the exponential decrease of the wave amplitude owing to resonant absorption.

When the effect of dissipation is steadily decreased in comparison to the effect of nonstationarity (the value of Λ becomes larger), the solutions become more and more oscillatory and their amplitudes grow very rapidly in the dissipative layer. This is clearly shown in Figure 1, in which the real and imaginary part of the function, $G(\tau)$, are plotted for different values of Λ. This kind of oscillatory behavior for the resistive global mode was found numerically by Poedts & Kerner (1991). This picture completes the temporal evolution for a plasma system that is externally driven at the real part of the frequency of the ideal quasi-mode. In an initial phase, the plasma response yields phase mixing and the amplitudes of the fields grow. After a while, the system has attained a stationary state in which all physical quantities oscillate harmonically at the frequency imposed by the external driver. The energy supplied by the external source is balanced exactly by the Ohmic dissipation in the resonant layer. The behavior in the dissipative layer is analytically described by Goossens et al. (1995), and their solutions are presented in this section with $\Lambda = 0$. When the driver is switched off, the global mode damps out. In this final phase, the plasma response phase mixes essentially to the form of the resistive eigenmode with the oscillatory behavior in the resistive layer, where the finest perturbation scales are limited by the resistivity: the smaller the resistivity, the finer the radial length scales in the resonance layer and the higher the amplitudes. Hence, kinetic effects or effects caused by finite electron inertia could become important (Rankin, Samson, & Frycz 1993; Mann, Wright, & Cally 1995; Wright & Allan 1996) accompanied with the breakdown of the one fluid MHD, unless nonlinear effects can broaden the resonance (Rankin et al. 1993; Poedts & Boynton 1996)

3. A SIMPLE NUMERICAL SCHEME

To determine the complex eigenfrequencies of global modes (corresponding to the ideal quasi-modes) in resistive MHD, there are several possibilities. The first, but probably the most difficult one, is to use a numerical code that integrates the resistive MHD equations in the whole volume of the equilibrium state to resolve the resistive spectrum of the system. This is done, for example, by Poedts & Kerner (1991) for a one-dimensional cylindrical configuration. Such a code should be pollution free, which means that it does not generate any spurious eigenfrequencies. In addition, it also has to be able to handle steep gradients and narrow layers. Poedts & Kerner calculated the resistive spectrum of a plasma column surrounded by a perfectly conducting wall with vacuum in between (as model for a tokamak). The quasi-mode that they found originates from an external kink instability that is stabilized by the perfectly conducting wall with oscillation frequency in the ideal Alfvén continuous spectrum.

A second method, which was introduced by Balet et al. (1982) in the study of MHD waves also in fusion plasmas, makes use of the observation that the global mode is a natural oscillation mode of the system. In their MHD study on Alfvén wave heating of low-β plasmas, Balet et al. considered the situation in which they imposed an initial displacement of the plasma column and then let it oscillate freely. They made a Fourier analysis of the radial displacement at different radii. For each radius, a maximum of the amplitude was found; and this maximum occurs around the same frequency (with an uncertainty of about 1%). Balet et al. viewed this as an indication of a global motion of the plasma column. They also noted that their (ideal MHD) spectral code does not show any evidence of an eigenmode corresponding to this frequency.

In this way, Steinolfson & Davila (1993) and Ofman et al. (1995) determined the oscillation frequencies of global modes in a one-dimensional slab model for coronal loops. This method is rather involved from a computational point of view and does not give any information on the accuracy of the eigenvalue and the eigenvector. In addition, it uses a property of the quasi-mode that has to be proven a priori. The logical way is to compute the quasi-mode as an eigenmode of the dissipative system (and then to identify the dominant contribution in the response in time as the quasi-mode). The numerical scheme that we present is mostly based on the two main characteristics of the quasi-mode. First, the quasi-mode is a global eigenmode of the resistive MHD operator (or other dissipative MHD operators). Second, it is a global discrete eigenmode coupled to a localized Alfvén wave, where the oscillating part of the frequency matches the local Alfvén frequency. At this position, there will be a quasi-singular layer where we know the behavior analytically.

Hence, the scheme searches for complex frequencies in such a way that the boundary conditions are satisfied when integrating the ideal counterparts of equation (4), while the resonance layer is crossed with the known analytical solution given in § 2.

