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BINARY–SINGLE-STAR SCATTERING. VII. HARD BINARY EXCHANGE CROSS SECTIONS FOR ARBITRARY MASS RATIOS: NUMERICAL RESULTS AND SEMIANALYTIC FITS

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ABSTRACT

We present the first comprehensive fitting formula for exchange reactions of arbitrary mass ratios. In a comparison with numerical results, this expression is shown to be accurate in the hard binary limit to within 25% for most mass ratios. The result will be useful in forming quantitative estimates for the branching ratios of various exchange reactions in astrophysical applications. For example, it can be used to construct quantitative formation scenarios for unusual objects in globular clusters, such as binaries containing a pulsar.

Subject headings: binaries: general — celestial mechanics, stellar dynamics — pulsars: general — stars: evolution

1. INTRODUCTION

In stellar dynamics, an exchange reaction is a particular type of interaction between a binary star and an incoming third star, where one of the components of the binary is expelled and its place is taken by the incomer.

In the long history of the three-body problem, the study of exchange reactions had a curious start. Though numerical examples were given by Becker (1920) long ago, it appears that the very *possibility* of such reactions remained controversial until later numerical work published in 1975 (Marchal 1990). This is all the more curious because in that very year Hills (1975) and Heggie (1975) separately published different treatments that, taking the existence of exchange for granted, attempted to determine how its probability depended on such parameters as the initial speed of the incomer, and the masses of the stars.

Since that time, a considerable number of studies of exchange reactions have been carried out. In this paper we concentrate on the problem of determining the *cross section* for exchange, in the limit in which the initial speed of the incoming star is very low, but for all possible masses. Our study will therefore be largely complementary to previous work, which has often adopted different restrictions, e.g., encounters at zero impact parameter (Hills & Fullerton 1980), the case in which one component of the binary has very low mass (Hills & Dissly 1989), or the initial eccentricity is zero (Hills 1991, 1992). Other work of this kind will be mentioned in § 5.1, for comparison with our own data.

This paper is the seventh in a series discussing many aspects of three-body scattering in the point-mass approximation (Hut & Bahcall 1983; Hut 1983; 1993; Heggie & Hut 1993; Goodman & Hut 1993; McMillan & Hut 1996), but this is the first to deal with stars of unequal mass. For the case of equal masses, much further information on exchange cross sections will be found in earlier papers of this series, especially Papers II and IV, as well as the atlas of hard binary scattering cross sections provided by Hut (1984).

The present paper is arranged as follows. In the following section we describe the numerical software we have used for generating cross sections. Because of its highly automated yet flexible construction, this is a topic of interest in its own right. In § 3 we analyze the problem analytically, in order to understand the dependence of the exchange cross section on the masses, especially in various asymptotic regimes. It turns out to be possible to write down a single expression that accommodates all asymptotic regimes. Section 4 synthesizes all our numerical data and asymptotic theory to provide a comprehensive and simple formula that is believed to be approximately valid for all masses. In the concluding section, it is tested against previous work by Sigurdsson & Phinney (1993), Hills (1992), and Rappaport, Putney, & Verbunt (1989), and then we summarize our results.

2. NUMERICAL COMPUTATION OF EXCHANGE CROSS SECTIONS

2.1. *Software for Three-Body Scattering*

The first sets of binary–single-star scattering experiments were reported by Hills (1975) and Heggie (1975). In the former, most encounters took place at zero impact parameter. The first direct determination of accurate cross sections and reaction rates for binary–single-star scattering was made by Hut & Bahcall (1983). For each type of total or differential cross section, a detailed search of impact parameter space was performed as a pilot study, before production runs were started. The problem with the choice of impact parameter (lateral offset from a head-on collision, as measured at infinity) is this: allowing too large an impact parameter can imply a large waste of computer time on uninteresting orbits; while choosing too small an impact parameter will yield a systematic underestimate of some cross sections, since some encounters of interest will be missed.

The first *automatic* determinations of cross sections and reaction rates for binary–single-star scattering are described by McMillan & Hut (1996, hereafter Paper VI). Rather than relying on human inspection of pilot calculations, their software package includes an automatic feedback system that ensures near-optimal coverage of parameter space while guaranteeing completeness. We refer the interested reader to Paper VI for further details on the STARLAB software package. References to earlier papers on three-body scattering can be found in the recent papers by Hills (1992), Heggie & Hut (1993), Hut (1993), and Sigurdsson & Phinney (1993).

2.2. *Numerical Results*

First, we explain our notation. Let m_1 and m_2 be the masses of the components of the binary, and m_3 that of the incoming third star. Then we define $M_{12} = m_1 + m_2$, $M_{123} = m_1 + m_2 + m_3$, etc. When the incoming third body and the binary are still at a very large distance, let the semimajor axis of the binary be a and the relative speed of the third body and the barycenter of the binary be V. Then we can scale V by the critical value, V_c, for which the total energy of the triple system, in the rest frame of its barycenter, is zero, i.e., let

$$v = V/V_c \,, \tag{1}$$

where

$$V_c^2 = Gm_1 m_2 M_{123}/(M_{12} m_3 a) \,. \tag{2}$$

All of our runs have been carried out with $v = 0.1$.

We also scale the cross sections themselves, in two ways. Let Σ be the cross section for an exchange process. First, following Paper I in this series, we may scale out the gravitational focusing of the third body as it approaches the binary, and also the physical cross section of the binary itself, by defining the dimensionless cross section

$$\sigma = \frac{v^2\Sigma}{\pi a^2} \,. \tag{3}$$

This definition has the disadvantage that σ becomes very large when $m_3 \gg M_{12}$, and we have found it convenient to scale V not by V_c but by a typical speed reached by the third body when it makes a close approach to the binary. To be precise, we define a speed V_g by $V_g^2 = GM_{123}/(2a)$, and let $\bar{v} = V/V_g$. Evidently V_g is the relative speed of the third body if it falls from rest at infinity to a distance $4a$ from a body of mass M_{12}. We have chosen this slightly odd numerical factor so that $V_g = V_c$ in the case of equal masses. Thus, we define a new dimensionless cross section $\bar{\sigma} = V^2\Sigma/(\pi a^2 V_g^2)$, i.e.,

$$\bar{\sigma} = \frac{2V^2\Sigma}{\pi G M_{123}\, a} \,, \tag{4}$$

which is related to the previous definition by $\bar{\sigma} = \sigma \bar{v}^2/v^2$.

The results of our runs are displayed in Table 1, with estimates of 1 σ errors. In general, we have attempted to ensure that the total exchange cross section for each component of the binary is calculated to better than about 10%, though there are clearly cases in which the cross section must be so small that the resulting computational effort would be prohibitive. This is also illustrated in Figure 1, whose main purpose is to show the coverage of the plane of mass ratios in our numerical experiments. We covered all mass ratios up to a maximum of 2 dex (between any pair of