A good starting value is given by the real eigenfrequency of the discrete eigenmode in the case that parameters are set in such a way that there is no resonance. A change in the parameter values may cause resonance whenever the frequency of the global discrete eigenmode lies within the range of the continuous spectrum; the eigenfrequency will shift into the lower half of the complex frequency plane. In this way, an easy general parametric study is possible. In what follows, we will illustrate our reasoning for cold plasmas and compare with known results.

As a first example, we consider a cold ($p = 0$) cylindrical configuration (as model for a coronal loop) with a straight uniform magnetic field. The density profile is given in dimensionless form by

$$\rho(r) = 0.1 + 0.9e^{-r^4} .$$

Steinolfson & Davila (1993) and Wright & Rickard (1995) employed the same variation in density in their MHD cavity (in Cartesian coordinates). The latter authors explained very well the physical origin of the quasi-mode, which has been the guidance for our numerical scheme to determine the quasi-mode frequencies. At the axis $r = 0$, the displacement ξ_r should remain finite. Since we are looking for trapped fast body waves, we require that $\xi_r = 0$ at $r = 5$ (note that there are two turning points; the resonance lies in the evanescent tail).

In the case that the poloidal wavenumber m equals zero, there is no singularity in the ideal counterparts of equation (4). The frequencies of the trapped fast body waves are real.

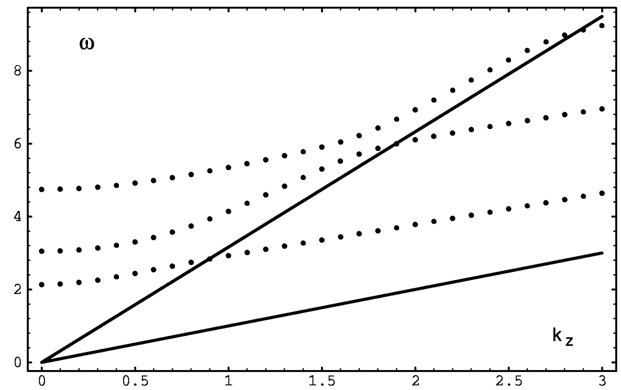

FIG. 2.— Eigenfrequencies of the first three fast body waves (*dotted lines*) in a cylindrical cold plasma, and the lower and upper bound of the Alfvén continuum (*full lines*) in function of the axial wavenumber k_z, in case of no resonance.

They are shown in Figure 2 for the first three harmonics in function of the axial wavenumber k_z, together with the upper and lower bound of the Alfvén continuum. Note that the modes above the continuum are not trapped (but they are not of our interest). For $k_z = 1.5$, the fundamental fast body wave has its frequency in the continuum. Hence, for $m = 1$ the resonance will damp the fast body wave and change its oscillation frequency. In order to find the damping and the change in oscillation frequency, we have steadily increased m from 0 to 1. Physically, this has no meaning but it is a mathematically straightforward manner to determine the complex eigenfrequency for the case in which $m = 1$. The shift into the complex frequency plane is shown in Figure 3 for different values of magnetic Reynolds number (10^7, 10^8, 10^9, 10^{10}). This picture shows that the eigenfrequency of the global mode tends to a limiting value in the limit of vanishing resistivity. This is in agreement with the results of Poedts & Kerner (1991). Our a priori assumption that the damping due to resonant absorption is weak, is also confirmed.

Let us note that we could not increase m much further beyond 1. The reason is that for larger values of m, the resonance position shifts toward the interval at which the Alfvén frequency is almost constant ($\Delta \to 0$). Here the theory stops to be valid because the resonance layer is not localized anymore.

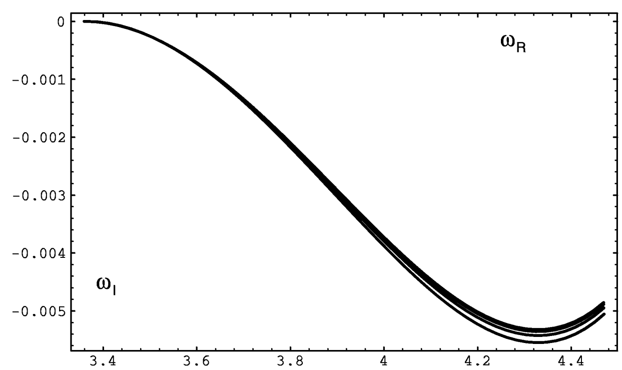

FIG. 3.—When m goes from 0 to 1, the frequency of the global mode shifts into the lower half plane. Different lines corresponding with different values of magnetic Reynolds number (10^7, 10^8, 10^9, 10^{10}) tend to a limiting line.