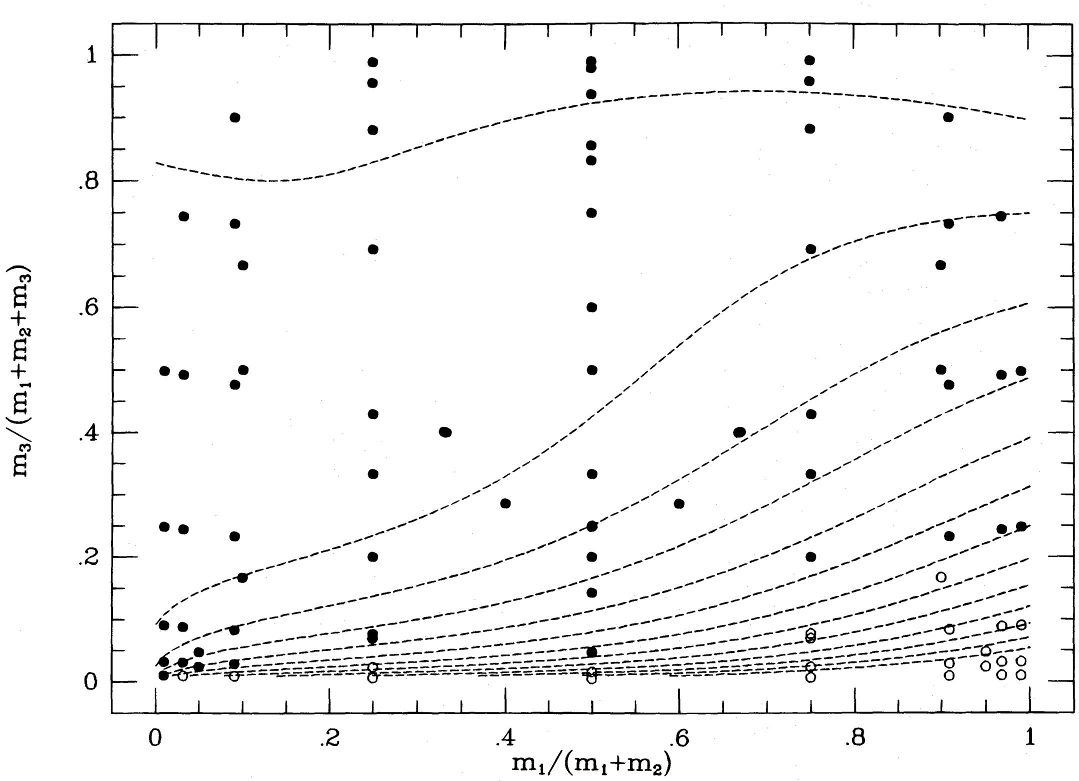

FIG. 1.—Coverage of the parameter space of mass ratios in the numerical experiments. Open circles represent experiments where the cross section was too small to be measurable. In this figure m_1 is the mass of the component that is ejected. Dashed lines are contours of the logarithm of the theoretical exchange cross section, $\log_{10} \bar{\sigma}$, given by eq. (17). The values of $\log_{10} \bar{\sigma}$ range from -5 at lower right in steps of 0.5 to 1.

TABLE 1

NUMERICAL EXCHANGE CROSS SECTIONS $\bar{\sigma}$

			Direct Exchange		Resonant Exchange	
m_1	m_2	m_3	Star 1	Star 2	Star 1	Star 2
0.500......	0.500	0.005	...	...	...	...
0.500......	0.500	0.017	...	...	...	...
0.500......	0.500	0.050	...	...	0.010 ± 0.010	...
0.500......	0.500	0.167	0.024 ± 0.012	0.024 ± 0.012	0.072 ± 0.024	0.114 ± 0.042
0.500......	0.500	0.250	0.015 ± 0.008	0.036 ± 0.016	0.326 ± 0.058	0.310 ± 0.060
0.500......	0.500	0.333	0.068 ± 0.017	0.093 ± 0.021	0.975 ± 0.135	1.065 ± 0.134
0.500......	0.500	0.500	0.357 ± 0.032	0.330 ± 0.028	1.962 ± 0.119	1.883 ± 0.121
0.500......	0.500	1.000	1.202 ± 0.076	1.165 ± 0.069	3.128 ± 0.176	2.969 ± 0.169
0.500......	0.500	1.500	1.657 ± 0.139	1.569 ± 0.128	3.625 ± 0.297	3.304 ± 0.278
0.500......	0.500	3.000	2.425 ± 0.263	2.867 ± 0.325	3.508 ± 0.427	3.539 ± 0.447
0.500......	0.500	5.000	3.134 ± 0.343	4.348 ± 0.460	4.390 ± 0.526	3.965 ± 0.525
0.500......	0.500	6.000	4.607 ± 0.372	3.796 ± 0.327	4.070 ± 0.422	4.396 ± 0.470
0.500......	0.500	15.000	6.619 ± 0.467	5.919 ± 0.433	5.001 ± 0.500	4.531 ± 0.433
0.500......	0.500	50.000	10.065 ± 0.669	9.925 ± 0.668	7.204 ± 0.737	7.505 ± 0.785
0.500......	0.500	99.000	8.594 ± 1.948	12.504 ± 2.805	14.804 ± 4.737	11.374 ± 4.133
0.333......	0.667	0.667	1.095 ± 0.092	0.339 ± 0.045	4.464 ± 0.300	0.582 ± 0.109
0.400......	0.600	0.400	0.383 ± 0.076	0.140 ± 0.037	1.972 ± 0.221	0.583 ± 0.121
0.250......	0.750	0.007	...	...	...	...
0.250......	0.750	0.025	...	...	...	...
0.250......	0.750	0.075	0.010 ± 0.010	...	0.040 ± 0.020	...
0.250......	0.750	0.083	...	...	0.099 ± 0.014	...
0.250......	0.750	0.250	0.276 ± 0.039	0.009 ± 0.005	2.189 ± 0.165	0.063 ± 0.024
0.250......	0.750	0.500	1.087 ± 0.094	0.065 ± 0.016	4.856 ± 0.302	0.115 ± 0.042
0.250......	0.750	0.750	1.359 ± 0.095	0.298 ± 0.042	5.721 ± 0.318	0.431 ± 0.101
0.250......	0.750	2.250	2.863 ± 0.088	1.840 ± 0.077	5.606 ± 0.167	1.358 ± 0.093
0.250......	0.750	7.500	4.349 ± 2.050	2.626 ± 1.700	6.129 ± 3.700	5.378 ± 3.650
0.250......	0.750	22.500	7.976 ± 0.517	7.251 ± 0.533	7.150 ± 0.600	4.501 ± 0.583
0.250......	0.750	99.000	13.916 ± 3.172	9.681 ± 2.261	14.478 ± 4.345	11.647 ± 4.554
0.100......	0.900	0.200	0.315 ± 0.050	...	3.038 ± 0.218	...
0.100......	0.900	1.000	2.058 ± 0.380	0.606 ± 0.358	4.697 ± 0.823	0.476 ± 0.364
0.100......	0.900	2.000	2.644 ± 0.243	1.004 ± 0.151	6.224 ± 0.538	1.258 ± 0.292
0.091......	0.909	0.009	...	...	...	...
0.091......	0.909	0.030	0.011 ± 0.011	...	0.109 ± 0.038	...
0.091......	0.909	0.091	0.051 ± 0.015	...	1.035 ± 0.104	...
0.091......	0.909	0.303	0.658 ± 0.045	0.009 ± 0.004	4.116 ± 0.165	0.007 ± 0.004
0.091......	0.909	0.909	2.031 ± 0.131	0.294 ± 0.051	5.710 ± 0.335	0.219 ± 0.070
0.091......	0.909	2.727	3.286 ± 0.214	1.417 ± 0.136	5.905 ± 0.387	1.216 ± 0.216
0.091......	0.909	9.091	5.257 ± 0.345	4.693 ± 0.382	6.671 ± 0.509	3.454 ± 0.436
0.050......	0.950	0.025	...	...	0.099 ± 0.049	...
0.050......	0.950	0.050	...	...	0.827 ± 0.186	...
0.032......	0.968	0.010	...	...	...	...
0.032......	0.968	0.032	0.008 ± 0.006	...	0.457 ± 0.072	...
0.032......	0.968	0.097	0.093 ± 0.028	...	2.009 ± 0.146	...
0.032......	0.968	0.323	1.000 ± 0.087	...	4.294 ± 0.250	0.012 ± 0.012
0.032......	0.968	0.968	2.264 ± 0.145	0.345 ± 0.058	5.445 ± 0.332	0.232 ± 0.069
0.032......	0.968	2.903	3.703 ± 0.237	2.210 ± 0.213	6.178 ± 0.409	1.629 ± 0.258
0.010......	0.990	0.010	...	...	0.091 ± 0.022	...
0.010......	0.990	0.033	0.008 ± 0.005	...	0.850 ± 0.097	...
0.010......	0.990	0.099	0.071 ± 0.020	...	2.095 ± 0.150	...
0.010......	0.990	0.330	0.981 ± 0.085	0.003 ± 0.002	4.002 ± 0.232	0.002 ± 0.002
0.010......	0.990	0.990	2.349 ± 0.143	0.271 ± 0.056	6.024 ± 0.337	0.244 ± 0.087