The present paper extends part of the results obtained by Cally (1986). Cally derived numerically dispersion relations for leaky and nonleaky oscillations in magnetic flux tubes (photospheric flux tubes, Hα fibrils, and coronal loops) with uniform internal and external media. When the discontinuity between the coronal loop and the external medium is replaced by a continuous transition, the nonleaky fast body waves (corresponding with Cally's B_+^+ modes) can resonantly couple to localized Alfvén waves. This will drastically change their dispersion relations. This change in eigenfrequency can easily be found with the method presented in this paper, as shown in the above described model for a coronal loop.

4. THE CORRESPONDENCE WITH THE IDEAL QUASI-MODE

In ideal MHD, the quasi-mode corresponds to a temporal dominant, global, exponentially decaying response to an initial perturbation. The frequency is found as a zero of the conjunct (which is independent of the space coordinate) in the unphysical lower half plane while solving the initial value problem with the Laplace transform approach. This technique is explained in an excellent paper by Sedláček (1971) about small amplitude electrostatic oscillations in a cold plasma with continuously varying density. The disadvantage of this method is that one has to be able to solve the differential equation analytically for the whole spatial region. Zhu & Kivelson (1988) modified the method in such a way that it requires an analytic expression only in the vicinity of the singularity of the differential equation.

To be sure of the validity of our current of thought and numerical scheme, the frequencies that we find should correspond to the zeros of the conjunct constructed by Zhu & Kivelson (1988). They considered a one-dimensional box model for the terrestrial magnetosphere: a cold plasma contained in a rectangular box embedded in a uniform magnetic field. One can take a fixed boundary condition for the plasma displacement normal to the magnetopause. For the inner boundary condition (in the inner magnetosphere), Zhu & Kivelson also took fixed boundary conditions. We consider the same Alfvén velocity profile in the x-direction viz.,

$$v_A^2(x) \sim \frac{1}{x} , \quad \text{with } x \in [0.1, 10] .$$

The z-direction is along the magnetic field, and the y-direction represents the azimuthal direction.

When the azimuthal wavenumber k_y equals zero, there is no resonance and the frequencies of the global compressional eigenmodes are real. They are shown in Figure 4 in function of the axial wavenumber k_z, together with the lower and upper bound of the Alfvén continuum for the first five global mode harmonics. For $k_z = 1$, these harmonics (and a lot more) have their frequency within the range of the Alfvén continuum. Hence, for $k_y \neq 0$ they will couple to localized Alfvén waves. Their behavior in the resonance layer is described by

$$D_\eta \frac{d(\xi_x)}{dx} = -C_2^* P' ,$$

$$D_\eta \frac{dP'}{dx} = C_3^* \xi_x , \tag{12}$$

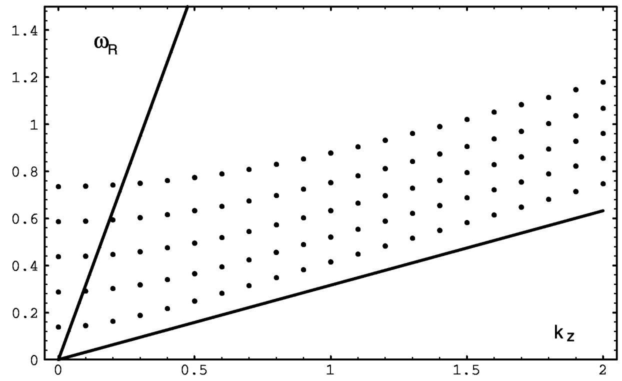

FIG. 4.—Eigenfrequencies of the first five global mode harmonics (*dotted lines*) in a one-dimensional box model for the terrestrial magnetosphere and the lower and upper bound of the Alfvén continuum in function of the axial wavenumber k_z, in case of no resonance.

where C_2^* and C_3^* are the Cartesian counterparts of C_2 and C_3, respectively.

Figure 5 shows the shift into the lower half of the frequency plane (*full line*) along with the results of Zhu & Kivelson (*dots*) for the first two global harmonics when k_y is steadily increased from 0 to 1. For small k_y, there is a discrepancy. This is owing to the fact that, while doing the local analysis around $x = x_A$, a term proportional to $\Delta s + 2i\omega_A \omega_I$ is neglected in comparison with the term proportional to k_y^2 in the right-hand side of equation (12), as described in § 2. In the case of cylindrical geometry, we did not take care of this problem because the azimuthal wavenumber m should be integer.

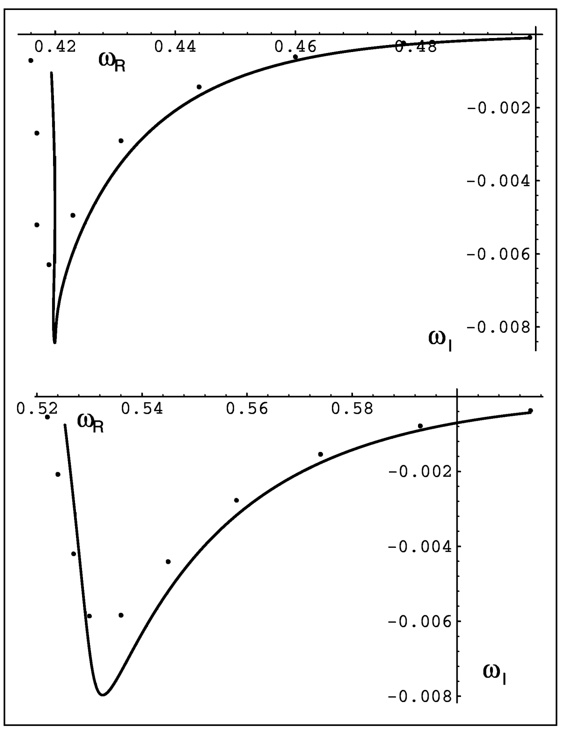

FIG. 5.—Shift of the eigenfrequencies of the first two global mode harmonics into the lower half plane (*full line*) when k_y goes from 0 to 1. For small k_y there is a discrepancy with the results of Zhu & Kivelson (*filled circles*) because of the neglect of a term proportional to $2i\omega_A \omega_I + s\Delta$, in comparison with a term proportional to k_y^2 in the analysis presented in § 2.

When keeping this term, equation (12) reduces close to the resonance point to

$$\left(2i\omega_A \omega_I + s\Delta - i\omega_A \eta \frac{d^2}{ds^2}\right) \times \frac{d\xi_x}{ds} = -\frac{1}{\rho_A v_A^2}(2i\omega_A \omega_I + s\Delta - k_y^2 v_A^2)P' \quad (13)$$

$$\frac{dP'}{ds} = 0 \, . \quad (14)$$

In function of the stretched variable $\tau = s/\delta_A$, the behavior in the dissipative layer is governed by

$$\left[\frac{d^2}{d\tau^2} + i \, \mathrm{sign}\,(\Delta)\tau - \Lambda\right] \times \frac{d\xi_x}{d\tau} = -\frac{i}{\rho_A v_A^2 |\Delta|}(2i\omega_A \omega_I + \tau\delta_A \Delta - k_y^2 v_A^2)P' \quad (15)$$

$$\frac{dP'}{d\tau} = 0 \, . \quad (16)$$

With the aid of the Appendix it is easy to verify that the solutions in the dissipative layer remain unaltered except for an additional linear term, which is proportional to δ_A. Hence, for small k_y this term has to be taken into account. As a result, we obtain

$$\xi_x = -\frac{k_y^2}{\rho_A \Delta} C_A^* G(\tau) - \delta_A C_A^* \tau + cte_{\xi_x} \, , \quad (17)$$

$$P' = C_A^* \, . \quad (18)$$

With the use of this solution to cross the quasi-singular layer, the agreement with the frequencies of the global modes found by Zhu & Kivelson is excellent, as shown in Figure 6.

5. DEPENDENCE ON THE WAVENUMBERS

Since the importance of global modes in coronal loop heating is well established (Ofman, Davila, & Shimizu 1996, and references therein), Ofman et al. (1995) investigated the scaling of the global mode resonant heating rate with the wavenumbers and the parametric dependence of the global mode frequency. They used a time-dependent code with an implicit integration scheme for the (full compressible, zero-β) resistive MHD equations. To determine the oscillation frequency of the quasi-mode, they used the physical fact that the quasi-mode manifests itself as a natural oscillation of the system: the frequency response to an initial disturbance with an arbitrary frequency spectrum will be sharply peaked at the frequency of the quasi-mode. By fast Fourier transforming the results from the time-dependent code at selected spatial locations, they could calculate the oscillating part of the frequency.