NOTE.—The columns headed "Star 1" and "Star 2" give the cross sections for exchange in which the particle of mass m_1 or m_2, respectively, is ejected.

stars in the system), in steps of 0.5 dex, as well as a few other cases.

It is helpful to distinguish two different kinds of exchange reaction (e.g., Heggie 1975), and Table 1 presents results for both. In *direct* exchange the encounter terminates promptly, and the orbits are uncomplicated; while in *resonant* exchange the three bodies form a temporary bound system, and the escaping particle emerges only after several interactions. The distinction between direct and resonant encounters is not always a clear one; the operational procedure used to classify our numerical results is presented in Paper VI.

A few other remarks about the data in Table 1 should be made at this point. First, for a few mass ratios, we have data from additional runs which are not shown here. Those shown are the data sets with the smallest errors. The additional data has been used, however, in the parameter fitting in § 4 below. Second, entries with an asterisk indicate experiments in which no events of the relevant kind were observed. Though it might be thought that it should be possible, on the basis of a large number of scattering experiments, to give an *upper bound* for the cross section of a process that produced no events, this is actually not rigorously possible because of the organization of the software, which decides on the range of impact parameters on the basis of *observed* events. If the software has no evidence of the range of impact parameters that can produce a given event, then it is possible that it can miss a range where the process is important. Therefore, this null data should simply be discarded.

3. ASYMPTOTIC THEORY OF EXCHANGE CROSS SECTIONS

In this section we address theoretically the problem of determining cross sections for the reactions discussed in this paper, i.e., exchange for hard binaries. Our aim will be to determine the way in which they scale with the parameters of the problem, especially in the extreme regimes of masses. Some of these extreme cases might seem physically implausible or unimportant, but the purpose of the theory is to try to account for trends in the numerical data and to suggest ways in which the numerical results might be extrapolated.

In general, we consider the approach of a third body whose speed "at infinity" relative to the barycenter of the binary was V, and denote the initial semimajor axis of the binary by a. We shall label the components of the binary such that m_1 is the mass of the component that is ejected as a result of the encounter, and in this section we shall generally add a subscript 1 to symbols for the cross section, in order to reinforce this convention. It must always be borne in mind that we are dealing with the case in which V is very small.

3.1. *The Case of a Massive Incoming Star*

First, we consider the regime in which both $m_1 \ll m_3$ and $m_2 \ll m_3$. In this case, the incomer is very massive, and we shall see that it is possible for a tidal encounter by the third body to unbind the binary. Since we are always considering the hard binary limit (i.e., $V \ll V_c$), the three stars cannot escape singly to infinity, and so an exchange interaction must occur. What is less obvious is how to decide which component escapes.

Let $R_p \gg a$ denote the distance of closest approach of the third body. (The subscript denotes pericenter.) At this distance, its speed relative to the barycenter of the binary is denoted by V_p and can be estimated from

$$V_p^2 \sim GM_{123}/R_p . \quad (5)$$

The duration of the encounter is of order R_p/V_p. During this period, the (tidal) acceleration of the relative motion of the binary components by the third body is of order $Gm_3 a/R_p^3$. Therefore, the change in their relative speed is of order $\Delta V_{12} \sim Gm_3 a/(R_p^2 V_p)$. This is enough to disrupt the binary if $(\Delta V_{12})^2 \gtrsim GM_{12}/a$, i.e., if

$$R_p^3 \lesssim m_3^2 a^3/(M_{123} M_{12}) . \quad (6)$$

Incidentally, we have assumed implicitly in all this that the duration of the encounter does not much exceed the period of the binary, for then the change in energy of the binary would be exponentially smaller than the estimate we have used (Heggie 1975). In fact, it is easy to show that the assumption on timescales is justified *post hoc* by equation (6).

If, as we assume throughout, the original speed of approach (V) of the third body is very small, the barycenter of the binary moves along a nearly parabolic orbit relative to m_3. As a result of the disruption of the binary, one component will now be moving in front of, and faster than, the barycenter, while the other will fall behind, and the probability that a given component is the one that moves in front is roughly the same for both components. Call this body m_f and the other component m_b. If $m_f \gg m_b$, then m_f is moving only slightly faster than the barycenter, but since the barycenter is moving on a slightly hyperbolic orbit, it follows that m_f will escape. If, on the other hand, $m_f \ll m_b$, then m_f will pick up almost all the extra velocity (ΔV_{12}) generated in the encounter, which is enough to ensure its escape. In either case, then, m_f is the component of the original binary that escapes, and, by the convention already stated, we identify this with m_1.

It might be surprising to find that the probability that a given component is the escaper does not diminish significantly when it is much more massive than the other component. This is, however, confirmed in Table 1 by such cases as $m_1 = 0.09, m_3 = 9.09$.

It remains only to estimate the cross section for this process. The impact parameter of the third body, p, when at a large distance on its incoming orbit, is related to other parameters of the encounter by approximate angular momentum conservation in its motion relative to the barycenter of the binary, which leads to $pV = R_p V_p$. Therefore, the cross section for these encounters is $\Sigma_1 \sim p^2 \sim (R_p V_p/V)^2$. [The factor $(V_p/V)^2$ accounts for the "gravitational focusing" that takes place as the third body approaches on its nearly parabolic orbit.] Using equation (5) and the expression on the right of equation (6) to estimate R_p^3, we find that $\Sigma_1 \sim (GM_{123} a/V^2)[m_3^2/(M_{123} M_{12})]^{1/3}$.

In view of our assumptions that m_1 and m_2 are very small compared with m_3, we see that an equally valid asymptotic expression is

$$\Sigma_1 \sim \frac{GM_{123} a}{V^2} \left(\frac{M_{123}}{M_{12}}\right)^{1/3} . \quad (7)$$

This will be more useful for combining with other asymptotic formulae later.

Finally, we have to dispose of the question of resonant exchange. In fact, if, after this first encounter, the third body is bound to the binary without disrupting it, thus forming a resonance, then on each subsequent encounter it is quite likely to become unbound again, and so the cross section for resonant exchange cannot greatly exceed that just estimated for direct exchange. Again, this is confirmed by data in Table 1, though it is noticeable that the cross section for ejection is now somewhat larger for the less massive star.

This discussion has been taken fairly slowly so as to introduce the various steps and considerations involved in the estimate of the exchange cross section in a certain regime. In subsequent discussions, only the distinctively different aspects will be treated in comparable detail.