Ofman et al. (1995) considered a one-dimensional slab model for a coronal loop as follows: cold plasma ($p = 0$), straight uniform magnetic field in the z-direction, and a density profile symmetric with respect to $x = 0$:

$$\rho_0(x) = 0.1 + 0.9e^{-x^4} \, ,$$

with zero boundary conditions. The boundaries were taken far enough (from the center, $x = 0$) so that they do not influence the trapped waves inside the loop. They had to use

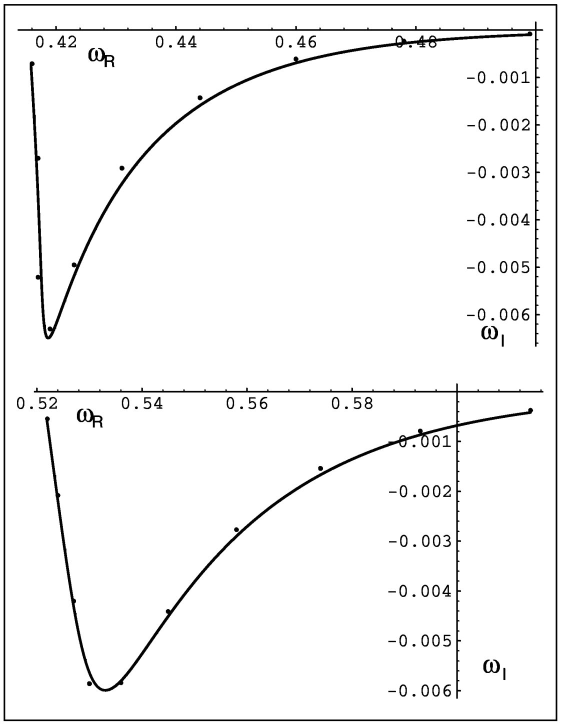

FIG. 6.—Shift of the eigenfrequencies of the first two global mode harmonics into the lower half plane (*full line*) when k_y goes from 0 to 1. Using the corrected solution in the dissipative layer to cross the quasi-singular layer in the numerical scheme the agreement with the results of Zhu & Kivelson (*filled circles*) is excellent.

a nonuniform grid spacing in order to well resolve the dissipative layer for rather low magnetic Reynolds numbers. Figure 7 shows the results of our eigenfrequency calculations for the same set of parameters as used by Ofman et al. (1995) in their Figure 7. If we compare these with their numerical values, we see an amazing maximal difference of only 0.4%. Therefore, there is no doubt that the global modes that Ofman et al. found in their fast Fourier transforms of the free oscillations are eigenmodes of linear dissipative MHD. It also indicates how dominantly the quasi-mode is present.

Ofman et al. (1995) compared their results for the variation of frequency as a function of the wavenumbers with the

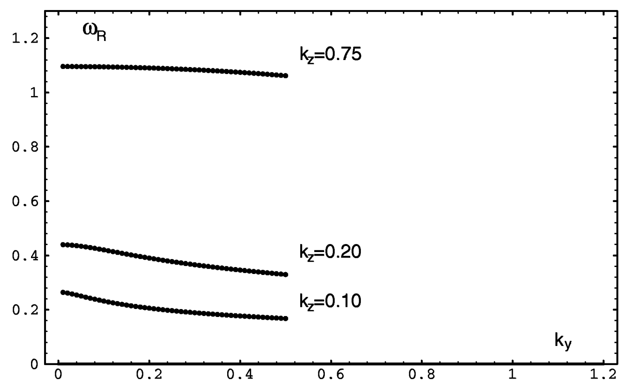

FIG. 7.—Dependence of the oscillation frequency of the global mode on the wavenumbers in the one-dimensional slab model for a coronal loop.

analytic expression obtained by Ionson (1978). For a fixed k_y, the oscillation frequency of the quasi-mode depends approximately linearly on k_z. Ionson's analytic expression gives a good representation of the numerical results for $k_y = 0.75$ by Ofman et al. A careful reader might be surprised by this agreement, since Ionson derived his result for a decaying surface mode on a "thin" interface with two uniform plasmas on both sides, while Ofman et al. considered trapped fast body waves in an Alfvén speed well. Since both the mathematical equation (in the limit of a cold plasma) and the Alfvén speed profile are similar, one can expect that there would be agreement for a certain value of k_y. The linear dependence of the oscillation frequency on k_z can also be seen in Figure 2, where the eigenfrequencies of the fast waves are plotted in function of k_z. This linear dependence was found in a cylindrical geometry (see § 3), so that we can state that this scaling law found in a Cartesian one-dimensional slab model for a coronal loop by Ofman et al. remains valid in the corresponding cylindrical model. But if we compare Figure 7 (for the slab model) and Figure 3 (for the cylindrical model) concerning the azimuthal dependence, we see that the oscillation frequency of the quasi-mode in the slab model decreases with increasing k_y, while in the cylindrical model it increases with increasing m (remember that in Fig. 3, m increases from 0 to 1 along the line). We have to note that Figure 7 is produced with values of $k_z \leq 0.75$, whereas Figure 3 has $k_z = 1.5$. But even for $k_z = 1.5$ in the slab model, the quasi-mode oscillation frequency decreases (very slowly) with increasing k_y. For higher values of k_z (> 2) it becomes a very slowly increasing function. Hence, the geometry can play a fundamental role in the scaling laws, so that we can conclude that one has to be careful when comparing scaling laws found in simplified models with observations.

6. SUMMARY

In this paper, quasi-modes are computed in models for solar coronal loops and the terrestrial magnetosphere as eigenmodes of the linear dissipative MHD equations. In this way, we demystify the "global mode calculations" of Steinolfson & Davila (1993) and Ofman et al. (1995). The eigenmode computation is carried out with a simple numerical scheme, which is based on analytical solutions to the dissipative MHD equations in the quasi-singular resonance layer.

The scheme is mostly based on two main characteristics of the global mode. The global mode manifests itself as a natural coherent oscillation: it is a eigenmode of the resistive MHD operator. The global mode is a discrete eigenmode, weakly damped due to resonant absorption. Hence, it is a good starting point to determine the real eigenfrequencies of the discrete eigenmodes in the case in which there is no resonance, but in which they are lying within the range of the continuous spectrum. When the parameters are changed in such a way that resonance appears, one can easily follow the change in oscillation frequency and the damping of the global eigenmode due to resonant absorption.

For small values of the resistivity, the dissipation layer around the resonance point will be narrow. The behavior in this narrow resonance layer is found analytically. The analytical solution shows clearly the effect of nonstationarity ($\omega_I \neq 0$) and is used in the numerical scheme to cross the quasi-singular layer around the resonance point where the oscillating part of the frequency matches the local Alfvén frequency.

We have illustrated the current of thought and the numerical scheme with two different geometries of cold plasmas. Those illustrations show clearly the correspondence of the resistive global eigenmode damped by resonant absorption with the ideal quasi-mode. The ideal quasi-mode is found as a temporal dominant but exponentially decaying response while solving the initial value problem with the Laplace transform approach. In the limit of vanishing resistivity, the eigenfrequency tends to a limiting value: the quasi-mode frequency, which cannot be an eigenfrequency of the Hermitian ideal MHD operator. By steadily increasing the value of the azimuthal wavenumber starting off from zero (which in the configurations we have used corresponds to no resonance), the shift of the eigenfrequency of the global mode into the lower half plane is clearly demonstrated. The damping rate reaches a maximum at a certain azimuthal wavenumber and then decreases again, which is in agreement with results by Zhu & Kivelson (1988) and Allan, White, & Poulter (1986). It is also shown that the oscillation frequency of the quasi-modes in coronal loop models depends linearly on k_z (wavenumber parallel to the magnetic field), and that the dependence on the azimuthal wavenumber is totally different for slab models and cylindrical models.

In keeping with the underlying unity of resonant absorption in all areas of plasma physics, a parametric study is not done in this paper, but the dependence of the global mode frequency on other parameters (twist in the magnetic field, strength of the mass flow, density distribution, etc.,) in the different possible application fields (solar coronal loops, magnetospheric cavity, and tokamak plasmas) can now be investigated easily.