There are several aspects of this kind of encounter that will be useful later (§ 3.2). The energy given to the binary will be comparable to the energy required to break it up. Therefore, the energy extracted from the relative motion of m_3 and the barycenter of the binary must also be comparable to $Gm_1 m_2/a$, and so this can be taken as an estimate of the binding energy of the new binary. It follows that the semimajor axis of the new orbit, denoted by a_{23}, may be estimated by

$$Gm_1 m_2/a \sim Gm_3 m_2/a_{23} , \quad (8)$$

i.e., $a_{23} \sim am_3/m_1$. Also, the pericenter distance of the new binary is of order R_p, and so can be expressed as

$$R_p \sim a_{23}(m_1/m_3)(M_{123}/M_{12})^{1/3} . \qquad (9)$$

It follows that the eccentricity of the new binary (e') is high, since $1 - e' \sim R_p/a_{23}$. Finally, the initial separation of m_1 and m_2 is related to the new semimajor axis by

$$R_{12}/a_{23} \sim m_1/m_3 . \qquad (10)$$

The argument that the binding energy of the new binary must be comparable to that of the original one has analogs in other capture processes. One familiar example is tidal capture in globular star clusters (Fabian, Pringle, & Rees 1975), where the encounter must be close enough to remove the kinetic energy, T, of relative motion of the two stars. In this case, it follows that the binding energy of the new binary will be comparable to T. The analogy is not precise, however, because in this case the binding is not achieved by disruption of one of the stars.

3.2. *The Case of Ejection of a Massive Star*

We move on now to the regime in which $m_2 \ll m_1$ and $m_3 \ll m_1$. The first point to notice about this case is that it may be obtained from the previous case by time reversal. In the previous case, a heavy incoming third body becomes a component in the new binary, while in the present case it is the heavy component that is ejected. (Recall that the components of the binary are named in such a way that it is the first component which is exchanged.) While the ejection of the most massive component in a three-body interaction seems unlikely, the use of time reversal yields the special circumstances in which it can happen. In fact, the encounter must occur while the two components of the binary are close to pericenter (since $R_p \ll a_{23}$ in the previous case), and the incoming star must come even closer to the lighter component, m_2.

We shall return to some aspects of a direct calculation of the cross section toward the end of this section. Mainly, however, we shall exploit the fact that time reversal leads to a process whose cross section we have already estimated, and so we may use the theory of detailed balance to estimate the required result. This theory is set out in the Appendix, and leads to the following result for the inverse process. (Here we omit the subscript 1 on the cross section, since the identity of the escaper is sufficiently defined by other aspects of the notation.)

Let $(d\Sigma/dE_{23})(E_{12} \to E_{23})$ be the differential cross section for the formation of a binary having components m_2, m_3, and energy E_{23}, from a binary having components m_1, m_2, and energy E_{12}, by exchange. Then this is related to the differential cross section for the inverse process by

$$\frac{d\Sigma}{dE_{12}}(E_{23} \to E_{12}) = \left(\frac{m_1}{m_3}\right)^{5/2}\left(\frac{M_{12}}{M_{23}}\right)^{1/2}\frac{V_3^2}{V_1^2}\left(\frac{E_{12}}{E_{23}}\right)^{-5/2} \times \frac{d\Sigma}{dE_{23}}(E_{12} \to E_{23}) . \qquad (11)$$

Here V_1 and V_3 are the speeds of the incoming single body in the two scatterings. The coefficient on the right-hand side comes from the phase space volumes associated with the reactants: for the incoming single body, this is proportional to V_i^2, and the factor $E_{ij}^{-5/2}$ is easily understood in a similar way in terms of the internal degrees of freedom of the binary (Hut 1985).

This relation involves differential cross sections, whereas equation (7) estimates a total cross section. However, we already estimated that the typical energy of the new binary in that case is $E_{23} \sim E_{12}$, and if we estimate $(d\Sigma/dE_{12})(E_{23} \to E_{12}) \sim \Sigma(E_{23} \to E_{12})/E_{12}$, we see that equation (11) yields the following estimate for the total cross section:

$$\Sigma(E_{23} \to E_{12}) \sim \left(\frac{m_1}{m_3}\right)^{5/2}\left(\frac{M_{12}}{M_{23}}\right)^{1/2}\frac{V_3^2}{V_1^2}\,\Sigma(E_{12} \to E_{23}) .$$

Substituting equation (7) in the right-hand side, we find, for the case $m_1, m_2 \ll m_3$, the estimate

$$\Sigma(E_{23} \to E_{12}) \sim \frac{GM_{123}a_{12}}{V_1^2}\left(\frac{M_{123}}{M_{12}}\right)^{1/3}\left(\frac{m_1}{m_3}\right)^{5/2}\left(\frac{M_{12}}{M_{23}}\right)^{1/2} ,$$

where a_{12} (heretofore denoted simply by a) is the semimajor axis of the binary with components 1 and 2. Now we use equation (8) to replace a_{12} by $a_{23}m_1/m_3$ and interchange the labeling of stars 1 and 2 to restore our customary labeling of the incomer. This yields the exchange cross section

$$\Sigma(E_{12} \to E_{23}) \sim \frac{GM_{123}a_{12}}{V_3^2}\left(\frac{M_{123}}{M_{23}}\right)^{1/3}\left(\frac{m_3}{m_1}\right)^{7/2}\left(\frac{M_{23}}{M_{12}}\right)^{1/2}$$

in the case m_2, $m_3 \ll m_1$. An equally valid asymptotic formula is obtained by replacing M_{12} by M_{123}, and then we see that a formula which is compatible with equation (7), and therefore valid in both the asymptotic regimes of §§ 3.1 and 3.2, is

$$\Sigma_1 \sim \frac{GM_{123}a}{V^2}\left(\frac{M_{23}}{M_{123}}\right)^{1/6}\left(\frac{m_3}{M_{13}}\right)^{7/2}\left(\frac{M_{123}}{M_{12}}\right)^{1/3} . \qquad (12)$$

Before we leave this regime, it is worth noting that the major part of the mass dependence here, i.e., the factor $(m_3/M_{13})^{7/2}$ is easily inderstood. We already saw (eq. [9], but with relabeling appropriate to time reversal) that the separation of the binary components at the time of approach of m_3 must be of order $R_{12} \sim a(m_3/m_1)(M_{123}/M_{23})^{1/3}$. Now, the probability of this (for a thermal distribution of binaries of a given energy) is of order $(R_{12}/a)^{5/2}$ when $R_{12} \ll a$ (see Paper IV). Next, the distance of closest approach of m_3 to m_2 must be of order $R_p \sim am_3/m_1$, by equation (10), and at this time the speed of the third body, which is gained mostly by falling to within a distance R_{12} of m_1, is given by $V_p^2 \sim GM_{123}/R_{12}$. It follows that the cross section $\Sigma_1 \sim (R_p^2 V_p^2/V^2)(R_{12}/a)^{5/2} \sim (GM_{123}a/V^2)(m_3/m_1)^{7/2}(M_{123}/M_{23})^{5/6}$. This is larger than our estimate in equation (12), because we have not taken into account the special circumstances of the encounter which allows m_3 to be captured by m_2, but does explain the major part of the mass dependence: all three stars must come within a separation that is of order the pericentric distance of a binary of high eccentricity. The factor $(m_3/m_1)^{7/2}$ is a measure of the phase-space volume available to such systems.

3.3. *The Case in Which a Massive Component Remains*

Finally, we turn to the regime in which both $m_1 \ll m_2$ and $m_3 \ll m_2$. We first discuss the case in which $m_3 \gg m_1$, i.e., an object of intermediate mass displaces a low-mass companion of an object of very high mass. (Recall our convention for labeling the components, which is that m_1 is the mass of the component that is ejected.) To begin, let us consider the possibility of direct exchange. Suppose m_3 approaches m_1 within a distance $R_p \ll a$. Then the speed of m_3 will be given by $V_p^2 \sim GM_{123}/a$ provided that the influence of m_1 is not dominant, i.e., provided that $R_p \gtrsim aM_{13}/M_{123}$. It follows that the speed imparted to m_1 is of order the escape speed provided that $R_p \sim aM_{13}/M_{123}$. Hence, the cross section for direct exchange is

$$\Sigma \sim \frac{GM_{123}a}{V^2}\left(\frac{m_3}{m_2}\right)^2 . \quad (13)$$

Now suppose only that the distance of closest approach between m_3 and m_1 is of order a. Then there is a significant probability that m_3 will become bound to the binary without ejecting m_1, i.e., a resonance will form. Thus, the cross section for resonance *formation* greatly exceeds the cross section for direct exchange, since our estimate of R_p for direct exchange requires that $R_p \ll a$. In fact, the cross section for formation of a resonance is simply

$$\Sigma_1 \sim \frac{GM_{123}a}{V^2} . \quad (14)$$

At this point in the evolution of the resonance, the binding energy of m_3 is of order the change in energy of m_1, which is of order $Gm_1 m_3/a$. Note that this estimate is valid no matter how small m_1 is; we are always considering the limit of extreme hardness, and even a small change in the energy of the third body can bind it to the binary if its energy at infinity was sufficiently small. On the other hand, the cross section for formation of a resonance leading to exchange cannot greatly exceed our estimate: if the closest distance of approach of m_3 greatly exceeds a, then a hierarchical triple system forms, and exchange is very improbable.

Now we must estimate the probability that the resonance will be resolved with the escape of star m_1. We think of the binding energy of this particle, E_1, as performing a random walk under the influence of repeated passages by star m_3. The typical change in E_1 is of order $Gm_3 m_1/a$. Because we are assuming that $m_3 \gg m_1$, it might be thought that the mean effect would be to systematically unbind m_1, by a kind of mass segregation. However, the mean change in E_1, taken over an ensemble of such systems, is actually of second order in the ratio of the masses: in the approximation of first-order perturbation theory, time reversal shows that for each change in E_1, there is a system in which the change has the opposite sign. Therefore, we shall assume that the mean change for a given system is negligible.

As an aside, it is worth mentioning here that the system is a kind of hierarchical triple. Usually this term is used in reference to a binary about which a third star revolves on a large elliptical orbit that is well separated spatially from the binary. In that case, the perturbation of the third body is weak because the orbit of the third body is large. In the present case, there is no such spatial separation, but still the perturbation of the third body by the binary is weak, and this is due to the low mass of one component of the binary.

We must estimate the probability that E_1, starting from the value of order $Gm_1 m_2/a$, may randomly walk, in steps of order $Gm_1 m_3/a$, to the value 0 without first reaching the value $Gm_1 m_2/a + Gm_1 m_3/a$ (as m_3 would then escape again). Now, the probability that a one-dimensional random walk exists from a given boundary is a linear function of the initial position, varying from unity at the boundary of interest to zero at the opposite boundary. In the present case, the boundaries are at $E_1 = Gm_1 m_2/a + Gm_1 m_3/a$ and 0, and so the probability of escape at the boundary $E_1 = 0$, starting at $E_1 = Gm_1 m_2/a$, is of order m_3/m_2. This can be estimated equally well as M_{13}/M_{123}, and so it follows, using equation (14), that the cross section for resonant exchange is

$$\Sigma_1 \sim \frac{GM_{123}a}{V^2}\left(\frac{M_{13}}{M_{123}}\right) .$$

It also follows that we obtain a form that is asymptotically correct in all regimes studied so far if we modify equation (12) to

$$\Sigma_1 \sim \frac{GM_{123}a}{V^2}\left(\frac{M_{23}}{M_{123}}\right)^{1/6}\left(\frac{m_3}{M_{13}}\right)^{7/2}\left(\frac{M_{123}}{M_{12}}\right)^{1/3}\left(\frac{M_{13}}{M_{123}}\right) , \quad (15)$$

or, if everything is normalized by the total mass of the triple system,

$$\Sigma_1 \sim \frac{GM_{123}a}{V^2}\left(\frac{M_{23}}{M_{123}}\right)^{1/6}\left(\frac{m_3}{M_{123}}\right)^{7/2} \times \left(\frac{M_{123}}{M_{12}}\right)^{1/3}\left(\frac{M_{13}}{M_{123}}\right)^{-5/2} .$$

We also note that the energy of the new binary will be comparable to that of the initial binary.

Observe that our estimate for resonant exchange is indeed larger than the estimate, equation (13), for direct exchange. The fact that this is true in the mass regime under discussion is also illustrated in Table 1 by such cases as $m_1 = 0.01, m_3 = 0.099$.

Finally, we turn to the case $m_3 \ll m_1 \ll m_2$, which is the time reversal of the case just considered. Though the foregoing argument for the formation of a resonance goes through, it is now very unlikely to be resolved by exchange. We can, however, estimate the cross section for resonant exchange by detailed balance, using the same method as in § 3.2. The result is that

$$\Sigma \sim \frac{GM_{123}a}{V^2}\left(\frac{m_3}{m_1}\right)^{7/2}\left(\frac{M_{23}}{M_{12}}\right)^{1/2}\left(\frac{M_{13}}{M_{123}}\right) ,$$

and we see that equation (15) is, once again, a valid asymptotic result in this regime. *We therefore adopt equation* (15) *as a cross section whose form is valid in all regimes.* Henceforth, we drop the subscript 1 on Σ and remind the reader that m_1 is the mass of the component of the original binary that is ejected.

4. SYNTHESIS OF NUMERICAL AND ANALYTICAL RESULTS

4.1. *A Test of the Asymptotic Formulae*

Though we already made a number of qualitative remarks relating the theory of § 3 to the numerical data in Table 1, it is now time for a more quantitative study. Our aim is not yet to provide a fitting formula for the data (which we take up in § 4.2), but nevertheless we shall generalize equation (15) slightly. Using the notation of equation

(4), we may rewrite equation (15) as

$$\bar{\sigma} \sim \left(1 + \frac{m_1}{M_{23}}\right)^{-1/6}\left(1 + \frac{m_1}{m_3}\right)^{-7/2}\left(1 + \frac{m_3}{M_{12}}\right)^{1/3} \times \left(1 + \frac{m_2}{M_{13}}\right)^{-1} .$$

In order to allow different multiplicative constants in the different asymptotic regimes, we generalize this to

$$\bar{\sigma} \sim a_1\left(a_2 + \frac{m_3}{M_{12}}\right)^{1/3}\left(a_3 + \frac{m_1}{m_3}\right)^{-7/2}\left(a_4 + \frac{m_1}{M_{23}}\right)^{-1/6} \times\left(a_5 + \frac{m_2}{M_{13}}\right)^{-1} , \quad (16)$$

where the a_i are constants. There is then one asymptotic form for the regime of § 3.1, one for that of § 3.2, and one each for the two regimes in § 3.3 (i.e., the regimes in which $m_1 \ll m_3$ and $m_1 \gg m_3$).

The formula in equation (16) has been fitted to the *logarithm* of the data in Table 1 using least squares, the standard deviation in the logarithm being estimated by the relative error. In cases where no cross section was measurable, the data point was assigned zero weight.