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APPENDIX

In this appendix, we derive analytical solutions of the equation in the following form:

$$\left[\frac{d^2}{d\tau^2} + i\ \mathrm{sign}\ (\Delta)\tau - \Lambda\right]\Psi(\tau) = \mathrm{RHC}\ , \tag{A1}$$

where RHC is the right-hand side constant. The boundary conditions are set by requiring that the solutions of the corresponding equations (8), (9), and (10) match the ideal solutions valid outside the dissipative layer. For the driven problem, Goossens et al. (1995) have determined the solutions in integral form using a sort of Laplace transform technique. Their analysis can be followed again, but here we present a shorter alternative approach. The value $\delta_A \ll s_A$ implies that the interval of validity of the simplified versions of the dissipative MHD equations embraces the dissipative layer and, in addition, contains two overlap regions to the left-hand side and the right-hand side of the dissipative layer, where ideal MHD is valid. By consequence, $\Psi(\tau)$ should vanish at infinity ($\sim 1/\tau$), and therefore its Fourier transform exists. Fourier transforming (in the sense of generalized functions) of equation (A1) gives

$$\sigma^2\tilde{\Psi} + \Lambda\tilde{\Psi} + \mathrm{sign}\ (\Delta)\frac{d\tilde{\Psi}}{d\sigma} = -2\pi\mathrm{RHC}\,\delta(\sigma)\ \mathrm{sign}\ (\Delta)\ , \tag{A2}$$

where $\tilde{\Psi}(\sigma)$ is the Fourier transform of $\Psi(\tau)$ with respect to τ, and $\delta(\sigma)$ stands for the delta distribution. With the boundary condition that $\tilde{\Psi}(\sigma)$ has to vanish at $|\sigma| \to \infty$, the only solution to equation (A2) is

$$\tilde{\Psi}(\sigma) = -2\pi\ \mathrm{sign}\ (\Delta)\mathrm{RHC}\, H(\sigma)e^{-(\sigma^3/3 + \Lambda\sigma)\ \mathrm{sign}\ (\Delta)}\ , \tag{A3}$$

where $H(\sigma)$ is the heaviside function. Finally, the physically relevant solution to equation (A1) is the inverse Fourier transform of (A3):

$$\Psi(\tau) = -\mathrm{RHC}\int_0^{\infty} e^{-\sigma^3/3 - \Lambda\sigma}e^{i\sigma\tau\,\sin\,(\Delta)}\,d\sigma\ . \tag{A4}$$

With this analysis it is now easy to recover the solutions formulated in (11).

REFERENCES

Allan, W., White, S. P., & Poulter, E. M. 1986, Planet. Space Sci., 34, 371
Balet, B., Appert, K., & Vaclavik, J. 1982, Plasma Phys., 24(8), 1005
Cally, P. S. 1986, Sol. Phys., 103, 277
Goedbloed, J. P. 1983, Rijnhuizen Report 83-145
Goossens, M., Ruderman, M. S., & Hollweg, J. V. 1995, Sol. Phys., 157, 75
Ionson, J. A. 1978, ApJ, 226, 650
Kivelson, M. G., & Southwood, D. J. 1985, Geophys. Res. Lett., 12, 49
Mann, I. R., Wright, A. N., & Cally, P. S. 1995, J. Geophys. Res., 100, 19441
Mullan, D. J., & Johnson, M. 1995, ApJ, 444, 350
Ofman, L., & Davila, J. M. 1996, ApJ, 456, L123
Ofman, L., Davila, J. M., & Shimizu, T. 1996, ApJ, 459, L39
Ofman, L., Davila, J. M., & Steinolfson, R. S. 1995, ApJ, 444, 471
Poedts, S., Beliën, A. J. C., & Goedbloed, J. P. 1994, Sol. Phys., 151, 271
Poedts, S., & Boynton, G. C. 1996, A&A, 306, 610
Poedts, S., Goossens, M., & Kerner, W. 1989, Sol. Phys., 123, 83
Poedts, S., & Kerner, W. 1991, Phys. Rev. Lett., 66, 22
———. 1992, J. Plasma Phys., 47(1), 139
Poedts, S., Kerner, W., & Goossens, M. 1989, J. Plasma Phys., 42(1), 27
Rankin, R., Samson, J. C., & Frycz, P. 1993, J. Geophys. Res., 98, 341
Ruderman, M. S., Tirry, W., & Goossens, M. 1995, J. Plasma Phys, 54(2), 129
Sakurai, T., Goossens, M., & Hollweg, J. V. 1991, Sol. Phys., 133, 227
Sedláček, Z. 1971, J. Plasma Phys., 5, 239
Steinolfson, R. S., & Davila, J. M. 1993, ApJ, 415, 354
Wright, A. N., & Allan, W. 1996, J. Geophys. Res., submitted
Wright, A. N., & Rickard, G. J. 1995, ApJ, 444, 458
Zhu, X., & Kivelson, M. G. 1988, J. Geophys. Res., 93, 8602