Since the aim of this exercise was to detect possible errors in the theoretical asymptotic form, we searched for deviations (between the results of this formula and the experimental data) that were significant ("2 σ"), large (above 1 in the natural logarithm), and in an extreme mass regime. We illustrate the results by considering in a little detail the set of data in which the discrepancies were largest. This was the case $m_2 \ll m_3 \simeq m_1/3$, as illustrated by Table 2. The numerical results show a trend, but not a significant one, and a constant value in the numerical data (as predicted by the formula) is not ruled out at the 20% level. A very similar set of results, but with smaller discrepancies and in the opposite sense, is found in the case $m_2 \ll m_1 = m_3$. Apart from these, the only systematic, large discrepancies occur at one or two data points where the masses are comparable, and the asymptotic formulae need not apply. In conclusion, then, there is no evidence that the theoretical formula is inconsistent, in the appropriate regimes, with trends in the numerical data.

4.2. *A Seminumerical Fitting Formula*

The theoretical results of § 3 are intended to provide the asymptotic dependence of the exchange cross section on the masses of the participants, but do not even attempt to provide numerical coefficients for these. The numerical data, on the other hand, apply to only discrete points in the parameter space of the masses. An obvious way of synthesizing the two kinds of result is to adopt a form with the same asymptotic properties as the analytical result, but with additional terms that are chosen to optimize the agreement with the numerical data. In a sense, that is what was done in the previous section, but with a limited degree of flexibility. Here we adopt a more general approach that could be extended more or less arbitrarily.

Again, we switch to $\bar{\sigma}$, defined by equation (4), and generalize equation (15) to

$$\bar{\sigma} = \left(\frac{M_{23}}{M_{123}}\right)^{1/6}\left(\frac{m_3}{M_{13}}\right)^{7/2}\left(\frac{M_{123}}{M_{12}}\right)^{1/3}\left(\frac{M_{13}}{M_{123}}\right) \times \exp\left(\sum_{m,n;m+n=0}^{m+n=N} a_{mn}\mu_1^m\mu_2^n\right) , \quad (17)$$

where the a_{mn} are constants,

$$\mu_1 = m_1/M_{12} , \quad \text{and} \quad \mu_2 = m_3/M_{123} . \quad (18)$$

These two parameters entirely span the possible ranges of mass ratios in the unit square $0 \le \mu_1, \mu_2 \le 1$. The exponential is used to constrain the function to be positive and also to avoid altering the asymptotic character of the leading expression. By taking larger values of the highest power N, it would be possible, in principle, to improve the fit to arbitrary accuracy.

We have fitted formulae with $N \le 5$, and at the largest value, the value of χ^2 is 133, which, considering the number of degrees of freedom, is still rather large. (The number of data points is 126, and the number of free coefficients is 21.) Nevertheless, we suggest the use of a cubic polynomial in the exponential in equation (17), with the coefficients given in Table 3. As these 10 coefficients are quoted to two decimal places, the maximum relative error in the evaluation of $\bar{\sigma}$ is about 5%. This is adequate in view of the accuracy of the fit, which is discussed further below.

One worry about using a single polynomial for any kind of interpolation is the possibility of large oscillations between data points, but in fact the polynomial with the above coefficients is well behaved. Its range is about five, i.e., $\bar{\sigma}$ varies by a factor of about 100 by the effect of this polynomial. Its most noticeable feature is a minimum value at $\mu_1 = 1$, $\mu_2 = 0$, i.e., the regime $m_1 \gg M_{23}$. The cross section is very small in this region anyway, because of the factor $(m_3/M_{13})^{7/2}$ in equation (17). This can also be seen in Figure 1, which shows contours of log $\bar{\sigma}$.

The fitting formula is quite successful. For half of our measurements, the result is accurate to better than 10%, and for about 75% of our measurements, it is better than 20%. Of the remaining measurements, there are some in which the disagreement exceeds two standard deviations, and they are shown in Figure 2. At each point, the label gives the relative error in the sense $(\bar{\sigma}_{th} - \bar{\sigma}_{num})/\bar{\sigma}_{th}$, where $\bar{\sigma}_{th}$ and $\bar{\sigma}_{num}$ are the theoretical value (i.e., from eq. [17]) and the numerical value (from Table 1), respectively. Thus, a

TABLE 2

EXAMPLE OF DISCREPANT RESULTS

			$\log_{10}\bar{\sigma}$	
m_1	m_2	m_3	Formula	Table 1
0.99......	0.0099	0.33	−0.65	-2.30 ± 0.25
0.97......	0.032	0.32	−0.65	-1.92 ± 0.43
0.91......	0.091	0.30	−0.65	-1.80 ± 0.15

TABLE 3

COEFFICIENTS FOR A SEMINUMERICAL EXCHANGE CROSS SECTION

n	$M = 0$	$M = 1$	$M = 2$	$M = 3$
0......	3.70	7.49	−15.49	3.07
1......	−1.89	−2.93	13.15	...
2......	−2.92	−5.23	...	...
3......	3.12	...	...	...

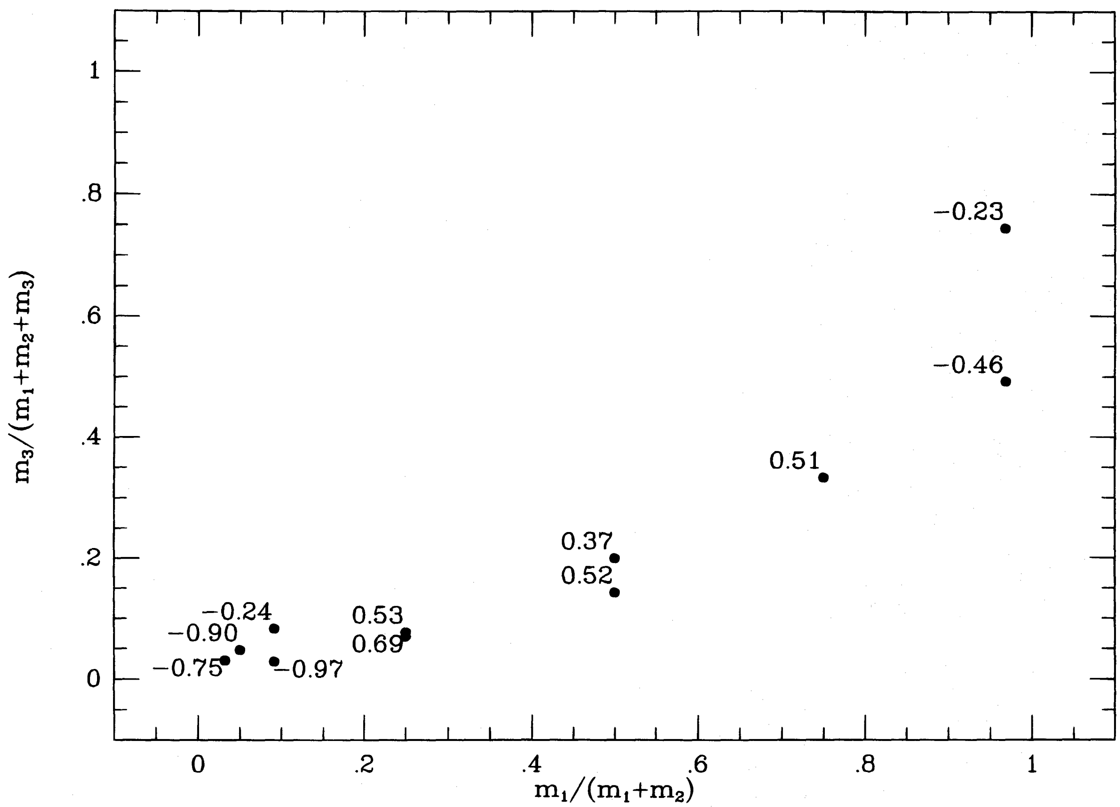

FIG. 2.—Data points where the fit of the semianalytical formula, eq. (17), is relatively poor. At each value of the mass ratio where the relative error exceeds both 20% and two standard deviations, the relative error is printed.

value of -1 would mean that $\bar{\sigma}_{\rm th} = 0.5\bar{\sigma}_{\rm num}$, while the extreme positive value of 0.69 means that $\bar{\sigma}_{\rm th} \simeq 3\bar{\sigma}_{\rm num}$. Note that the cross sections around this point are very low (Fig. 1), which has two consequences: first, that the standard deviation of the numerical cross section is comparable to the cross section itself, and, second, that exchange may be unimportant in this regime. Indeed, for all but two of the points plotted in Figure 2, the discrepancy between the formula and the numerical data is less than three standard deviations. The two exceptions are the points labeled 0.53 and -0.75. It is also reassuring that almost all the discrepant points are surrounded by data points (Fig. 1) for which the agreement between the numerical and semitheoretical results is satisfactory by the above criteria. Thus, the errors are localized. Nevertheless, it is evident that there are some significant and systematic trends in this data, and the possible effect of these discrepancies should be assessed in any application of the fitting formula.

5. DISCUSSION AND CONCLUSIONS

5.1. *Comparison with Other Authors*

Before summarizing our findings, our immediate task will be a quantitative comparison of our fitting formula with some existing results in the literature. The first of these that we shall examine is the paper of Sigurdsson & Phinney (1993). They give results for different speeds V, and we have chosen the data for the lowest speed, since the validity of our conclusions is restricted to the case of hard binaries. Data in their Tables 3A and 3B have been normalized, where necessary, to the cross section σ (eq. [3]) and collected in our Table 4. Also included are results of our fitting formula, equation (17), again converted to σ.

Unfortunately, Sigurdsson & Phinney do not give estimates of the errors of their results, and in most of their runs the initial eccentricity of the binary was chosen to be zero, and so only an informal comparison is possible. The agreement is often quite good and almost always within a factor of 2. In the cases where the disagreement is most serious, it is probably attributable to the different choice of initial eccentricity distribution. For example, in the equal mass case our fitting formula agrees with our numerical data to within 10%. In the case $m_2 \ll m_1 = m_3$, however, where their result for ejection of the low-mass component falls below that of our fitting formula, Sigurdsson & Phinney have missed a significant fraction of exchange encounters by

TABLE 4

COMPARISON WITH RESULTS OF SIGURDSSON & PHINNEY FOR σ

			Sigurdsson & Phinney		This Paper	
m_1	m_2	m_3	Star 1	Star 2	Star 1	Star 2
1.0......	0.8	1.0	1.5	2.8	1.88	3.37
1.0......	0.4	1.0	1.2	7.0	1.67	11.2
1.0......	0.2	1.0	1.6	15	1.85	23.9
1.0......	0.1	1.0	2.6	28	2.61	44.1
1.0......	0.05	1.0	5.0	50	4.35	80.8
1.0......	0.025	1.0	10	100	7.93	152
1.0......	0.0125	1.0	20	170	15.2	295
1.0......	1.0	0.4	0.060	0.060	0.128	0.128
1.0......	1.0	1.0	1.1	1.1	1.96	1.96
0.5......	0.35	1.0	5.5	9.8	8.07	14.4
1.0......	0.35	0.5	0.1	2.8	0.183	3.78

NOTE.—The columns headed "Star 1" and "Star 2" give the cross sections for exchange in which the particle of mass m_1 or m_2, respectively, is ejected.

too small a choice of the maximum impact parameter (E. S. Phinney 1995, private communication). The fact that each of their data points is a weighted average over a *range* of speeds V may complicate the comparison further.

Now, we turn to data presented by Hills (1992) for the case in which $m_1 = m_2$. The initial eccentricity in his experiments was again zero, but we find that our results agree with Hills' to within about a factor of 2 over the entire range for which he found exchange events, i.e., $0.3 \lesssim m_3/m_1 \lesssim 10^4$. Hills' result exceeds ours except at the lowest mass ratio in this range, and the discussion at the end of § 3.2 suggests that the different choices of initial eccentricity provide a plausible explanation for this last fact.

Finally, in this section we compare our results with those of Rappaport et al. (1989). They computed the exchange cross section, by numerical scattering experiments, for a sequence of binary pulsars. The sequence is characterized either by the orbital period or the mass of the low-mass companion of the neutron star. The cross section was computed for an environment containing a stellar population drawn from dynamical models of two globular clusters. Here we restrict attention to their binary of shortest period (3 days), since our results are restricted to the hard binary regime and to the model of ω Cen, for illustration. The results of Rappaport et al. (1989) give the dimensionless scattering cross sections $\Sigma/(\pi a^2) = 1.3$ and 55, for ejection of the neutron star and low-mass companion, respectively. Typical uncertainties are about 40% and 15%, respectively.

For our comparison we have used equation (17) for each of the 10 components in the stellar population listed by Rappaport et al. (1989) and have summed the contributions, account being taken of their relative number density and velocity dispersions. Expressed in terms of the quantity computed by them, our results are $\Sigma/(\pi a^2) = 1.37$ and 76.0. The agreement is acceptable, considering the typical errors in all results, and the fact that the result of Rappaport et al. (1989) applies to an initially circular binary. It also illustrates the utility of our results, as the cross section could be obtained for any reasonable stellar population with little extra work.

5.2. *Conclusions*

This paper is a contribution to the theory of three-body classical gravitational scattering. This is a large topic with an extensive literature, but it is the application to the dynamics of globular star clusters that has provided the focus for our work. From the point of view of this application, one of the most important processes is exchange, whereby an incoming star ejects one component of a binary and forms a new binary with the other component. In the context of star clusters, we are also mainly concerned with encounters with hard binaries, which are too energetic for an encounter to break up the system into three single stars. Finally, this application dictates the importance of understanding scattering in a system where the stars may have quite widely differing masses.

The main result of this paper is a semianalytical cross section for exchange, in the hard binary limit, for all possible masses. It has been derived partly from theoretical considerations and partly from extensive new numerical data on scattering events. The theory allowed us to estimate the dependence of the cross section on the masses, in various limiting cases, and the numerical data showed how the theoretical results can be parameterized so as to provide a better fit, including the cases where the masses are comparable.

The result is given in equation (17) with coefficients in Table 3, and we now summarize the way in which this information may be used. Suppose a star of mass m_3 approaches a binary with components of mass m_1, m_2, and that its speed, while still at a large distance from the binary, is V relative to the binary. Let the initial semimajor axis of the binary be a. Then the cross section for events in which the component of mass m_1 is ejected, leaving a binary consisting of the other two stars, can be computed in the following way. First, compute $M_{12} = m_1 + m_2$, and also M_{23}, M_{13}, and M_{123}, defined similarly. Next, compute $\bar{\sigma}$ from equation (17), where $N = 3$, μ_1 and μ_2 are defined in equations (18) and the coefficients are taken from Table 3. (Note that the exponential in eq. [17] is simply that of the cubic $a_{00} + a_{10}\mu_1 + a_{01}\mu_2 + \cdots + a_{12}\mu_1\mu_2^2 + a_{03}\mu_2^3$.) Then the required cross section is given by solving equation (4) for Σ. This result is approximately valid provided that the binary is hard, i.e., $v^2 \ll 1$, where v is defined by equation (1).

In astrophysical units, this can all be summarized in the formula

$$\Sigma = 1.39\left(\frac{a}{0.1\ \mathrm{AU}}\right)\left(\frac{10\ \mathrm{km\ s^{-1}}}{V}\right)^2\left(\frac{M_{123}}{M_\odot}\right)\left(\frac{M_{23}}{M_{123}}\right)^{1/6} \times\left(\frac{m_3}{M_{13}}\right)^{7/2}\left(\frac{M_{123}}{M_{12}}\right)^{1/3}\left(\frac{M_{13}}{M_{123}}\right) \times \exp(3.70 + 7.49\mu_1 - 1.89\mu_2 - 15.49\mu_1^2 - 2.93\mu_1\mu_2 - 2.92\mu_2^2 + 3.07\mu_1^3 + 13.15\mu_1^2\mu_2 - 5.23\mu_1\mu_2^2 + 3.12\mu_2^3)\ \mathrm{AU}^2\,, \quad (19)$$

where $\mu_1 = m_1/M_{12}$ and $\mu_2 = m_3/M_{123}$. The condition that the binary is hard is

$$0.011\left(\frac{V}{10\ \mathrm{km\ s^{-1}}}\right)^2\left(\frac{a}{0.1\ \mathrm{AU}}\right)\left(\frac{M_{12}m_3M_\odot}{m_1m_2M_{123}}\right) \ll 1\,. \quad (20)$$

(The last factor is dimensionless, and the factor $M_\odot$ may be omitted if the masses are expressed in units of the solar mass.)

For example, consider a binary pulsar consisting of a neutron star of mass 1.4 $M_\odot$ and a white dwarf companion of mass 0.2 $M_\odot$ in an orbit of period 10 days. Suppose it encounters a 10 $M_\odot$ black hole with a relative speed of 10 km s^{-1}. Then $a \simeq 0.11$ AU, $\mu_2 \simeq 0.862$, and the left-hand side of equation (20) evaluates to 0.06. To compute the cross section for ejection of the neutron star, we have $\mu_1 = 0.875$, and so equation (19) gives $\Sigma \simeq 120$ AU^2. Surprisingly, perhaps, the cross section is not much smaller than that for ejection of the low-mass companion; this is obtained by setting $\mu_1 = 0.125$ and is approximately 200 AU^2. Thus, of all encounters leading to exchange, roughly one-third lead to ejection of the companion of higher mass, for these parameters.

Some cautionary remarks are now in order. First, the cross section in equation (19) is a statistical result, and one of the averages that is implicitly performed in our work is over the initial eccentricity of the binary. It is assumed to have the "thermal" distribution $f(e) = 2e$. The limited evidence available (§ 5.1) suggests that this affects the result by at most a factor of 2, except in regimes where exchange is probable only when the eccentricity is high. From the discussion of § 3.2 (which also applies with some changes to § 3.3), these events are those in which $m_3 \ll m_1$; in such circumstances exchange is actually very rare anyway, and so this issue is unlikely to be important.

The second precaution concerns the error in the fitting formula. We have found that it usually agrees with numerical data to better than 20%, but that there are a few places where the error can apparently exceed 50%. These are illustrated in Figure 2, and it may be advisable to check whether events in these areas of parameter space are of importance in a given application.

Finally, no one who makes use of these results in applications will need to be reminded that they apply to the point-mass approximation. In some cases the results would be drastically different for stars of finite radius.

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APPENDIX

DETAILED BALANCE FOR EXCHANGE CROSS SECTIONS

The theory of detailed balance is described in some generality in Heggie (1975), though it is expressed in terms of rate functions, i.e., the integral of a cross section over a Maxwellian distribution of velocities, and does not explicitly deal with exchange reactions. Detailed balance is also described in terms of cross sections in Paper III in this series, though in the case of equal masses. Since the integration over a Maxwellian distribution is essentially a Laplace transform, it is possible to obtain the result for cross sections from the result in Heggie (1975), and this will be the starting point for the following treatment. We have verified that a direct derivation for cross sections for exchange reactions with different masses leads to the same result.

With some changes of notation, the result presented in Heggie (1975) can be written as

$$\frac{1}{2} n_1 n_2 n_3 \left(\frac{\pi}{kT}\right)^{3/2} (m_1 m_2)^3 |E_{12}|^{-5/2} \frac{dR}{dE'_{12}} (E_{12} \to E'_{12}) \exp\left(-\frac{E_{12}}{kT}\right) = \frac{1}{2} n_1 n_2 n_3 \left(\frac{\pi}{kT}\right)^{3/2} (m_1 m_2)^3 |E'_{12}|^{-5/2} \times \frac{dR}{dE_{12}} (E'_{12} \to E_{12}) \exp\left(-\frac{E'_{12}}{kT}\right), \quad \text{(A1)}$$

where $dR(E_{12} \to E'_{12})$ is the rate (per unit density of reactants) of reactions that change the binding energy of a binary from E_{12} to a value E'_{12} within a range of size dE'_{12}, T is the kinetic temperature, and n_i is the number density of stars with mass m_i; these cancel from this equation and are irrelevant in what follows. Equation (A1) is appropriate to encounters that do not lead to exchange, and the use of the labels 1 and 2, which identify the components of the binary, seems pedantic at this stage, but it becomes useful when we go on to discuss exchange.

Now, the rate function can be defined in terms of a differential cross section by

$$\frac{dR}{dE'_{12}} (E_{12} \to E'_{12}) = \left(\frac{2\pi kT M_{123}}{m_3 M_{12}}\right)^{-3/2} \int V_3 \exp\left(-\frac{m_3 M_{12} V_3^2}{2M_{123} kT}\right) \frac{d\Sigma}{dE'_{12}} (E_{12} \to E'_{12}) d^3 V_3 \,.$$

Then we can find the detailed balance relation for the differential cross section by substituting this integral into equation (A1) and inserting a delta function $\delta(m_3 M_{12}/(2M_{123}) V_3^2 + E_{12} - E)$, which isolates interactions involving systems with total energy E in their barycentric frame. (Primed variables are used on the right-hand side of eq. [A1].) Canceling symmetric functions of the masses, the result we obtain is

$$(m_1 m_2)^3 (m_3 M_{12})^{1/2} |E_{12}|^{-5/2} V_3^2 \frac{d\Sigma}{dE'_{12}} (E_{12} \to E'_{12}) = (m_1 m_2)^3 (m_3 M_{12})^{1/2} |E'_{12}|^{-5/2} V_3'^2 \frac{d\Sigma}{dE_{12}} (E'_{12} \to E_{12}) \,.$$

Now, we observe that a similar relation holds for exchange reactions, provided that the masses are correctly identified. Thus,

$$(m_1 m_2)^3 (m_3 M_{12})^{1/2} |E_{12}|^{-5/2} V_3^2 \frac{d\Sigma}{dE'_{23}} (E_{12} \to E'_{23}) = (m_2 m_3)^3 (m_1 M_{23})^{1/2} |E'_{23}|^{-5/2} V_1'^2 \frac{d\Sigma}{dE_{12}} (E'_{23} \to E_{12}) \,.$$

We can now drop the primes, as the start and end states are sufficiently identified by the subscripts, and we deduce that

$$\frac{d\Sigma}{dE_{12}} (E_{23} \to E_{12}) = \left(\frac{m_1}{m_3}\right)^{5/2} \left(\frac{M_{12}}{M_{23}}\right)^{1/2} \frac{V_3^2}{V_1^2} \left(\frac{E_{12}}{E_{23}}\right)^{-5/2} \frac{d\Sigma}{dE_{23}} (E_{12} \to E_{23}) \,,$$

which is equation (11) in this paper.

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